The Hodge-Arakelov Theory of Elliptic Curves: Global Discretization of Local Hodge Theories

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Introduction

§1. Statement of the Main Results

The main result of this paper is a *Comparison Theorem* (cf. Theorem A below) for elliptic curves, which states roughly that:

> The space of "polynomial functions" of degree (roughly) $< d$ on the *universal extension of an elliptic curve* maps isomorphically via restriction to the space of (set-theoretic) functions on the *d-torsion points* of the universal extension.

This rough statement is essentially precise for (smooth) elliptic curves over fields of characteristic zero. For elliptic curves in mixed characteristic and degenerating elliptic curves, this statement may be made precise (i.e., the restriction map becomes an isomorphism) if one *modifies the "integral structure" on the space of polynomial functions* in an appropriate fashion. Similarly, in the case of elliptic curves over the complex numbers, one can ask whether or not one obtains an *isometry* if one puts natural archimedean metrics on the spaces involved. In this paper, we also compute precisely *what modification to the integral structure/metrics* in all of these cases (i.e., at p-adic and archimedean primes, and for degenerating elliptic curves) *is necessary to obtain an isomorphism* (or something very close to an isomorphism).

In characteristic zero, the universal extension of an elliptic curve may be regarded as the *de Rham cohomology of the elliptic curve*, with coefficients in the sheaf of invertible functions on the curve. On the other hand, the torsion points of the elliptic curve may be regarded as a portion of the *étale cohomology of the elliptic curve*. Thus, one may regard this Comparison Theorem as a sort of isomorphism between the de Rham and étale cohomologies of the elliptic curve, given by considering *functions* on each of the respective cohomology spaces. When regarded from this point of view, *this Comparison Theorem may be thought of as a sort of discrete or Arakelov-theoretic analogue of the usual comparison theorems between de Rham and étale/singular cohomology* in the complex and p-adic cases. This analogy with the "classical" local comparison theorems can be made very precise, and is one of the main topics of Chapter IX.

> Using this point of view, we apply the Comparison Theorem to construct a *global/Arakelov-theoretic analogue* for elliptic curves over number fields of the Kodaira-Spencer morphism of a family of elliptic curves over a geometric base.

This *arithmetic Kodaira-Spencer morphism* is studied in Chapter IX.

Suppose that E is an *elliptic curve over a field* K *of characteristic zero.* Let d be a positive integer, and $\eta \in E(K)$ a torsion point of order not dividing d. Write

$$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(d \cdot [\eta])$$

for the line bundle on E corresponding to the divisor of multiplicity d with support at the point η. Write

$$E^\dagger \to E$$

for the *universal extension of the elliptic curve*, i.e., *the moduli space of pairs* $(\mathcal{M}, \nabla_{\overline{M}})$ *consisting of a degree zero line bundle* $\mathcal{M}$ *on* E, *together with a connection* $\nabla_{\overline{M}}$. Thus, $E^\dagger$ is an affine torsor on E under the module ω_E of invariant differentials on E. In particular, since $E^\dagger$ is (Zariski locally over E) the spectrum of a polynomial algebra in one variable with coefficients in the sheaf of functions on E, it makes sense to speak of the "relative degree over E" – which we refer to in this paper as the *torsorial degree* – of a function on $E^\dagger$. Note that (since we are in characteristic zero) the subscheme ${}_dE^\dagger \subseteq E^\dagger$ of d-torsion points of $E^\dagger$ maps isomorphically to the subscheme ${}_dE \subseteq E$ of d-torsion points of E. Then in its simplest form, the main result of this paper states the following:

Theorem $\mathrm{A}^{\mathrm{simple}}$**.** *Let* E *be an elliptic curve over a field* K *of characteristic zero. Write* $E^\dagger \to E$ *for its universal extension. Let* d *be a positive integer, and* $\eta \in E(K)$ *a torsion point whose order does not divide* d. *Write* $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(d \cdot [\eta])$. *Then the natural map*

$$\Gamma(E^\dagger, \mathcal{L})^{<d} \to \mathcal{L}|_{{}_dE^\dagger}$$

given by restricting sections of $\mathcal{L}$ *over* $E^\dagger$ *whose torsorial degree is* $< d$ *to the* d*-torsion points of* $E^\dagger$ *is a bijection between* K*-vector spaces of dimension* d^2.

The remainder of the main theorem essentially consists of specifying precisely *how one must modify the integral structure of* $\Gamma(E^\dagger, \mathcal{L})^{<d}$ *over more general bases* in order to obtain an isomorphism at the finite and infinite primes of a number field, as well as for degenerating elliptic curves.

To state the main theorem in its more general form, it is natural to work over a *fine noetherian log scheme* $S^{\log}$ (cf. [Kato]). Over the base $S^{\log}$, we consider what we call a *log elliptic curve*

$$C^{\log} \to S^{\log}$$

(cf. Chapter III, Definition 1.1), i.e., the result of pulling back, via some morphism $S^{\log} \to (\overline{\mathcal{M}}_{1,0}^{\log})_{\mathbf{Z}}$, the *universal log curve* $\mathcal{C}^{log} \to (\overline{\mathcal{M}}_{1,0}^{\log})_{\mathbf{Z}}$. Here, $(\overline{\mathcal{M}}_{1,0}^{\log})_{\mathbf{Z}}$ is the log moduli stack of elliptic curves over $\mathbf{Z}$, equipped with its natural log structure defined by the divisor at infinity, and $\mathcal{C} \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ is the unique (proper) semi-stable curve of genus 1, which we equip with the log structure defined by the divisor in $\mathcal{C}$ which is the pull-back of the divisor at infinity of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$.

In the general version of the main theorem, we must use line bundles – which play the role of the line bundle $\mathcal{L}$ in Theorem $\mathrm{A}^{\mathrm{simple}}$ – equipped with a *metric* as in [Zh] in a neighborhood of the divisor at infinity $D \subseteq S$. These metrized line bundles "$\overline{\mathcal{L}}$" live on a "Zhang-theoretic version" $E_{\infty,S} \to S_\infty$ of $C^{\log} \to S^{\log}$ (cf. Chapter IV, §4,5), and are discussed in detail in Chapter V, §1. For *smooth elliptic curves*, these line bundles are *exactly the same* as the line bundle "$\mathcal{L}$" of Theorem $\mathrm{A}^{\mathrm{simple}}$.

Next, it turns out that at finite primes the universal extension $E^\dagger \to E$ is not quite adequate; instead, one must modify it by *multiplying the ω_E-portion of $E^\dagger$ by a factor of d* (cf. Chapter V, §2). The resulting object $E^\dagger_{[d]} \to E$ is an ω_E-torsor which coincides with $E^\dagger$ over bases on which d is invertible. In a neighborhood of the divisor at infinity, one must consider a version of $E^\dagger_{[d]}$ which is compatible with Zhang's theory of metrized line bundles. This version of $E^\dagger_{[d]}$ is denoted by

$$E^\dagger_{\infty,[d]}$$

(cf. Chapter V, §2). *For smooth elliptic curves in characteristic zero, $E^\dagger_{\infty,[d]}$ is the same as $E^\dagger$.*

In the general form of Theorem A, we would like to consider *sections of the metrized line bundle $\overline{\mathcal{L}}$ over $E^\dagger_{\infty,[d]}$ of torsorial degree $< d$.* It turns out, however, that if one just considers the usual global section or push-forward functor in the naive sense, then one does not get the desired isomorphism at finite or degenerating primes. In order to get the desired isomorphism at such primes, one must modify the integral structure at those primes. The resulting *push-forward functor* is denoted

$$(f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\infty,\mathrm{et}\}$$

(cf. Chapter VI, Definition 1.3). For *smooth elliptic curves in characteristic zero*, this push-forward is the same as the usual one. Finally, by restricting such sections of $\overline{\mathcal{L}}$ over $E^\dagger_{\infty,[d]}$ to the d-torsion points ${}_dE^\dagger_\infty$ in $E^\dagger_{\infty,[d]}$, one obtains an *evaluation map*

$$\Xi\{\infty,\mathrm{et}\} : (f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\infty,\mathrm{et}\} \to (f_S)_*(\overline{\mathcal{L}}|_{({}_dE^\dagger_\infty)})$$

(cf. Chapter V, Proposition 2.2; Chapter VI, Theorem 3.1 (1)). For *smooth elliptic curves in characteristic zero*, this is the same as the map considered in Theorem $\mathrm{A}^{\mathrm{simple}}$.

We are now ready to state the *main theorem of this paper* (cf. Chapter VIII, §0, Theorem A):

Theorem A. (The Hodge-Arakelov Comparison Isomorphism) *Let $d, m \geq 1$ be integers such that m does* **not** *divide d. Suppose that $S^{\log}$ is a fine noetherian log scheme, and let*

$$C^{\log} \to S^{\log}$$

be a **log elliptic curve** *over $S^{\log}$ such that the divisor at infinity $D \subseteq S$ (i.e., the pull-back of the divisor at infinity of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ via the classifying morphism $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$) is a Cartier divisor on S. Also, let us assume that étale locally on the completion of S along D, the pull-back of the Tate parameter q to this completion admits a d-th root, and that we are given a* **torsion point**

$$\eta \in E_{\infty,S}(S_{\infty})$$

of order precisely *m which defines line bundles $\overline{\mathcal{L}}_{\mathrm{st},\eta}$, $\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta}$ (cf. Chapter V, §1). If d is* **odd** *(respectively,* **even***), then let $\overline{\mathcal{L}} \stackrel{\mathrm{def}}{=} \overline{\mathcal{L}}_{\mathrm{st},\eta}$ (respectively, $\overline{\mathcal{L}} \stackrel{\mathrm{def}}{=} \overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta}$). Then:*

(1) (Compatibility with Base-Change) *The formation of the push-forward (cf. Chapter VI, Definition 1.3)*

$$(f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<d}\{\infty, \mathrm{et}\}$$

(along with its natural filtration by torsorial degree) commutes with base-change (among bases $S^{\log}$ satisfying the hypotheses given above).

(2) (Zero Locus of the Determinant) *Assume that S is* $\mathbf{Z}$-flat. *The scheme-theoretic zero locus of the determinant* $\det(\Xi\{\infty, \mathrm{et}\})$*, i.e., the determinant of the* **evaluation map** *(cf. Chapter V, Proposition 2.2; Chapter VI, Theorem 3.1, (1))*

$$\Xi\{\infty, \mathrm{et}\} : (f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<d}\{\infty, \mathrm{et}\} \to (f_S)_*(\overline{\mathcal{L}}|_{({}_dE^{\dagger}_{\infty})})$$

is given by the **divisor**

$$d \cdot [\eta \bigcap ({}_dE)]$$

(where ${}_dE$ is the kernel of multiplication by d on E_d). In fact, the divisor of poles of the inverse morphism to $\Xi\{\infty, \mathrm{et}\}$ is contained in the divisor $[\eta \bigcap ({}_dE)]$.

(3) (Analytic Torsion at the Divisor at Infinity) *For each ι, there is a sequence of elements*

$$\mathbf{a}_\iota = \{(\mathbf{a}_\iota)_0, \ldots, (\mathbf{a}_\iota)_{d-1}\}; \quad (\mathbf{a}_\iota)_j \approx \frac{j^2}{8d}$$

of $\mathbf{Q}_{\geq 0} \cdot \log(q)$, where $(\mathbf{a}_\iota)_j$ goes roughly (as a function of j) as $\frac{j^2}{8d}$ (cf. Chapter VI, Theorem 3.1, (2)), such that the subquotients of the natural filtration on the domain of $\Xi\{\infty, \mathrm{et}\}$ admits natural isomorphisms:

$$(F^{j+1}/F^j)((f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\infty, \mathrm{et}\}) \longrightarrow \frac{1}{j!} \cdot \exp(-(\mathbf{a}_\iota)_j) \cdot (f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}}) \otimes_{\mathcal{O}_S} \tau_E^{\otimes j}$$

(where τ_E is the dual of ω_E) for $j = 0, \ldots, d-1$. Moreover, the sections of $\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}}$ that realize these bijections have **q-expansions** *in a neighborhood of infinity that are given* **explicitly** *in Chapter V, Theorem 4.8.*

(4) (Integrality Properties at the Infinite Prime) *Suppose that $D = \emptyset$, and S is of finite type over $\mathbf{C}$. Let us then write $\mathcal{L}$, Ξ, ${}_dE^\dagger$, $E^\dagger_{[d]}$ for $\overline{\mathcal{L}}$, $\Xi\{\infty, \mathrm{et}\}$, ${}_dE^\dagger_\infty$, $E^\dagger_{\infty,[d]}$. Then one may equip $\mathcal{L}$ with a (smooth, i.e., $\mathcal{C}^\infty$) metric $| \sim |_{\mathcal{L}}$ whose* **curvature is translation-invariant** *on the fibers of $E \to S$. Moreover, such a metric is unique up to multiplication by a (smooth) positive function on $S(\mathbf{C})$. Then $| \sim |_{\mathcal{L}}$ defines a metric on the vector bundle $(f_S)_*(\mathcal{L}|_{{}_dE^\dagger})$ (i.e., the range of Ξ), namely, the L^2-metric for "$\mathcal{L}$-valued functions on ${}_dE^\dagger$" (where we assume that the total mass of ${}_dE^\dagger$ is 1). Since Ξ is an isomorphism, this metric thus induces a metric on $(f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d}$ (i.e., the domain of Ξ), which we denote by*

$$|| \sim ||_{\mathrm{et}}$$

and refer to as the **étale metric**. *On the other hand, by using the canonical real analytic splitting of $E^\dagger_{[d]}(\mathbf{C}) \to E(\mathbf{C})$ (i.e., the unique splitting which is a continuous homomorphism), we may split sections of $(f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d}$ into components which are real analytic sections of $\mathcal{L} \otimes \tau_E^{\otimes r}$ (where $r < d$) over $E(\mathbf{C})$. Since τ_E gets a natural metric by square integration over E, these components have natural*

L^2-norms determined by integrating their $| \sim |^2_{\mathcal{L}}$ over the fibers of $E(\mathbf{C}) \to S(\mathbf{C})$. This defines what we refer to as the **de Rham metric**

$$|| \sim ||_{\mathrm{DR}}$$

on $(f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d}$. The relationship between the étale and de Rham metrics may be described using three "models":

> **(A.) The Hermite Model:** *This model states that if we fix $r < d$, and let $d \to \infty$, then over any compact subset of $S(\mathbf{C})$, the étale metric $|| \sim ||_{\mathrm{et}}$ on $F^r((f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d})$ converges (up to a factor $\leq e^{\pi+r}$, ≥ 1) to the metric $|| \sim ||_{\mathrm{DR}}$, as well as to a certain metric "$|| \sim ||_{\mathrm{HM}_d}$" defined by considering* **Hermite polynomials scaled by a factor of (constant)$\cdot\sqrt{\mathbf{d}}$** *in the derivatives of the theta functions $\in (f_S)_*(\mathcal{L}|_E) = F^1((f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d})$.*

> **(B.) The Legendre Model:** *This model states (roughly) that over any compact subset of $S(\mathbf{C})$, a certain* **average** *– which we denote $|| \sim ||_{w,\boldsymbol{\mu}_a}$ – of translates of the étale metric $|| \sim ||_{\mathrm{et}}$ on $(f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d}$ is equal (provided $d \geq 25$), up to a factor of $(\mathrm{constant})^d$, to the de Rham metric $|| \sim ||_{\mathrm{DR}}$, as well as to a certain metric "$|| \sim ||_{\mathrm{Tch}}$" defined by considering* **discrete Tchebycheff polynomials scaled by a factor of d** *in the derivatives of the theta functions $\in (f_S)_*(\mathcal{L}|_E) = F^1((f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d})$. These discrete Tchebycheff polynomials are discrete versions of the Legendre polynomials, and in fact, if we let $d \to \infty$ with the said scaling by d, then the discrete Tchebycheff polynomials converge uniformly to the* **Legendre polynomials**.

> **(C.) The Binomial Model:** *This model involves the explicit q-expansions (where we write $E = \mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$, and q is a holomorphic function which is defined, at least locally, on $S(\mathbf{C})$) referred to in (3) above, which are essentially* **binomial coefficient polynomials (scaled by 1)** *in the derivatives of the theta functions $\in (f_S)_*(\mathcal{L}|_E) = F^1((f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d})$. If we divide these functions by appropriate powers of q, then the norm $|| \sim ||_{\mathrm{qCG}}$ for which these functions divided by powers of q are orthonormal satisfies the following property: If $d \geq 12$, and $\mathrm{Im}(\tau) \geq 200\{\log^2(d) + n \cdot \log(d) + n \cdot \log(n)\}$ (where $q = \exp(2\pi i\tau)$), then:*
>
> $$n^{-1} \cdot e^{-32d} \cdot || \sim ||_{\mathrm{qCG}} \leq || \sim ||_{\mathrm{et}} \leq e^{4d} \cdot || \sim ||_{\mathrm{qCG}}$$

Here, the factor of n^{-1} that appears is the exact **archimedean analogue** *of the poles that appeared at finite primes in (2) above.*

Finally, for each of these three models, the **combinatorial/arithmetic portion of the analytic torsion** *(i.e., the portion not arising from letting the elliptic curve E degenerate – cf. (3) above for the portion arising from degeneration of the elliptic curve) induced on $(F^{r+1}/F^r)((f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d})$ by the metrics $|| \sim ||_{\mathrm{DR}}$; $|| \sim ||_{\mathrm{HM}_d}$; $|| \sim ||_{\mathrm{Tch}}$; $|| \sim ||_{w,\boldsymbol{\mu}_a}$; $|| \sim ||_{\mathrm{qCG}}$ (in their respective domains of applicability) as $r \to d$, goes (modulo factors of the order* $(\mathrm{constant})^d$*) as*

$$\approx (r!)^{-1} \approx (d!)^{-1}$$

which is precisely what you would expect by applying the **product formula** *to the computation of the "analytic torsion" in the* **finite prime case***, which consists of a factor of precisely* $(r!)^{-1}$ *(cf. Chapter V, Theorem 3.1; Chapter VI, Theorem 4.1; Chapter VII, Proposition 3.4).*

In particular, Theorem A, (3), tells us that the *necessary modification to the integral structure of the degree j portion of the push-forward* at the p-adic and degenerating primes is a factor of

$$\frac{1}{j!} \cdot q^{\frac{-j^2}{8d}}$$

The factor $q^{\frac{-j^2}{8d}}$ involving the q-parameter is of *fundamental importance in the theory of this paper*. Because (as a function of j) this factor varies like a "Gaussian $\exp(-x^2)$" and gives rise to *poles* (relative to the usual push-forward) in the modified push-forward that we consider in Theorem A, we refer to the modification in integral structure arising from this factor as the *Gaussian poles*. The factor of $\frac{1}{8d}$ in the exponent may be justified by the following calculation: On the one hand, if one thinks in terms of degrees of vector bundles on $(\overline{\mathcal{M}}_{1,0})_{\mathbf{C}}$, the degree of the usual push-forward goes roughly as

$$\sum_{j=0}^{d-1} \deg(\tau_E^{\otimes j}) = \sum_{j=0}^{d-1} j \cdot \deg(\tau_E) \approx \frac{1}{2}d^2 \cdot \deg(\tau_E) = -\frac{1}{2}d^2 \cdot \frac{1}{12}\log(q) = -\frac{d^2}{24} \cdot \log(q)$$

where "$\log(q)$" is a symbol that stands for the element in $\mathrm{Pic}((\overline{\mathcal{M}}_{1,0})_{\mathbf{C}})$ defined by the divisor at infinity. (It turns out that the contribution to the degree by the line bundle $\overline{\mathcal{L}}$ is negligible.) On the other hand, the sum of the degrees resulting from the Gaussian poles is

$$\sum_{j=0}^{d-1} \frac{j^2}{8d} \cdot \log(q) \approx \frac{1}{3}d^3 \cdot \frac{1}{8d} \cdot \log(q) = \frac{d^2}{24} \cdot \log(q)$$

In other words, the factor of $\frac{1}{8d}$ is just enough to make the total degree 0. Since the restriction of $\overline{\mathcal{L}}$ to the torsion points is (essentially) a "torsion line bundle" (i.e., some tensor power of it is trivial), the degree of the range of the evaluation map is zero – i.e., *the factor of* $\frac{1}{8d}$ *is just enough to make the degrees of the domain and range of the evaluation map of Theorem A equal* (which is natural, since we want this evaluation map to be an isomorphism). In fact,

> *It turns out that the proof of the scheme-theoretic portion of Theorem A in Chapter VI, §3,4, is based on precisely this sort of "summation of degrees" argument.*

In this proof, however, in order to get an exact isomorphism, it is necessary to compute all the degrees involved precisely. This computation requires a substantial amount of work (involving, for instance, the theory of [Zh]) and is carried out in Chapters IV, V, VI.

On the other hand, the key point of the archimedean portion of Theorem A is the comparison of the *étale and de Rham metrics* $|| \sim ||_{\mathrm{et}}$, $|| \sim ||_{\mathrm{DR}}$. Unfortunately, we are unable to prove a simple sharp result that they always coincide. Instead, we choose three natural "domains of investigation" – which we refer to as *models* – where we compare these two metrics using a particular system of functions which are well-adapted to the domain of investigation in question. One of the most important features of these three models is that they each have natural *scaling factors* associated to them. The three models, along with their natural scaling factors, and natural domains of applicability are as follows:

Hermite Model (scaling factor $= d^{\frac{1}{2}}$) : nondegenerating E, fixed $r < d$

Legendre Model (scaling factor $= d$) : nondegenerating E, varying $r < d$

Binomial Model (scaling factor $= 1$) : degenerating E

It is interesting to observe that the exponents appearing in these scaling factors, i.e., $0, \frac{1}{2}, 1$, which we refer to as *slopes*, are precisely the *same as the slopes that appear when one considers the action of Frobenius on the crystalline cohomology of an elliptic curve at a finite prime* – cf. the discussions at the end of Chapter VII, §3, 6, for more on this analogy.

Finally, we apply Theorem A to construct an *arithmetic analogue of the Kodaira-Spencer morphism*, as follows. In the following discussion, we write

$$\Pi_S \stackrel{\mathrm{def}}{=} \pi_1((S^{\log})_{\mathbf{Q}}, \overline{s})$$

for the *algebraic fundamental group* of the base $S^{\log}$. Thus, if $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of a number field K, then $\Pi_S = \mathrm{Gal}(\overline{K}/K)$ is the *absolute Galois group of* K. In this discussion, we make the technical assumption that if d is *even*, then Π_S *acts trivially on the* 2-*torsion* of the elliptic curve in question. Then it follows from Mumford's theory of algebraic theta functions that Π_S acts naturally (up to poles annihilated by 4) on the *range* of the Comparison Isomorphism of Theorem A. Thus, Theorem A tells us that, by transport of structure, we get an action of Π_S on the *domain* of the Comparison Isomorphism which is (roughly) *integral at all primes* (finite and infinite). Since this domain is equipped with a natural *Hodge filtration*, we thus get a morphism

$$\kappa_E^{\text{arith}} : \Pi_S \to \mathfrak{Filt}(\mathcal{H}_{\text{DR}})(S)$$

where $\mathfrak{Filt}(\mathcal{H}_{\text{DR}})(S)$ is a certain *flag variety of filtrations* of the domain of the Comparison Isomorphism, which we denote by $\mathcal{H}_{\text{DR}}$ (since it is a sort of de Rham cohomology). This morphism, which we refer to as the *arithmetic Kodaira-Spencer morphism* of the given family of elliptic curves, has *remarkable integrality properties* (cf. the discussion preceding Chapter IX, Definition 3.4). In Chapter IX, §1,2, we explain in detail how this arithmetic Kodaira-Spencer morphism is a *very precise analogue of the classical Kodaira-Spencer morphism for families of complex and p-adic elliptic curves* (cf. also [Katz2]). In the construction of all of these Kodaira-Spencer morphisms, the main idea consists, as depicted in the following diagram:

<u>Kodaira-Spencer morphism</u>:

motion in base-space $\mapsto$ *induced deformation of Hodge filtration*

of the idea that the Kodaira-Spencer morphism is the map which associates to a "motion" in the base-space of a family of elliptic curves, the deformation in the Hodge filtration of the de Rham cohomology of the elliptic curve induced by the motion. We refer to Chapter IX for more details.

§2. Technical Roots: the Work of Mumford and Zhang

Let K be an *algebraically closed field of characteristic* 0. Let E be an *elliptic curve over* K. Let $\mathcal{L}$ be the line bundle of Theorem A^{simple}. Then instead of considering sections of $\mathcal{L}$ over $E^{\dagger}$, one can consider sections of $\mathcal{L}$ over E. Such sections may be restricted to $\mathcal{L}|_{_dE}$. Moreover, by the theory of *algebraic theta functions* (cf. [Mumf1,2,3]), the restriction $\mathcal{L}|_{_dE}$ of $\mathcal{L}$ to the d-torsion points ${}_dE \subseteq E$ has a *canonical trivialization* $\mathcal{L}|_{_dE} \cong \mathcal{L}|_e \otimes_K \mathcal{O}_{_dE}$ (where $e \in E(K)$ is the zero element) — at least when d is *odd*. Thus, by composing the restriction morphism with this trivialization, we obtain a morphism (as in [Mumf1,2,3]):

$$\Gamma(E, \mathcal{L}) \hookrightarrow \mathcal{L}|_{_dE} \cong \mathcal{L}|_e \otimes_K \mathcal{O}_{_dE}$$

i.e., one may think of sections of $\mathcal{L}$ over E as *functions* on ${}_dE$. These functions are Mumford's "algebraic theta functions."

Now let us observe that $\dim_K(\Gamma(E, \mathcal{L})) = d$, while $\dim_K(\mathcal{L}|_e \otimes_K \mathcal{O}_{dE}) = d^2$. That is to say, *Mumford's theory only addresses a fraction (more precisely: $\frac{1}{d}$) of the functions in* $\mathcal{L}|_e \otimes_K \mathcal{O}_{dE}$. Thus, it is natural to ask:

> *Is there a natural extension of Mumford's theory that allows one to give meaning to all the functions of $\mathcal{L}|_e \otimes_K \mathcal{O}_{dE}$ as some sort of "global" sections of $\mathcal{L}$?*

Theorem $\mathrm{A}^{\mathrm{simple}}$ provides a natural, affirmative answer to this question: i.e., it states these functions may be interpreted naturally as the *sections of $\mathcal{L}$ over the universal extension $E^\dagger$ of torsorial degree $< d$.*

In more classical terms, to consider the universal extension amounts essentially to considering *the derivatives of (classical) theta functions* (cf., e.g., [Katz1], Appendix C). For instance, if one takes $K = \mathbf{C}$, and writes

$$\theta_\tau(z) \stackrel{\mathrm{def}}{=} \sum_{n \in \mathbf{Z}} e^{\pi i \tau \cdot n^2} \cdot e^{2\pi i z \cdot n}$$

for the "*standard theta function*" (where $z \in \mathbf{C}$, $\tau \in \mathfrak{H} \stackrel{\mathrm{def}}{=} \{w \in \mathbf{C} \mid \mathrm{Im}(w) > 0\}$), then up to the operation of taking the Fourier expansion, this theta function is essentially a "*Gaussian $e^{\pi i \tau \cdot n^2}$*," and its derivatives $P(\frac{\partial}{\partial z}) \cdot \theta_\tau(z)$ (where $P(-)$ is a polynomial with coefficients in $\mathbf{C}$) are given by *polynomial multiples of (which are equivalent to derivatives of) the Gaussian*:

$$P(2\pi i \cdot n) \cdot e^{\pi i \tau \cdot n^2}$$

Just as theta functions are the "fundamental functions on an elliptic curve" (more precisely: generate the space of sections of $\mathcal{L}$ over E), these derivatives are the "fundamental functions on the universal extension of the elliptic curve" (more precisely: generate the space of sections of $\mathcal{L}$ over $E^\dagger$). This point of view is discussed in more detail in Chapter III, §5, 6, 7; Chapter VII, §6. As one knows from elementary analysis, the most natural polynomial multiples/derivatives of a Gaussian are those given by the *Hermite polynomials*. It is thus natural to expect that the Hermite polynomials should appear naturally in the portion of the theory of this paper concerning the behavior of the comparison isomorphism at archimedean primes. This intuition is made rigorous in the theory of Chapters IX, X. In fact, more generally:

> *The essential model that permeates the theory of this paper is that of the Gaussian and its derivatives. This model may be seen especially in the "Gaussian poles," as well as in the "Hermite model" at the infinite prime (cf. §1).*

In the classical theory over $\mathbf{C}$, the most basic derivative of the theta function is the so-called *Weierstrass zeta function* (cf. [Katz1], Appendix C). It should thus not be surprising to the reader that various generalizations of the Weierstrass zeta function – which we refer to as *Schottky-Weierstrass zeta functions* (cf. Chapter III, §6, 7) – play a fundamental role in this paper.

So far, in the above discussion, we concentrated on *smooth* elliptic curves. On the other hand, when one wishes to consider *degenerating elliptic curves*, Zhang has constructed a theory of *metrized line bundles* on such degenerating elliptic curves (cf. [Zh]). In this theory, one can consider the *curvatures* of such metrized line bundles, as well as intersection numbers between two metrized line bundles in a fashion entirely similar to Arakelov intersection theory. Using Zhang's theory of metrized line bundles, it is not difficult to *extend Mumford's theory of algebraic theta functions* in a natural fashion to metrized ample line bundles on degenerating elliptic curves (cf., e.g., Chapter IV, §5, for more details). Unfortunately, however, just as Mumford's theory only addresses sections over the original elliptic curve (as opposed to over the universal extension, as discussed above), Zhang's theory also only deals with the theory of metrized line bundles over the original (degenerating) elliptic curve. Thus, it is natural to ask whether one can generalize Theorem $\mathrm{A}^{\mathrm{simple}}$ to the case of degenerating elliptic curves in such a way that the resulting generalization of the portion of $\Gamma(E^\dagger, \mathcal{L})$ arising from sections over E is compatible with Zhang's theory of metrized line bundles (and their sections) over E. In other words, it is natural to ask:

> *Can one "de Rham-ify" the theory of [Zh], so that it addresses the "metric" behavior of sections of $\mathcal{L}$ not only over E, but over $E^\dagger$, as well?*

An affirmative answer to this question is given by the theory of *Gaussian poles*, or "*analytic torsion at the divisor at infinity*" – cf. Theorem A, (3) in §1; Chapters V, VI.

Another way to view the relation to Zhang's theory is the following. One consequence of the theory of [Zh] is the construction of a natural "metric" (or integral structure) on the space ω_E of invariant differentials on a (degenerating) elliptic curve. If we regard ω_E as a line bundle on the compactified moduli space of elliptic curves, then Zhang's "admissible metric" on ω_E essentially amounts to the (metrized) line bundle $\omega_E(-\frac{1}{12}\cdot\infty)$ (where ∞ is the divisor at infinity of the moduli space), i.e., the line bundle ω_E with integral structure at infinity modified by tensoring with $\mathcal{O}(-\frac{1}{12}\cdot\infty)$. Moreover, it follows from Zhang's theory that there is a natural *trivialization*

$$\omega_E(-\frac{1}{12}\cdot\infty) \cong \mathcal{O}$$

of this metrized line bundle over the moduli space. The 12-th tensor power of this trivialization is the cuspidal modular form usually denoted "Δ" ([KM], Chapter 8, §8.1). Similarly, Theorem A, (3), states that by allowing "Gaussian poles" in the sections of $\Gamma(E^\dagger, \mathcal{L})$, one gets a natural isomorphism between $\Gamma(E^\dagger, \mathcal{L})$ (with this modified integral structure) and a vector bundle which is trivial (in characteristic zero) over some finite log étale covering of the compactified moduli (log) stack of elliptic curves. That is to say,

> *One may regard the theory of Gaussian poles/analytic torsion at the divisor at infinity in Theorem A, (3), as a sort of "GL_2-analogue" of the isomorphism of line bundles (i.e., $\mathbf{G}_{\mathrm{m}}$-torsors) $\omega_E(-\frac{1}{12}\cdot\infty) \cong \mathcal{O}$ — or, alternatively, a GL_2-analogue of the modular form Δ.*

Note: The reason that we mention "GL_2" is that the vector bundle on the "étale side" of the comparison isomorphism of Theorem A arises naturally (at least in characteristic zero) from a representation (defined by the Galois action on the d-torsion points) of the fundamental group of the moduli stack of elliptic curves into GL_2, whereas the isomorphism $\omega_E(-\frac{1}{12}\cdot\infty) \cong \mathcal{O}$ naturally corresponds to an *abelian* representation (i.e., a representation into $\mathbf{G}_{\mathrm{m}}$ which is, in fact, of order 12) of this fundamental group – cf. [KM], Chapter 8, §8.1.

§3. Conceptual Roots: the Search for a Global Hodge Theory

§3.1. From Absolute Differentiation to Comparison Isomorphisms

Let K be either a *number field* (i.e., a finite extension of $\mathbf{Q}$) or a *function field in one variable* over some coefficient field k (which we assume to be algebraically closed in K). Let S be the unique one-dimensional regular scheme whose closed points s correspond naturally (via Zariski localization of S at s) to the set of all discrete valuations of K (where in the function field case we assume that the elements of $k^\times$ are units for the valuations). We shall call S the *complete model* of K. Of course, in the number field case, it is natural to "formally append" to S the set of archimedean valuations of K.

Let

$$E \to S$$

be a *one-dimensional, generically proper semi-abelian scheme* over S, i.e., $E_K \stackrel{\mathrm{def}}{=} E \otimes_S K$ is an *elliptic curve* over K with semi-stable reduction everywhere. Then E defines a *classifying morphism*

$$\alpha : S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$$

to the compactified moduli stack of elliptic curves over $\mathbf{Z}$. It is natural to endow S with the log structure arising from the set of closed points at which $E \to S$ has bad reduction. Then α extends to a morphism $\alpha^{\log} : S^{\log} \to (\overline{\mathcal{M}}_{1,0}^{\log})_{\mathbf{Z}}$ in the logarithmic category.

Now, *in the function field case, if we differentiate* α, *we obtain the* **Kodaira-Spencer morphism** *of* $E \to S$:

$$\kappa_E : \omega_E^{\otimes 2} \cong \alpha^* \Omega_{(\overline{\mathcal{M}}^{\log}_{1,0})_{\mathbf{Z}}} \to \Omega_{S^{\log}/k}$$

(where ω_E is the restriction to the identity section of $E \to S$ of the relative cotangent bundle of E over S). Since ω_E is naturally the pull-back via α of an *ample line bundle on* $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$, and κ_E is typically nonzero (for instance, it is always nonzero if K is of characteristic zero and $E \to S$ is not isotrivial (i.e., trivial after restriction to a finite covering of S)), the existence of the Kodaira-Spencer morphism κ_E gives rise to a *bound* on the height of $E \to S$ by the degree of $\Omega_{S^{\log}/k}$. The thrust of a family of conjectures due to Vojta (cf. [Lang], [Vojta]) is that *this bound* (or at least, a bound roughly similar to this bound) in the geometric case (i.e., the case when K is a function field) *also holds in the "arithmetic case"* (i.e., the case when K is a number field). Thus,

> *In order to prove Vojta's Conjecture in the arithmetic case, it is natural to attempt to construct some sort of arithmetic analogue of the Kodaira-Spencer morphism.*

Indeed, this point of view of approaching the verification of some inequality by first trying to construct "the theory underlying the inequality" is reminiscent of the approach to proving the Weil Conjectures (which may be thought of as *inequalities* concerning the number of rational points of varieties over finite fields) by attempting to construct a "Weil cohomology theory" for varieties over finite fields which has enough "good properties" to allow a natural proof of the Weil Conjectures.

Of course, if one tries to construct any sort of naive analogue of the Kodaira-Spencer morphism in the arithmetic case, one immediately runs into a multitude of fundamental obstacles. In some sense, these obstacles revolve around the fact that the ring of rational integers $\mathbf{Z}$ does not admit "a field of coefficients" $\mathbf{F}_1 \subseteq \mathbf{Z}$. If such a field of absolute constants existed, then one could consider "*absolute differentials* $\Omega_{\mathbf{Z}/\mathbf{F}_1}$," or

$$\text{“}\Omega_{\mathcal{O}_K/\mathbf{F}_1}\text{”}$$

Moreover, since moduli spaces tend to be rather absolute and fundamental objects, it is natural to imagine that if one had a field of absolute constants "$\mathbf{F}_1$," then $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ should *descend naturally* to an object $(\overline{\mathcal{M}}_{1,0})_{\mathbf{F}_1}$ over $\mathbf{F}_1$, so that one could *differentiate the classifying morphism* $\alpha : S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ *in the arithmetic case, as well, to obtain an* **arithmetic Kodaira-Spencer morphism**

$$\text{“}\kappa_E : \omega_E^{\otimes 2} \cong \alpha^* \Omega_{(\overline{\mathcal{M}}^{\log}_{1,0})_{\mathbf{F}_1}} \to \Omega_{S^{\log}/\mathbf{F}_1}\text{”}$$

and then use this arithmetic Kodaira-Spencer morphism to prove Vojta's Conjecture concerning the heights of elliptic curves. (Note: In this case, Vojta's Conjecture is also referred to as "Szpiro's Conjecture.")

Unfortunately, this sort of "absolute field of constants $\mathbf{F}_1$" does not, of course, exist in any naive sense. Thus, it is natural to look for a more indirect, abstract approach. In the geometric case, when $k = \mathbf{C}$, the algebraic curve S defines a *Riemann surface* S^{an}. Let us write $U_S \subseteq S$ for the open subobject where the log structure of $S^{\log}$ is trivial. Then the *first singular cohomology module of the fibers of* $E \to S$ naturally forms a local system

$$H^1_{\mathrm{sing}}(E/S, \mathbf{Z})$$

on the Riemann surface U_S^{an}. One the other hand, the *first de Rham cohomology module* of the fibers of $E \to S$ forms a rank two vector bundle

$$H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$$

on U_S^{an}. This de Rham cohomology admits a *Hodge filtration*, which may be thought of as a natural exact sequence:

$$0 \to \omega_E \to H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E) \to \tau_E \to 0$$

Moreover, this vector bundle $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ on U_S^{an} admits a connection ∇_{DR} — *called the Gauss-Manin connection* — which allows one to differentiate sections of $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$. Using this connection ∇_{DR} to *differentiate the Hodge filtration* gives rise to a natural morphism

$$\Theta_{S^{\log}/k} \to \tau_E^{\otimes 2}$$

(where $\Theta_{S^{\log}/k}$ is the dual to $\Omega_{S^{\log}/k}$) which is dual to the Kodaira-Spencer morphism κ_E. Thus,

> *Another way to think of our search for "$\mathbf{F}_1$" or "a notion of absolute differentiation" is as the search for an arithmetic analogue of the Gauss-Manin connection ∇_{DR} on the de Rham cohomology $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$.*

This is the first step towards raising our search for an arithmetic Kodaira-Spencer morphism to a more abstract level.

Next, let us recall that the *de Rham isomorphism* defines a natural isomorphism

$$H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E) \cong H^1_{\mathrm{sing}}(E/S, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{O}_{U_S^{\mathrm{an}}}$$

(in the complex analytic category) over U_S^{an}. Moreover, the sections of $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ defined (via this isomorphism) by sections of $H^1_{\mathrm{sing}}(E/S, \mathbf{Z})$ are *horizontal* for ∇_{DR}. Thus, we conclude that:

To construct the Gauss-Manin connection ∇_{DR}, it is enough to know the de Rham isomorphism between de Rham and singular cohomology.

The de Rham isomorphism is a special case of the general notion of a *Comparison Isomorphism* between de Rham and singular/étale cohomology. In the last few decades, this sort of Comparison Isomorphism has been constructed over p-adic bases, as well (cf., e.g., [Falt1,2], [Hyodo]). In the arithmetic case, we would like to construct some sort of analogue of the Kodaira-Spencer morphism over S which also has natural integrality properties at the archimedean places, as well (since we would like to use it conclude inequalities concerning the height of the elliptic curve E_K). Put another way, we would like to construct some sort of arithmetic Kodaira-Spencer morphism in the context of *Arakelov theory.* Thus, in summary, the above discussion suggests that:

In order to construct this sort of arithmetic Kodaira-Spencer morphism, a natural approach is to attempt to construct some sort of **Comparison Isomorphism in the "Arakelov theater,"** *analogous to the well-known complex and p-adic Comparison Isomorphisms between de Rham and étale/singular cohomology.*

The construction of such an Arakelov-theoretic Comparison Theorem is the main goal of this paper. *To a certain extent, this goal is achieved by Theorem A* (cf. §1). For a detailed explanation of the sense in which the Comparison Isomorphism of Theorem A is analogous to the well-known complex and p-adic Comparison Isomorphisms, we refer to Chapter IX. Unfortunately, however, for various technical reasons, the arithmetic Kodaira-Spencer morphism that naturally arises from Theorem A is not well enough understood at the time of writing to allow its application to a proof of Vojta's Conjectures (for more on these "technical reasons," cf. §5.1 below; Chapter IX, Example 3.5, and the Remark following Example 3.5). In the remainder of the present §3, we would like to explain in detail how we were led to Theorem A as a global, Arakelov-theoretic analogue of the well-known "local Comparison Isomorphisms."

§3.2. A Function-Theoretic Comparison Isomorphism

In §3.1, we saw that one way to think about absolute differentiation or an absolute/arithmetic Kodaira-Spencer morphism is to regard such objects as natural consequences of a *"global Hodge theory,"* or *Comparison Isomorphism* between the de Rham and étale cohomologies of an elliptic curve. The question then arises:

Just what form should such a Global Comparison Isomorphism — i.e., in suggestive notation

$$H^1_{\mathrm{DR}}(E) \otimes ?? \ \cong \ H^1_{\mathrm{et}}(E) \otimes ??$$

— take?

For instance, over $\mathbf{C}$, such a comparison isomorphism exists naturally over $\mathbf{C}$, i.e., when one takes ?? $= \mathbf{C}$. In the p-adic case, one must introduce rings of *p-adic periods* such as B_{DR}, B_{crys} (cf., e.g., [Falt2]) in order to obtain such an isomorphism. Thus, we would like to know over if there is some sort of natural "ring of global periods" over which we may expect to obtain our global comparison isomorphism.

In fact, in the comparison isomorphism obtained in this paper (cf. §1, Theorem A), unlike the situation over complex and p-adic bases, we do not work over some "global ring of periods." Instead, the situation is somewhat more complicated. Roughly speaking, what we end up doing is the following:

> *In the Hodge-Arakelov Comparison Isomorphism, we obtain a comparison isomorphism between the de Rham and étale cohomologies of an elliptic curve by considering functions on the de Rham and étale cohomologies of the elliptic curve and then constructing an isomorphism between the two resulting* **function spaces** *which is (essentially) an* **isometry** *with respect to natural metrics on these function spaces at all the primes of the base.*

Indeed, for instance over a number field, the de Rham cohomology and étale cohomology are finite modules over very different sorts of rings (i.e., the ring of integers of the number field in the de Rham case; the profinite completion of $\mathbf{Z}$, or one of its quotients in the étale case), and it is difficult to imagine the existence of a natural "global arithmetic ring" containing both of these two types of rings. (Note here that unlike the case with Shimura varieties, the adèles are not a natural choice here for a number of reasons. Indeed, to consider the adèles here roughly amounts to simply forming the direct product of the various local (i.e., complex and p-adic) comparison isomorphisms, which is not very interesting in the sense that such a simple direct product does not result in any natural *global structures*.) Thus:

> *The idea here is to abandon the hope of obtaining a global* **linear** *isomorphism between the de Rham and étale cohomology* **modules**, *and instead to look for an isomorphism (as mentioned above) between the corresponding function spaces which does not necessarily arise from a linear morphism between modules.*

In the present §, we explain how we were led to look for such a "function-theoretic comparison isomorphism," while in §3.3 below, we examine the meaning of the nonlinearity of this sort of comparison isomorphism.

In order to understand the motivating circumstances that naturally lead to the introduction of this sort of function-theoretic point of view, we must first return to the discussion of the case over the complexes in §3.1 above. Thus, in the following discussion, we use the notation of the discussion of the complex case in §3.1. One more indirect way to think about the existence of the Kodaira-Spencer morphism is the following. Recall the exact sequence

$$0 \to \omega_E \to H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E) \to \tau_E \to 0$$

which in fact exists naturally over S^{an}. The Gauss-Manin connection ∇_{DR} acts on the middle term of this exact sequence (as a connection with logarithmic poles at the points of bad reduction), but does not preserve the image of ω_E. One important consequence of this fact is that:

> *(If one imposes certain "natural logarithmic conditions" on the splitting at points of bad reduction, then) this exact sequence does not split.*

Indeed, if this exact sequence split, then one could use this splitting to obtain a *connection on* ω_E induced by ∇_{DR}. Moreover, if the "natural logarithmic conditions" are satisfied, it would follow that this connection on ω_E has zero monodromy at the points of bad reduction, i.e., that the connection is regular over all of S^{an}. But since ω_E is the pull-back to S of an ample line bundle on $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$, it follows (so long as $E \to S$ is not isotrivial) that $\deg(\omega_E) \neq 0$, hence that the line bundle ω_E cannot admit an everywhere regular connection. That is, we obtain a contradiction.

Another (essentially equivalent) way to think about the relationship between the fact that the above exact sequence does not split and the existence of the Kodaira-Spencer morphism is the following. If one considers the $\omega_E^{\otimes 2}$*-torsor* of splittings of the above exact sequence (together with the "natural logarithmic conditions" at the points of bad reduction), we obtain a class

$$\eta \in H^1_{\mathrm{c}}(S, \omega_E^{\otimes 2})$$

where the subscript "c" stands for "cohomology with compact support." (The reason that we get a class with compact support is because of the "natural logarithmic conditions" at the points of bad reduction.) On the other hand, if we apply the functor $H^1_{\mathrm{c}}(-)$ to the Kodaira-Spencer morphism, we obtain a morphism

$$H^1_{\mathrm{c}}(S, \omega_E^{\otimes 2}) \to H^1_{\mathrm{c}}(S, \Omega_{S^{\mathrm{log}}/k}) = H^1_{\mathrm{c}}(S, \Omega_{S/k}) \cong k$$

Moreover, the image of η under this morphism can easily be shown to be the element of $H^1_{\mathrm{c}}(S, \Omega_{S/k}) \cong k$ which is the degree of the classifying morphism $\alpha : S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{C}}$, i.e., $\deg(\alpha) \in \mathbf{Z} \subseteq \mathbf{C} = k$, which is nonzero (so long as $E \to S$ is not isotrivial). This implies that η is nonzero.

Thus, in summary,

> *An indirect way to "witness the existence of the Kodaira-Spencer morphism" is to observe that the above exact sequence does not split, i.e., that the* $\omega_E^{\otimes 2}$*-torsor of splittings of this sequence is nontrivial.*

This point of view is discussed in more detail in [Mzk2], Introduction, §2.3. Also, we observe that this nonsplitting of the above exact sequence may also be regarded as a sort of "stability of the (vector bundle plus connection) pair" $(H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E), \nabla_{\mathrm{DR}})$. This type of stability of a bundle equipped with connection is referred to as *"crys-stability"* in [Mzk2] — cf. [Mzk2], Introduction, §1.3; [Mzk2], Chapter I, for more details.

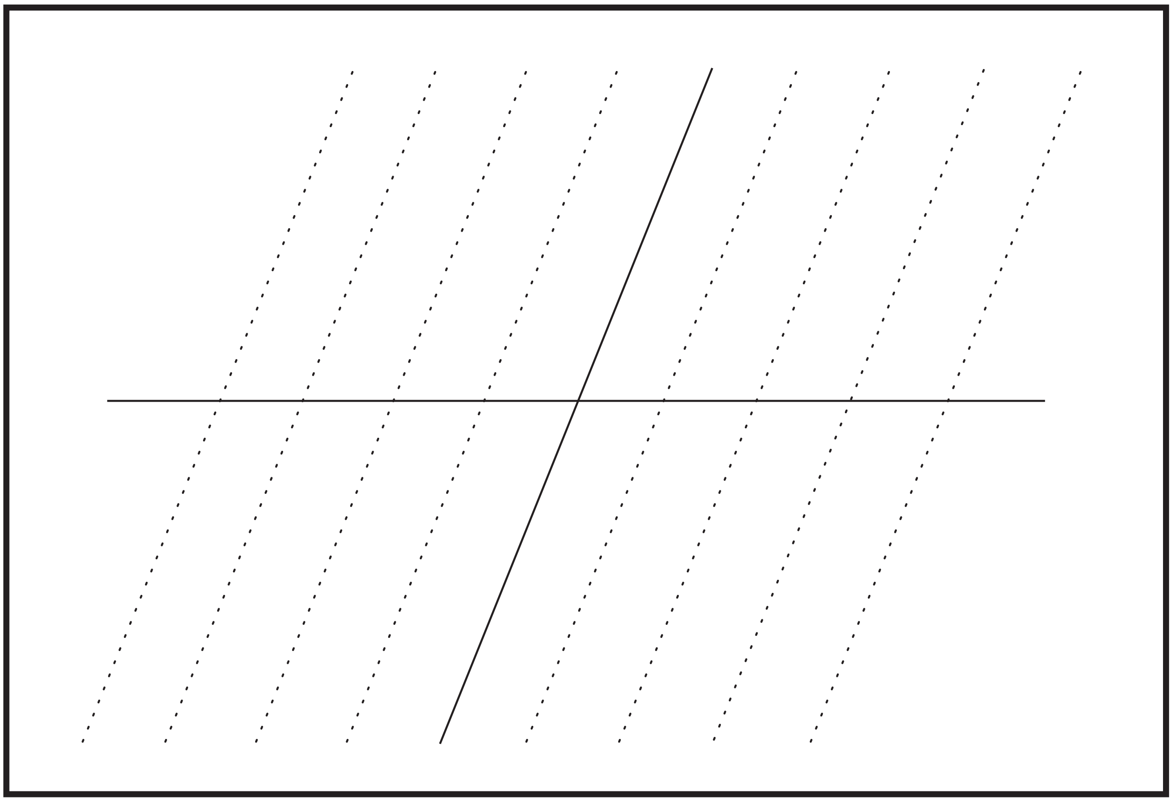

Fig. 1: The split case.

Now let us return to the *arithmetic case*. In this case, $S = \mathrm{Spec}(\mathcal{O}_K)$ (where $\mathcal{O}_K$ is the ring of integers of a number field K). Moreover, we have a natural exact sequence of $\mathcal{O}_K$-modules:

$$0 \to \omega_E \to H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E) \to \tau_E \to 0$$

which extends naturally to an *exact sequence of $\mathcal{O}_K$-modules with Hermitian metrics at the infinite primes*, i.e., an exact sequence of *arithmetic vector bundles on $\overline{S}$* (where $\overline{S}$ denotes the formal union of S with the set of infinite primes of K) in the sense of Arakelov theory. Then one can consider *whether or not this exact sequence of arithmetic vector bundles splits*. Moreover, just as in the complex case, one can think of this issue as the issue of whether or not a certain *Arakelov-theoretic* $\omega_E^{\otimes 2}$*-torsor* splits. (The notion and basic properties of torsors in Arakelov-theory are discussed in Chapter I.) Thus,

> *One way to regard the issue of constructing an arithmetic Kodaira-Spencer morphism is as the issue of constructing a theory that proves that/explains why this Arakelov-theoretic* $\omega_E^{\otimes 2}$*-torsor does not split.*

Indeed, the nonsplitting of this torsor is very closely related to the Conjectures of Vojta and Szpiro — in fact, the existence of (for instance, an infinite number of) counterexamples to these conjectures would imply (in an infinite number of cases) the splitting of this torsor (cf. Chapter I, Theorem 2.4; Chapter I, §4).

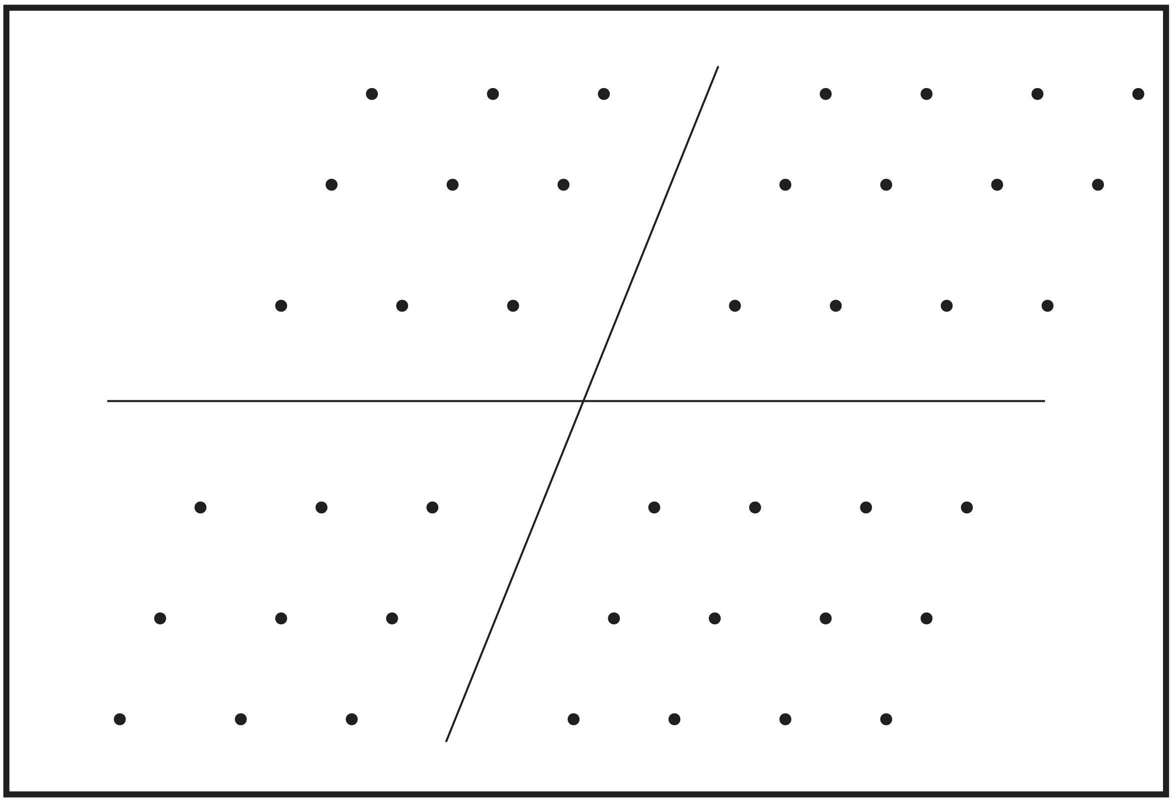

Fig. 2: The non-split case.

From a more elementary point of view, the nonsplitting of this torsor may be thought of in the following fashion. First of all, an arithmetic vector bundle over $\overline{S}$ may be thought of as an ($\mathcal{O}_K$-) *lattice* in a real or complex vector space. Thus, for instance, in the case $K = \mathbf{Q}$, one may think of $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ as a lattice in $\mathbf{R}^2$, while the arithmetic line bundles ω_E and τ_E may be thought of as lattices in $\mathbf{R}$. In the case of arithmetic line bundles, to say that the *degree* of the arithmetic line bundle (cf. Chapter I, §1) is large (respectively, small) amounts to saying that the points of the lattice are rather densely (respectively, sparsely) distributed. Note that since ω_E is the pull-back to $\overline{S}$ of an ample line bundle $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$, its degree tends to be rather *large*. Thus, to say that the torsor in question splits is to say that $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ looks rather like the lattice of Fig. 1, i.e., it is dense in one direction and sparse in another (roughly orthogonal) direction. We would like to show that $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ looks more like the lattice in Fig. 2, i.e.:

> We would like to show that the lattice corresponding to $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ is *roughly equidistributed* in all directions.

Since we are thinking about *comparison isomorphisms*, it is thus tempting to think of the comparison isomorphism as something which *guarantees that the "distribution of matter" in the lattice* $H^1_{\mathrm{DR}}(E/S, \mathcal{O}_E)$ *is as even in all directions as the "distribution of matter" in the étale cohomology of* E. Also, it is natural to think of the "distribution of matter issues" involving the étale cohomology, or torsion points, of E as being related to the action of Galois. (The action of Galois on the torsion points is discussed in Chapter II.) Thus, in summary:

> *It is natural to expect that the global comparison isomorphism should be some sort of equivalence between "distributions of matter" in the de Rham and étale cohomologies of an elliptic curve.*

Typically, in analytic number theory, probability theory, and other field of mathematics where "distributions of matter" must be measured precisely, it is customary to measure them by thinking about functions — i.e., so-called *"test functions"* — (and the resulting function spaces) on the spaces where these distributions of matter occur. It is for this reason that the author was led to the conclusion that:

> *The proper formulation for a global comparison isomorphism should be some sort of isometric (for metrics at all the primes of a number field) isomorphism between spaces of functions on the de Rham and étale cohomologies of an elliptic curve.*

This is precisely what is obtained in Theorem A.

§3.3. The Meaning of Nonlinearity

In §3.2 above, we saw that one of the *central ideas of this paper* is that to obtain a "global Hodge theory," one must *sacrifice linearity = additivity*, and instead look for isometric isomorphisms between spaces of functions on the de Rham and étale cohomology. Put another way, this approach amounts to abandoning the idea that the de Rham and étale cohomologies are *modules*, and instead thinking of them as *(nonlinear) geometric objects*. Thus, one appropriate name for this approach might be *"geometric motive theory."*

This approach contrasts sharply with typical approaches to the "theory of motives" or to "global Hodge theories" which tend to *revolve around additivity/linearization*, and involve such "linear" techniques as the introduction of derived categories that contain motives. In some sense, since we set out to develop a global Hodge theory in the context of *Arakelov theory*, the nonlinearity of the Comparison Isomorphism of Theorem A is perhaps not so surprising: Indeed, many objects in Arakelov theory which are analogues of "linear" objects in usual scheme theory become "nonlinear" when treated in the context of Arakelov theory. Perhaps the most basic example of this phenomenon is the fact that in Arakelov theory, the space of global sections of an arithmetic line bundle is not closed under addition. This makes it difficult and unnatural (if not impossible) to do homological algebra (e.g., involving derived categories) in the context of Arakelov theory.

Another way to think about the nonlinearity of the Hodge-Arakelov Comparison Isomorphism is that it is natural considering that the fundamental algebraic object that encodes the "symmetries of the Gaussian," namely, the *Heisenberg algebra* — i.e., the Lie algebra generated by 1, x, y, and the relations $[x, y] = 1$, $[x, 1] = [y, 1] = 0$ (the *étale counterpart* of which is the *theta groups* of Mumford) — is closely related to *nonlinear* geometries. In the field of noncommutative geometry, the object that represents these symmetries is known as the *noncommutative torus*. Since the Gaussian and its derivatives lie at the technical heart at the theory of this paper, it is thus not surprising that the nonabelian nature of the symmetries of the Gaussian should manifest itself in the theory. In fact, in the portion of the theory of this paper at archimedean primes, it turns out that the Comparison Isomorphism in some sense amounts to a function-theoretic splitting of the exact sequence $0 \to d \cdot \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/d \cdot \mathbf{Z} \to 0$, i.e., a function-theoretic version of a bijection

$$(\mathbf{Z}/d \cdot \mathbf{Z}) \times (d \cdot \mathbf{Z}) \cong \mathbf{Z}$$

This sort of splitting is somewhat reminiscent of the "splitting" inherent in regarding $\mathbf{Z}$ as being some sort of "polynomial algebra over $\mathbf{F}_1$." Moreover, this archimedean portion of the theory is also (not surprisingly) closely related to the derivatives of the Gaussian and the symmetries encoded in the Heisenberg algebra. Thus, in summary:

> *It is as if the symmetries/twist inherent in the inclusion "$\mathbf{F}_1 \subseteq \mathbf{Z}$" are precisely the symmetries/twist encoded in the noncommutative torus (of noncommutative geometry).*

We refer to the discussion of Chapter VIII, §0, for more details on this point of view.

§3.4. Hodge Theory at Finite Resolution

So far, we have discussed the idea that the appropriate way to think about Comparison Isomorphisms is to regard them as (isometric) isomorphisms between spaces of functions on the de Rham and étale cohomologies of an elliptic curve. The question then arises: How does one define such a natural isomorphism? The *key idea* here is the following:

> *Comparison Isomorphisms should be defined as* **evaluation maps**, *given by evaluating functions on the universal extension of an elliptic curve — which is a sort of "$H^1_{\mathrm{DR}}(E, \mathcal{O}_E^\times)$," i.e., a kind of de Rham cohomology of the elliptic curve — at the torsion points of the universal extension (which may naturally be identified with the étale cohomology of the elliptic curve).*

In fact, one of the main observations that led to the development of the theory of this paper is the following:

> *The classical Comparison Isomorphisms over complex and p-adic bases may be formulated precisely as evaluation maps of certain functions on the universal extension at the torsion points (or the "singular cohomology analogue of torsion points") of the universal extension.*

This *key observation* is discussed in detail in Chapter IX, §1, 2. In the complex case, it amounts to an essentially trivial reformulation of the classical theory. Perhaps the best way to summarize this reformulation is to state that the *subspace of functions* on the "singular cohomology analogue of torsion points" *arising from the theta functions* on the elliptic curve is itself a sort of *"function-theoretic" representation of the Hodge filtration* induced (by the de Rham isomorphism) on the singular cohomology with complex coefficients of an elliptic curve. In fact, this observation more than any other played an essential role in convincing the author that (roughly speaking) "theta functions naturally define the Comparison Isomorphism" (hence that any global Comparison Isomorphism should involve theta functions). In the p-adic case, this "key observation" amounts to what is usually referred to as the *p-adic period map* (cf., e.g., [Coln], [Colz1,2], [Font], [Wint]; the beginning of Chapter IX, §2, of the present paper) of elliptic curves (or abelian varieties).

Thus, in summary, the complex, p-adic, and Hodge-Arakelov Comparison Isomorphisms may all be formulated along very similar lines, i.e., as evaluation maps of functions on the universal extension at the torsion points of the universal extension. Of course, the difference between the Hodge-Arakelov Comparison Isomorphism and its local (i.e., complex and p-adic) counterparts is that unlike in the local case, where the spaces of torsion points involved are "completed at some prime," in the Hodge-Arakelov case, we work with a *discrete set of torsion points*. It is for this reason that we find it natural to think of the theory of the present paper as a *"discretization"* of the well-known local comparison isomorphisms. Another way that one might think of the theory of the present paper is as a *"Hodge theory at finite resolution"* (where we use the term "resolution" as in discussions of the number of "pixels" (i.e., "picture elements, dots") of a computer screen).

At this point, the reader might feel motivated to pose the following question:

If the Hodge-Arakelov Comparison Isomorphism is indeed a comparison isomorphism analogous to the complex and p-adic comparison isomorphisms, then what sorts of "global periods" does it give rise to?

For instance, in the case of the complex comparison isomorphism, the most basic period is the period of the Tate motive, i.e., of H^1 of $\mathbf{G}_{\mathrm{m}}$, namely, $2\pi i$. In the p-adic case, the corresponding period is the copy of $\mathbf{Z}_p(1)$ that sits naturally inside B_{crys}. The analogous "period" resulting from the theory of the present paper, then, is the following: Let U be the standard multiplicative coordinate on $\mathbf{G}_{\mathrm{m}}$. Then $U-1$ forms a section of some ample line bundle on $\mathbf{G}_{\mathrm{m}}$, hence may be thought of as a sort of "theta function" (cf. especially, the Schottky uniformization of an elliptic curve, as in [Mumf4], §5). Then, roughly speaking, *the "discretized Hodge theory" of the present paper amounts essentially* — from the point of view of *periods — to thinking of the period "$2\pi i$" as*

$$\lim_{n\to\infty} \ n \cdot (U-1)|_{U=\exp(2\pi i/n)}$$

i.e., the evaluation of a theta function at an n-torsion point, for some large n. For the *elliptic curve analogue (at archimedean primes) of this representation of* $2\pi i$, we refer especially to Chapter VII, §5, 6, of the present paper.

In fact, another way to interpret the theory of the present paper is the following. First, let us observe that the classical complex comparison isomorphism (i.e., the de Rham isomorphism) is centered around "differentiation" and "integration," i.e., *calculus* on the elliptic curve. Moreover, in some sense, the most fundamental aspect of calculus as opposed to algebraic geometry on the elliptic curve is the use of *real analytic functions* on the elliptic curve. In the present context, however, we wish to keep everything "arithmetic" and "global" over a number field. Thus, instead of performing calculus on the underlying real analytic manifold of a (complex) elliptic curve, we approximate this classical sort of calculus by *performing calculus on a finite (but "large") set of torsion points* of the elliptic curve. That is to say:

We regard the set of torsion points as an **approximation** *of the underlying real analytic manifold of an elliptic curve.*

Indeed, this notion of **"discrete torsion calculus"** is one of the key ideas of this paper. For instance, the universal extension of a complex elliptic curve has a *canonical real analytic splitting* (cf. Chapter III, Definition 3.2), which is fundamental to the Hodge theory of the elliptic curve (cf., e.g., [Mzk2], Introduction, §0.7, 0.8). Since this splitting passes through the torsion points of the universal extension (and, in fact, is equal to the closure of these torsion points in the complex topology), it is thus natural to regard the torsion points of the universal extension as a "discrete torsion calculus approximation" to the canonical real analytic splitting (cf. Remark 1 following Chapter III, Definition 3.2). This "discrete torsion calculus" point of view may also be seen in the use of the operator "δ" in Chapter III, §6,7 (cf. also Chapter V, §4), as well as in the discussion of the "discrete Tchebycheff polynomials" in Chapter VII, §3.

§3.5. Relationship to Ordinary Frobenius Liftings and Anabelian Varieties

Finally, before proceeding, we present one more approach to thinking about "absolute differentiation over $\mathbf{F}_1$." Perhaps the most *naive* approach to defining the *derivative of a number* $n \in \mathbf{Z}$ (cf. [Ihara]) is to fix a prime number p, and then to compare n with its *Teichmüller representative* $[n]_p \in \mathbf{Z}_p$. The idea here is that Teichmüller representatives should somehow represent something analogous to a "field of constants" inside $\mathbf{Z}_p$. Thus, we obtain a correspondence

$$p \mapsto \frac{1}{p}(n - [n]_p)$$

Unfortunately, if one starts from this naive point of view, it seems to be very difficult to prove interesting global results concerning this correspondence, much less to apply it to proving interesting results in diophantine geometry.

Thus, it is natural to attempt to recast this naive approach in a form that is more amenable to globalization. To do this, let us first note that to consider Teichmüller representatives is very closely related to considering the natural *Frobenius morphism*

$$\Phi_A : A \to A$$

on the ring of Witt vectors $A \stackrel{\mathrm{def}}{=} W(\overline{\mathbf{F}}_p)$. In fact, the Teichmüller representatives in A are precisely the elements which satisfy the equation:

$$\Phi_A(a) = a^p$$

Put another way, if a is a *unit*, then it may be thought of as an element $\in \mathbf{G}_{\mathrm{m}}(A)$. Moreover, $\mathbf{G}_{\mathrm{m}}$ is equipped with its own natural *Frobenius action* $\Phi_{\mathbf{G}_{\mathrm{m}}}$, given by $U \mapsto U^p$ (where U is the standard multiplicative coordinate on $\mathbf{G}_{\mathrm{m}}$). Thus, the Teichmüller representatives are given by those elements of $a \in \mathbf{G}_{\mathrm{m}}(A)$ such that

$$\Phi_A(a) = \Phi_{\mathbf{G}_{\mathrm{m}}}(a)$$

In fact, this sort of situation where one has a natural Frobenius action on a p-adic (formal) scheme, and one considers natural p-adic liftings of points on this scheme modulo p which are characterized by the property that they are taken to their Frobenius (i.e., Φ_A) conjugates by the given action of Frobenius occurs elsewhere in arithmetic geometry. Perhaps the most well-known example of this situation (after $\mathbf{G}_{\mathrm{m}}$) is *the Serre-Tate theory of liftings of ordinary abelian varieties*. Recently, this theory has been generalized to the case of moduli of hyperbolic curves ([Mzk1,2]). We refer to the Introductions of [Mzk1,2] for more on this phenomenon. In the theory of [Mzk1,2], this sort of natural Frobenius action on a p-adic (formal) scheme is referred to as an *ordinary Frobenius lifting*.

The theory of ordinary Frobenius liftings is itself a special (and, in some sense, the simplest) case of *p-adic Hodge theory.* Thus, in summary, from this point of view, the naive approach discussed above (involving the correspondence $p \mapsto \frac{1}{p}(n - [n]_p)$) may be thought of as the approach given by *"looking at the p-adic Hodge theory of* $\mathbf{G}_\mathrm{m}$ *at each prime* p*."* In particular, the relationship between the approach of this paper and the above naive approach may be thought of as the difference between **discretizing** *the various local p-adic Hodge theories into a global Arakelov-theoretic theory* (as discussed in §3.1 – 3.4 above) and *looking at the full* **completed** *p-adic Hodge theories individually.*

In fact, there is another important difference between the approach of this paper and the above naive approach — namely, the *difference between* $\mathbf{G}_\mathrm{m}$ *and* $\overline{\mathcal{M}}_{1,0}^{\log}$ (the log moduli stack of elliptic curves). That is to say, unlike the example discussed above, which is essentially concerned with rational points of $\mathbf{G}_\mathrm{m}$, the theory of this paper concerns "absolute differentiation for points of $\overline{\mathcal{M}}_{1,0}^{\log}$." At the present time, the author does not know of an analogous approach to "globally discretizing" the local Hodge theories of $\mathbf{G}_\mathrm{m}$ (i.e., of doing for $\mathbf{G}_\mathrm{m}$ what is done for $\overline{\mathcal{M}}_{1,0}^{\log}$ in this paper). Also, it is interesting to observe that, unlike many theories for elliptic curves which generalize in a fairly straightforward manner to abelian varieties of higher dimension, it is not so clear how to generalize the theory of this paper to higher-dimensional abelian varieties (cf. §5.2 below). Thus, it is tempting to conjecture that perhaps the existence of the theory of the present paper in the case of $\overline{\mathcal{M}}_{1,0}^{\log}$ is somehow related to the *anabelian nature of* $\overline{\mathcal{M}}_{1,0}^{\log}$ (cf. [Mzk3], [IN]). That is to say, one central feature of anabelian varieties is a certain *"extraordinary rigidity"* exhibited by their p-adic Hodge theory (cf. [Groth]; the Introduction to [Mzk3]). In particular, it is tempting to suspect that this sort of rigidity or *coherence* is what allows one to discretize the various local Hodge theories into a coherent global theory. Another interesting observation in this direction is that the *theta groups* that play an essential role in this paper are essentially the same as/intimately related to the quotient

$$\pi_1(E - \mathrm{pt.})/[\pi_1(E - \mathrm{pt.}), [\pi_1(E - \mathrm{pt.}), \pi_1(E - \mathrm{pt.})]]$$

of the fundamental group $\pi_1(E - \mathrm{pt.})$ of an elliptic curve with one point removed (which is itself an anabelian variety). This sort of quotient of the fundamental group plays a central role in [Mzk3].

Finally, we remark that one point of view related to the discussion of the preceding paragraph is the following: One fundamental obstacle to "differentiating an integer $n \in \mathbf{Z}$ (or $\mathbf{Q}$-rational point of $\mathbf{G}_\mathrm{m}$) over $\mathbf{F}_1$" is that the residue fields $\mathbf{F}_p$ at the different points of Spec($\mathbf{Z}$) differ, thus making it difficult to compare the value of n at distinct points of Spec($\mathbf{Z}$). On the other hand, the theory of the present paper — which involves differentiating $\mathbf{Z}$-valued points of the moduli stack of log elliptic curves $\overline{\mathcal{M}}_{1,0}^{\log}$ — gets around this problem effectively by taking the set of d-torsion points as one's *absolute constants* that do not vary even as the residue field varies. Note that relative to the discussion of §3.4, this set of torsion points should be regarded as a discrete analogue/approximation to the underlying real analytic manifold (which, of course, remains constant) of a family of complex elliptic curves.

§4. Guide to the Text

In this §, we give brief summaries of the contents of the various chapters of this paper.

In Chapter I, we discuss the notion of a *"torsor"* in Arakelov theory, give various criteria for when such torsors split, and finally, explain how such torsors arise naturally in arithmetic geometry. In Chapter II, we consider the issue of the extent to which the *Galois action on the torsion points* of an elliptic curve over a number field is transitive. We remark that *neither of these two chapters is logically necessary for the proof of the main results of this paper.* (Thus, the reader who is only interested in the proof of Theorem A (of §1), for instance, may skip these two chapters.) The reason for the inclusion of these two chapters is that they help to clarify the background of the Hodge-Arakelov Comparison Isomorphism, as explained in the discussion of §3.2 above.

The bulk of the remainder of the text is devoted to the proof of the Hodge-Arakelov Comparison Isomorphism. In Chapter III, we review basic facts concerning the *universal extension of an elliptic curve*, and especially its relation to the Schottky uniformization of an elliptic curve. In Chapter IV, we review Mumford's theory of *algebraic theta functions* ([Mumf1,2,3]) and Zhang's theory of *metrized line bundles* ([Zh]) and derive various consequences of these theories as they relate to the theory of the present paper. In Chapter V, we construct the *evaluation map* that gives rise to the Hodge-Arakelov Comparison Isomorphism and verify various basic properties of this evaluation map. Finally, in Chapter VI, we prove the *scheme-theoretic portion* of the Comparison Isomorphism by means of (among other things) a rather involved computation of degrees (cf. the discussion immediately following the statement of Theorem A in §1).

The next two chapters (VII and VIII) are devoted to the theory of the Comparison Isomorphism at *archimedean primes.* In Chapter VII, we discuss various *natural systems of orthogonal functions*, which provide natural coordinate systems in the spaces of functions that appear in the study of the Comparison Isomorphism. Many of these systems of orthogonal functions are closely related to the well-known classical systems of *Legendre* and *Hermite.* Finally, in Chapter VIII, we apply the theory of Chapter VII to complete the proof of the archimedean portion of the Hodge-Arakelov Comparison Isomorphism.

In the final chapter of the text, Chapter IX, we explain how the Hodge-Arakelov Comparison Isomorphism may be applied to construct an *arithmetic Kodaira-Spencer morphism* for elliptic curves over number fields. We also explain how the Hodge-Arakelov Comparison Isomorphism and the arithmetic Kodaira-Spencer morphism derived from it are related to more classical comparison isomorphisms and Kodaira-Spencer-type morphisms over complex and p-adic bases.

§5. Future Directions

§5.1. Gaussian Poles and the Theta Convolution

In some sense, the most fundamental outstanding problem left unsolved in this paper is the following:

How can one get rid of the **Gaussian poles** *(cf. §1)?*

For instance, if one could get rid of the Gaussian poles in Theorem A, there would be *substantial hope of applying Theorem A to Vojta's Conjectures* (cf. Chapter IX, Example 3.5, and the Remark following Example 3.5). For instance, if we take $m = 2d$ in Theorem A (so that the metrized line bundle $\overline{\mathcal{L}}$ is *symmetric*), then one gets an action of $\{\pm 1\}$ on the domain and range of the Comparison Isomorphism (which is itself compatible with this action). If one then decomposes the domain without Gaussian poles into eigenspaces for this action (and uses the fact that everything is compatible with the action of the theta groups involved), then one obtains natural $\omega_E^{\otimes 2j}$-torsors (where $j \geq 1$ is an integer) that split if and only if these eigenspaces of the domain split into direct sums of nonpositive powers of ω_E. Thus, if one could somehow get the Galois (i.e., Π_S) action resulting from the Hodge-Arakelov Comparison Isomorphism to exist without the Gaussian poles, then *violations to Vojta's Conjecture* would result in a complete splitting of the domain (without Gaussian poles) into nonpositive powers of ω_E. Since, moreover, this Galois action is not likely to preserve the Hodge filtration, one would then most likely obtain morphisms from various powers of ω_E into strictly smaller powers of ω_E, which would result in a *contradiction* (since ω_E tends to have positive degree).

The above sketch of an argument (i.e., that one might be able to apply Theorem A to Vojta's Conjectures if only one could get rid of the Gaussian poles) provides, in the opinion of the author, strong motivation for investigating the issue of whether or not one can somehow eliminate the Gaussian poles from Theorem A. Moreover, the following argument indicates how this might be done: As discussed in §2, in some sense, one may regard the theory of this paper as the theory of the *Gaussian and its derivatives.* The classical example of the theory of the Gaussian and its derivatives is the theory of *Hermite functions.* The Hermite functions, which are various derivatives of the Gaussian, are not themselves polynomials, but rather of the form: (polynomial) $\cdot$ (Gaussian). Thus, it is natural to *divide the Hermite functions by the Gaussian*, which then gives us polynomials which are called the *Hermite polynomials.* In the theory of this paper, the original Gaussian corresponds (relative to taking the Fourier expansion) to the algebraic theta functions of Mumford (i.e., before we consider derivatives); *the "unwanted" Gaussian that remains in the Hermite functions corresponds to the Gaussian poles.* Moreover, since multiplication and division after taking the Fourier expansion correspond to *convolution*, it is natural to imagine that the image of the "domain without Gaussian poles" of the Comparison Isomorphism (of Theorem A) should correspond to those functions on the torsion points that are in the image of (the morphism given by) *convolution with the (original) theta function* — which we refer to as the *"theta convolution"* for short. Thus, it is natural to conjecture that:

> *By studying the theta convolution, one might be able to construct a Galois action like the one needed in the argument above, i.e., a "Galois action without Gaussian poles."*

In [STC], we study this theta convolution, and obtain, in particular, a *theta-convoluted comparison isomorphism*, which has the property that, in a neighborhood of the divisor at infinity, when one works with an *étale* (i.e., isomorphic to $\mathbf{Z}/d\mathbf{Z}$) Lagrangian subgroup, and a *multiplicative* (i.e., isomorphic to $\boldsymbol{\mu}_d$) restriction subgroup, then the Gaussian poles *vanish*, as desired (cf. [STC], especially Remark 1 following Theorem 10.1, for more details). In fact, this point of view is already implicit in the present paper in the theory of the metrics "$|| \sim ||_{w,\boldsymbol{\mu}_a}$" of Chapter VIII, §1,2. Even in the case of the theta-convoluted comparison isomorphism, however, the Gaussian poles fail to vanish (in a neighborhood of the divisor at infinity) if either the restriction subgroup fails to be multiplicative or the Lagrangian subgroup fails to be étale.

In these cases, it is tempting to suspect that (at least at p-adic primes) perhaps by working over some sort of base like "B_{crys}" — over which $\boldsymbol{\mu}_d$ and $\mathbf{Z}/d\cdot\mathbf{Z}$ become isomorphic — these technical problems involving étale restriction subgroups and multiplicative Lagrangian subgroups may be overcome. In fact, this point of view is already implicit in the discussion of Chapter IX, §2, of the present paper. It is the hope of the author to discuss these issues in a sequel to the present paper and [STC] devoted to the *p-adic aspects of the theta convolution* (cf. the Introduction to [STC]).

§5.2. Higher Dimensional Abelian Varieties and Hyperbolic Curves

Once results such as Theorem A (of §1) have been established for elliptic curves, it is natural to attempt to generalize such results to *higher dimensional abelian varieties* and *hyperbolic curves*. Unfortunately, however, even in the case of the abelian varieties, where one expects the generalization to be relatively straightforward, one immediately runs into a number of problems. For instance, if one considers sections of an ample line bundle $\mathcal{L}$ over the universal extension of an abelian variety of dimension g, the dimension over the base field of the space of global sections of torsorial degree $< d$ is:

$$\binom{d-1+g}{g} \cdot d^g < d^{2g}$$

(where we assume that the dimension of the space of global sections of $\mathcal{L}$ over the abelian variety itself is equal to d^g) if $g > 1$. Thus, the naive generalization of Theorem A$^{\mathrm{simple}}$ cannot possibly hold (i.e., since the two spaces between which one must construct an isomorphism have different dimensions). Nevertheless, it is the hope of the author that someday this sort of technical problem may be overcome, and that the theory of this paper may be generalized to arbitrary abelian varieties. If such a generalization could be realized, it would be interesting if, for instance, just as the various models (Hermite, Legendre, Binomial) that occur in the archimedean theory of the present paper correspond naturally to the possible slopes of the action of Frobenius (on the first crystalline cohomology module of an elliptic curve) at finite primes (cf. the discussion of this phenomenon in §1), it were the case that the corresponding models in the archimedean theory for arbitrary abelian varieties correspond to the possible *Newton polygons* of the action of Frobenius (on the first crystalline cohomology module of an abelian variety of the dimension under consideration) at finite primes.

Another natural direction in which to attempt to generalize the theory of this paper would be to extend it to a global/Arakelov-theoretic Hodge theory of hyperbolic curves. Indeed, the "complex Hodge theory of hyperbolic curves," which revolves around the Köbe uniformization of (the Riemann surfaces corresponding to) such curves by the upper half-plane, has already been extended to the p-adic case ([Mzk1,2]). Moreover, the theory of [Mzk1,2] may also be regarded as the "hyperbolic curve analogue" of Serre-Tate theory. Thus, it is natural to attempt to construct a "global Arakelov version" of [Mzk1,2], just as the theory of the present paper in some sense constitutes a globalization of the Serre-Tate theory/p-adic Hodge theory of elliptic curves (cf. Chapter IX, §2).

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Chapter I: Torsors in Arakelov Theory

§0. Introduction

In this Chapter, after reviewing various well-known facts from Arakelov theory, we discuss the notion of a *torsor* in the Arakelov context and prove various results (Theorems 2.4, 3.2; Corollary 2.5) concerning the existence of splittings of such torsors. Finally, in §4, we discuss how such torsors arise naturally in arithmetic geometry, and, more specifically, how they relate to *the main goal of this paper, which is to understand the structure of the arithmetic torsor defined by the universal extension of an elliptic curve.*

§1. Arakelov Theory in Geometric Dimension Zero

In this §, we review basic facts concerning Arakelov theory in the case of geometric dimension zero. Our references will be [GS] and [Szp]. Arakelov theory provides a convenient language for expressing many well-known results in elementary number theory.

We begin by introducing notation. Let K be a *number field*, i.e., a finite extension of $\mathbf{Q}$. Let $\mathcal{O}_K$ be its ring of integers. Let $d \stackrel{\mathrm{def}}{=} [K : \mathbf{Q}]$ (i.e., the degree of K over $\mathbf{Q}$). Let r_1 (respectively, r_2) be the number of real (respectively, complex) places of K (so $d = r_1 + 2r_2$). Let Σ be the *set of embeddings* $K \hookrightarrow \mathbf{C}$. If $\sigma \in \Sigma$, we write $\overline{\sigma}$ for the embedding obtained by composing σ with complex conjugation. Also, if $\sigma \in \Sigma$, we shall write ϵ_σ for $[K_\sigma : \mathbf{R}]$. Let $D_K \in \mathbf{Z}_{>0}$ be the absolute value of the *discriminant* of K over $\mathbf{Q}$.

Definition 1.1. We shall call (cf. [GS], §2.4.1) a pair $\overline{M} = (M, h)$ an *arithmetic vector bundle over* K *(of rank* r*)* if M if a finite, flat $\mathcal{O}_K$-module (such that $\dim_K(M \otimes_{\mathcal{O}_K} K) = r$), and $h = \{h_\sigma\}_{\sigma \in \Sigma}$ is a set of Hermitian metrics h_σ on $M_\sigma \stackrel{\mathrm{def}}{=} M \otimes_{\mathcal{O}_K} K_\sigma$ such that the metric $h_{\overline{\sigma}}$ is the complex conjugate of the metric h_σ. We shall refer to an arithmetic vector bundle of rank one as an *arithmetic line bundle.*

Example 1.2. The $\mathcal{O}_K$-module $\mathcal{O}_K$ equipped with the metrics given by the absolute value $| \ |_\sigma$ defined by $\sigma \in \Sigma$ defines a natural arithmetic line bundle. We shall denote this arithmetic line bundle by $\overline{\mathcal{O}}_K$.

Example 1.3. Consider the $\mathcal{O}_K$-module $\omega_K \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathbf{Z}}(\mathcal{O}_K, \mathbf{Z})$ (cf. [Szp], §1.3). Observe that the trace defines a natural element $\mathrm{Tr} \in \omega_K$. Moreover, there exists a unique metric on $(\omega_K)_\sigma$ such that $|\mathrm{Tr}|_\sigma = \epsilon_\sigma$. We shall denote the resulting arithmetic line bundle by $\overline{\omega}_K$.

Note that if $\overline{L}$ and $\overline{M}$ are arithmetic vector bundles, then one can form various new arithmetic vector bundles as follows: $\overline{L} \otimes_{\overline{\mathcal{O}}_K} \overline{M}$; $\mathrm{Hom}_{\overline{\mathcal{O}}_K}(\overline{M}, \overline{L})$; $\overline{M}^\vee \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\overline{\mathcal{O}}_K}(\overline{M}, \overline{\mathcal{O}}_K)$; $\overline{M}^* \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\overline{\mathcal{O}}_K}(\overline{M}, \overline{\omega}_K)$; (for $i \in \mathbf{Z}_{\geq 0}$) $\wedge^i \overline{M}$. (Note that our notation differs from that of [GS].) All have naturally defined metrics, which we leave to the reader to make explicit. If $\overline{L}$, $\overline{M}$, and $\overline{P}$ are arithmetic line bundles, then we define an *exact sequence of arithmetic line bundles*

$$0 \to \overline{L} \to \overline{M} \to \overline{P} \to 0$$

to be an exact sequence of the underlying $\mathcal{O}_K$-modules such that: at each σ, (i) the metric on L_σ is induced by restricting the metric of M_σ; (ii) the metric on P_σ is induced by the metric on the orthogonal complement to L_σ in M_σ.

Let $\overline{M}$ be an *arithmetic vector bundle*. Then we would like to define a number of invariants associated to $\overline{M}$. Let us write

$$\mathrm{H}^0(\overline{M}) \stackrel{\mathrm{def}}{=} \{m \in \overline{M} \text{ such that } |m|_\sigma \leq 1 \ \forall \sigma \in \Sigma\}$$

and

$$\mathrm{h}^0(\overline{M}) \stackrel{\mathrm{def}}{=} \log(\#\mathrm{H}^0(\overline{M})) \in \mathbf{R}$$

Next, we could like to define the degree $\deg(\overline{M})$ of $\overline{M}$. First of all, if $\overline{M}$ is an *arithmetic line bundle*, then for any $m \in M$, we define (cf. [GS], §2.4.1)

$$\deg(\overline{M}) \stackrel{\mathrm{def}}{=} \log(\#(M/\mathcal{O} \cdot m)) - \sum_{\sigma \in \Sigma} \log(|m|_\sigma) \in \mathbf{R}$$

This real number is easily seen (by means of the *product formula* of elementary number theory) to be independent of the choice of $m \in M$. Note that if $\overline{M}$ is an *arithmetic line bundle*, and $\mathrm{h}^0(\overline{M}) > 0$ (so $\mathrm{H}^0(\overline{M})$ has at least one *nonzero* element), then it follows from the definition of $\mathrm{H}^0(\overline{M})$ that $\deg(\overline{M}) \geq 0$.

More generally, for $\overline{M}$ of arbitrary rank r, we let $\deg(\overline{M}) \stackrel{\mathrm{def}}{=} \deg(\wedge^r \overline{M})$. If $\overline{L}$ is an arithmetic line bundle, then $\deg(\overline{M} \otimes \overline{L}) = \deg(\overline{M}) + r \cdot \deg(\overline{L})$. Moreover, $\deg(-)$ is additive on exact sequences. Finally, we would like to define the *Euler characteristic* $\chi(\overline{M})$ of $\overline{M}$:

$$\chi(\overline{M}) \stackrel{\mathrm{def}}{=} \deg(M) + r \cdot \chi(\overline{\mathcal{O}}_K)$$

where

$$\chi(\overline{\mathcal{O}}_K) \stackrel{\text{def}}{=} r_2 \cdot \log(2) - \frac{1}{2} \log(D_K)$$

(In [Szp], §1.3, a different, but equivalent definition of $\chi(-)$ is given in terms of logarithms of volumes.) The invariant $\chi(-)$ is additive on exact sequences.

Proposition 1.4. *We have:* $\deg(\overline{\omega}_K) = -2 \cdot \chi(\overline{\mathcal{O}}_K) = \log(D_K) - 2r_2 \cdot \log(2)$.

Proof. See [Szp], Lemma 1.4. ○

Next, we would like to review the main result (Theorem 2) of [GS].

Theorem 1.5. *There exists a universal constant $C \in \mathbf{R}_{>0}$ which is independent of K such that the following inequalities hold:*

$$-C \cdot (nd) \cdot (\log(nd) + 1) \leq \mathrm{h}^0(\overline{M}) - \mathrm{h}^0(\overline{M}^*) - \chi(\overline{M}) \leq C \cdot (nd) \cdot (\log(nd) + 1)$$

for any arithmetic vector bundle $\overline{M}$ of rank $n > 0$ over a number field K.

Proof. This is essentially Theorem 2 of [GS]. It is easy to see (using the fact that $N! \leq N^N$ and Stirling's formula (see, e.g., [Ahlf], Chapter 5, §2.5 – cf. also Chapter VII, Lemma 1.5) to estimate the size of the Γ function) that the number "$C(r_1, r_2, n)$" of *loc. cit.* can be bounded uniformly (with respect to n, $\overline{M}$, K) by $C \cdot (nd) \cdot (\log(nd) + 1)$, for some universal $C \in \mathbf{R}_{>0}$. ○

§2. Definition and First Properties of Torsors

Next, we would like to define the notion of an *arithmetic torsor* over an arithmetic vector bundle. Thus, let $\overline{M}$ be an *arithmetic vector bundle* over K.

Definition 2.1. We shall call a pair $\overline{T} = (T, s)$ an *arithmetic torsor over* $\overline{M}$, or, simply, an *$\overline{M}$-torsor* if T has the structure of an M-torsor $T \to \mathrm{Spec}(\mathcal{O}_K)$ on $\mathrm{Spec}(\mathcal{O}_K)$, and $s = \{s_\sigma\}_{\sigma \in \Sigma}$ is a collection of rational points $s_\sigma \in T(K_\sigma)$ such that $s_{\overline{\sigma}}$ is the complex conjugate of s_σ. We shall refer to the $\{s_\sigma\}$ as the *approximands* of the arithmetic torsor $\overline{T}$.

Let $\overline{T}$ be an $\overline{M}$-torsor. Then we shall be especially interested in the issue of when $\overline{T}$ "splits," or "is trivial."

Definition 2.2. We shall say that $\overline{T}$ *splits* if there exists a $t \in T(\mathcal{O}_K)$ such that for each $\sigma \in \Sigma$, the element $t - s_\sigma \in M_\sigma$ satisfies $|t - s_\sigma| \leq 1$. In this case, t will be called a *splitting of* $\overline{T}$, and we shall write $t \in \overline{T}(\overline{\mathcal{O}}_K)$. If $\epsilon > 0$ and $|t - s_\sigma| \leq \epsilon$ for all $\sigma \in \Sigma$, then we shall call t a *splitting of proximity* ϵ, or, simply, an ϵ*-splitting.* If $t - s_\sigma = 0$ for all $\sigma \in \Sigma$, then we shall say that the torsor $\overline{T}$ is *trivial.*

Example 2.3. Suppose that $0 \to \overline{M} \to \overline{P} \to \overline{\mathcal{O}}_K \to 0$ is an exact sequence of arithmetic vector bundles (over K). Then the splittings $\overline{\mathcal{O}}_K \to \overline{P}$ of this exact sequence form an $\overline{M}$-torsor $\overline{T}$ in a natural way: Indeed, if we forget about metrics, then the splittings $\mathcal{O}_K \to P$ form an M-torsor $T \to \mathrm{Spec}(\mathcal{O}_K)$. Thus, it remains to define the "splittings at the infinite prime," i.e., the s_σ. We take s_σ to be the splitting defined by the orthogonal complement to M_σ in P_σ. This gives us an $\overline{M}$-torsor $\overline{T}$. In fact, it is easy to show that every $\overline{M}$-torsor arises in this way. Moreover, $\overline{T}$ is trivial if and only if $\overline{P} \cong \overline{M} \oplus \overline{\mathcal{O}}_K$.

Next, we would like to use Theorem 1.5 above to show that if $\overline{L}$ is a line bundle of large degree, then every $\overline{L}$-torsor splits.

Theorem 2.4. *There exists a universal constant $C_d \in \mathbf{R}$ depending only on $d = [K : \mathbf{Q}]$ with the following property: Let $\overline{L}$ be an arithmetic line bundle over K such that the inequality*

$$\deg(\overline{L}) \ \geq \ \log(D_K) \ + \ C_d$$

is satisfied. Then any $\overline{L}$-torsor $\overline{T}$ splits.

Proof. Let $\lambda \in \mathbf{R}_{>0}$. Let us write $\overline{\mathbf{Z}}(\lambda)$ for the arithmetic line bundle over $\mathbf{Q}$ defined as follows: The underlying $\mathbf{Z}$-module of $\overline{\mathbf{Z}}(\lambda)$ is $\mathbf{Z}$, while the norm of $1 \in \mathbf{Z}$ is given by $|1| = \lambda^{-1}$. Thus, if λ is much greater than 1, $\overline{\mathbf{Z}}(\lambda)$ will have lots of global sections (i.e., its h^0 will be large). Since the underlying $\mathbf{Z}$-module of $\overline{\mathbf{Z}}(\lambda)$ is $\mathbf{Z}$, one can regard $\mathrm{H}^0(\overline{\mathbf{Z}}(\lambda))$ as a (finite) subset of $\mathbf{Z}$. Let N_λ be the product of all the positive elements of this subset. Thus, $N_\lambda \in \mathbf{Z}$, and every nonzero element of $\mathrm{H}^0(\overline{\mathbf{Z}}(\lambda))$ divides N_λ.

Next, let C be the universal constant of Theorem 1.5. Let $C'_d \stackrel{\mathrm{def}}{=} 2C(d+1)(1+\log(d+1)) + 1 \in \mathbf{R}$. Let λ be a real number > 1 and such that $\log(\lambda) \geq C'_d$. Let C_d be a positive real number such that $C_d \geq C'_d$ and $C_d > \log(N_\lambda)$. In the following, we would like to show that this C_d has the property stated in the theorem.

Now let $\overline{L}$ be an arithmetic line bundle over K satisfying the inequality in the statement of the theorem. Then we wish to define a new arithmetic line bundle $\overline{L}_\lambda$ over K as follows: The underlying $\mathcal{O}_K$-module L_λ will be the $\mathcal{O}_K$-submodule $N_\lambda \cdot L \subseteq L$. The metric on $(L_\lambda)_\sigma = L_\sigma$ is defined by taking λ times the metric on L_σ. Thus, we have an inclusion $\phi : \overline{L}_\lambda \hookrightarrow \overline{L}$ which is "globally integral" in the sense that the following two conditions are satisfied: (i) $\alpha \in L_\lambda$ maps to an element of L; (ii) $\alpha \in (L_\lambda)_\sigma$ with absolute value ≤ 1 maps to an element of L_σ with absolute value ≤ 1. It is then immediate from Definition 2.1 that every $\overline{L}$-torsor $\overline{T}$ arises from an $\overline{L}_\lambda$-torsor $\overline{T}_\lambda$ by pushing forward via ϕ.

Now let us consider the $\overline{L}$-torsor $\overline{T}$ arising (cf. Example 2.3) from an exact sequence

$$0 \to \overline{L} \to \overline{M} \to \overline{\mathcal{O}}_K \to 0$$

By the remark at the end of the last paragraph, it follows that this exact sequence can always be obtained by pushing forward (via ϕ) an exact sequence

$$0 \to \overline{L}_\lambda \to \overline{M}_\lambda \to \overline{\mathcal{O}}_K \to 0$$

Next, let us "push-forward" this exact sequence via "$\mathrm{Spec}(\overline{\mathcal{O}}_K) \to \mathrm{Spec}(\overline{\mathbf{Z}})$" so as to obtain an exact sequence of arithmetic vector bundles over $\mathbf{Q}$. Then, let us pull-back this exact sequence via the morphism of $\overline{\mathbf{Z}}$-modules $\overline{\mathbf{Z}} \hookrightarrow \overline{\mathcal{O}}_K$ given by the structure of $\mathcal{O}_K$ as a $\mathbf{Z}$-algebra. This gives us an exact sequence

$$0 \to \overline{L}'_\lambda \to \overline{P}'_\lambda \to \overline{\mathbf{Z}} \to 0$$

Finally, let us tensor this exact sequence with $\overline{\mathbf{Z}}(\lambda)$. This gives us an exact sequence

$$0 \to \overline{L}'_\lambda(\lambda) \to \overline{P}'_\lambda(\lambda) \to \overline{\mathbf{Z}}(\lambda) \to 0$$

We would like to use Theorem 1.5 to compute the number of global sections of the arithmetic vector bundles in this exact sequence.

First, let us observe that

$$\begin{aligned}\chi(\overline{L}'_\lambda(\lambda)) &= -\log(N_\lambda) + \chi(\overline{L}) \\ &= -\log(N_\lambda) + \deg(\overline{L}) + \chi(\overline{\mathcal{O}}_K)\end{aligned}$$

In particular, $\deg(\overline{L}'_\lambda(\lambda)) = -\log(N_\lambda) + \deg(\overline{L})$. Thus,

$$\begin{aligned}\deg(\mathrm{Hom}_{\overline{\mathbf{Z}}}(\overline{L}'_\lambda(\lambda), \overline{\mathbf{Z}})) &= \log(N_\lambda) - \deg(\overline{L}) + \log(D_K) - 2r_2 \cdot \log(2) \\ &\leq \log(N_\lambda) + \log(D_K) - \deg(\overline{L}) \\ &< C_d + \log(D_K) - \deg(\overline{L})\end{aligned}$$

(where the last inequality follows from the fact that we chose C_d to be $> \log(N_\lambda)$). It thus follows that $\mathrm{Hom}_{\overline{\mathbf{Z}}}(\overline{L}'_\lambda(\lambda), \overline{\mathbf{Z}}))$ is an arithmetic line bundle over K whose degree is *negative*. Thus, by [Szp], Lemma 1.1, this arithmetic line bundle has no global sections. Similarly, since $\chi(\overline{\mathbf{Z}}(\lambda)) = \deg(\overline{\mathbf{Z}}(\lambda)) = \log(\lambda)$, we conclude that $\mathrm{Hom}_{\overline{\mathbf{Z}}}(\overline{\mathbf{Z}}(\lambda), \overline{\mathbf{Z}})$ has no global sections. It thus follows that $\mathrm{Hom}_{\overline{\mathbf{Z}}}(\overline{P}'_\lambda(\lambda), \overline{\mathbf{Z}})$ also has no global sections.

Now we are ready to apply Theorem 1.5. We thus obtain that

$$\begin{aligned}
\mathrm{h}^0(\overline{P}'_\lambda(\lambda)) &\geq \chi(\overline{P}'_\lambda(\lambda)) - C(d+1)(1+\log(d+1)) \\
&> \log(\lambda) + \chi(\overline{L}'_\lambda(\lambda)) - C'_d + C(d+1)(1+\log(d+1)) \\
&\geq (\log(\lambda) - C'_d) + \chi(\overline{L}'_\lambda(\lambda)) + Cd(1+\log(d)) \\
&\geq \mathrm{h}^0(\overline{L}'_\lambda(\lambda))
\end{aligned}$$

Thus, there exists a section $s \in \mathrm{H}^0(\overline{P}'_\lambda(\lambda))$ whose image s' in $\overline{\mathbf{Z}}(\lambda)$ is *nonzero*. Thus, $s' \in \mathbf{Z}$ divides N_λ. Then, unraveling all of the complicated definitions reveals that if we divide the section s by the integer s', we obtain a section $s'' \in M$ whose image in $\mathcal{O}_K$ is 1. Thus, it remains to analyze s'' at the infinite primes. Let us denote (for $\sigma \in \Sigma$) by δ_σ (respectively, δ''_σ) the orthogonal projection of s (respectively, s'') to $(\overline{L}'_\lambda(\lambda))_\sigma$ (respectively, L_σ). Then since s is "integral at σ," it follows that

$$|\delta_\sigma|^2_\sigma \leq |\delta_\sigma|^2_\sigma + \lambda^{-2} \cdot (s')^2 \leq 1$$

On the other hand, $|\delta''_\sigma|_\sigma = (s')^{-1} \cdot |\delta_\sigma|_\sigma$. Thus, it follows immediately that $|\delta''_\sigma|_\sigma \leq 1$. In other words, s'' defines a splitting of the torsor $\overline{T}$. This completes the proof of the theorem. ○

Remark. Theorem 2.4 is the number-theoretic analogue of the following well-known fact concerning line bundles on algebraic curves: If X is a smooth, geometrically connected, proper curve over a field k, and $\mathcal{L}$ is a line bundle on X such that $\deg(\mathcal{L}) \geq \deg(\omega_{X/k}) + 1$, then $\mathrm{H}^1(X, \mathcal{L}) = 0$.

Corollary 2.5. *There exists a universal constant $C_d \in \mathbf{R}$ depending only on $d = [K : \mathbf{Q}]$ with the following property: Let $\overline{E}$ be an arithmetic vector bundle of rank r over a number field K which is equipped with a* filtration *of arithmetic vector bundles*

$$0 = \overline{E}_0 \subseteq \overline{E}_1 \subseteq \overline{E}_2 \subseteq \ldots \subseteq \overline{E}_{r-1} \subseteq \overline{E}_r = \overline{E}$$

such that for all $i = 0, \ldots, r$, the subquotients $\overline{L}_i \stackrel{\mathrm{def}}{=} \overline{E}_i/\overline{E}_{i-1}$ are arithmetic line bundles which satisfy the inequality

$$\deg(\overline{L}_i) \;\geq\; \log(D_K) + \; d \cdot \log(r) + \; C_d$$

Then any $\overline{E}$-torsor $\overline{T}$ splits.

Proof. Given an $\overline{E}$-torsor, we may assume that the $\overline{E}$-torsor arises (by push-forward) from some torsor over $\overline{E}(r^{-1}) \stackrel{\text{def}}{=} \overline{E} \otimes_{\overline{\mathbf{Z}}} \overline{\mathbf{Z}}(r^{-1})$. By applying Theorem 2.4 to the push-forward of this $\overline{E}(r^{-1})$-torsor via the surjection $\overline{E}(r^{-1}) \to \overline{L}_r(r^{-1})$ – where we note that by assumption,

$$\deg(\overline{L}_r(r^{-1})) \geq \log(D_K) + \; C_D$$

so it is alright to apply Theorem 2.4 to $\overline{L}_r(r^{-1})$ – we obtain a splitting of the resulting $\overline{L}_r(r^{-1})$-torsor. Thus, to split the $\overline{E}(r^{-1})$-torsor in question, it remains to split a certain $\overline{E}_{r-1}(r^{-1})$-torsor. Continuing in this fashion (i.e., pushing forward the $\overline{E}_{r-1}(r^{-1})$-torsor via the surjection $\overline{E}_{r-1}(r^{-1}) \to \overline{L}_{r-1}(r^{-1})$, splitting the resulting $\overline{L}_{r-1}(r^{-1})$-torsor by Theorem 2.4, etc.) we see that relative to the $\overline{E}(r^{-1})$-torsor in question, we obtain a splitting s_E of the underlying $E(r^{-1})$-torsor whose distance at an archimedean prime σ from the approximand (cf. Definition 2.1) at σ is

$$\leq 1 + 1 + \ldots + 1 = r$$

(i.e., a total of r "1"'s), where the r "1"'s arise from the distance from the approximands of the torsors over $\overline{L}_1(r^{-1}), \overline{L}_2(r^{-1}), \ldots, \overline{L}_r(r^{-1})$ that we split in the process of constructing s_E. On the other hand, to say that s_E has distance $\leq r$ from the approximands of the $\overline{E}(r^{-1})$-torsor in question means precisely (by the definition of the "(r^{-1})") that if we regard s_E as a splitting of the original E-torsor, the distance of s_E at an archimedean place σ from the approximand (of the original $\overline{E}$-torsor) at σ is ≤ 1, i.e., that s_E defines a splitting of the original $\overline{E}$-torsor, as desired. ◯

§3. Splittings with Bounded Denominators

Unfortunately, Theorem 2.4 is only useful when considering collections of torsors all of which are defined over number fields of bounded degree (over $\mathbf{Q}$). Ideally, however, we would like to be able to consider collections of torsors over various number fields of arbitrarily large degree. In this case, we are not able to obtain as sharp a result as Theorem 2.4, and instead must settle for *splittings with denominators*, as follows:

Let $\overline{M}$ be an *arithmetic vector bundle* over K. Let $\overline{T} = (T, s)$ be an $\overline{M}$*-torsor*. Thus, T is an M-torsor. Since $\mathrm{Spec}(\mathcal{O}_K)$ is affine, there exists a splitting of T, i.e., an $\mathcal{O}_K$-isomorphism $\alpha : T \cong M$. Now let $t \in T(K) = T_K(K)$ be a splitting of the M_K-torsor T_K

(where the subscript K denotes "$\otimes_{\mathcal{O}_K} K$"). Then we may define the *ideal of denominators* of t as follows:

$$\mathcal{I}_t \stackrel{\text{def}}{=} \{a \in \mathcal{O}_K \mid a \cdot \alpha_K(t) \in M\}$$

(where $\alpha_K : T_K \cong M_K$). Since, for any $m \in M$, we have $a \cdot \alpha_K(t) \in M \iff a \cdot \alpha_K(t+m) \in M$, it follows that $\mathcal{I}_t$ does not depend on the choice of α. Note that $\mathcal{I}_t$ will always be a nonzero ideal in $\mathcal{O}_K$. Also, let us write

$$\mathrm{Den}(t) \stackrel{\text{def}}{=} \log(\mathcal{O}_K/\mathcal{I}_t) + \sum_{\sigma} \max\{0, \log(|t - s_\sigma|_\sigma)\}$$

where the sum ranges over all $\sigma \in \Sigma$. Note that t defines an (integral) splitting (cf. Definition 2.2) $\in \overline{T}(\overline{\mathcal{O}}_K)$ if and only if $\mathrm{Den}(t) = 0$.

Definition 3.1. For $t \in T_K$, we shall refer to the ideal $\mathcal{I}_t \subseteq \mathcal{O}_K$ as the *ideal of denominators of* t. Morever, we define the *size of the denominators of* t to be the number $\mathrm{Den}(t) \in \mathbf{R}_{\geq 0}$. If $\lambda \in \mathbf{R}$, then we shall call a section $t \in T(K)$ a *splitting with denominators of size* $\leq \lambda$ if $\mathrm{Den}(t) \leq \lambda$. Finally, we shall say that t is *integral* at a prime $\mathfrak{p}$ (respectively, $\sigma \in \Sigma$) of K if $\mathfrak{p}$ does not lie in the support of $\mathcal{O}_K/\mathcal{I}_t$ (respectively, $|t - s_\sigma|_\sigma \leq 1$).

Theorem 3.2. *There exists a universal constant $C \in \mathbf{R}$ with the following property: Let $\overline{E}$ be an arithmetic vector bundle of rank r over a number field K which is equipped with a* filtration *of arithmetic vector bundles*

$$0 = \overline{E}_0 \subseteq \overline{E}_1 \subseteq \overline{E}_2 \subseteq \ldots \subseteq \overline{E}_{r-1} \subseteq \overline{E}_r = \overline{E}$$

such that for all $i = 0, \ldots, r$, the subquotients $\overline{L}_i \stackrel{\text{def}}{=} \overline{E}_i/\overline{E}_{i-1}$ are arithmetic line bundles which satisfy the inequality

$$\deg(\overline{L}_i) \; > \; \log(D_K)$$

(where $d \stackrel{\text{def}}{=} [K : \mathbf{Q}]$). Then any $\overline{E}$-torsor $\overline{T}$ admits a splitting with denominators of size

$$\leq \frac{1}{2}\log(D_K) + C \cdot rd(\log(rd) + 1)$$

(cf. Definition 3.1).

Proof. Just as in the proof of Theorem 2.4, for $\lambda \in \mathbf{R}_{>0}$, we write $\overline{\mathcal{O}}_K(\lambda)$ for the arithmetic line bundle over K defined as follows: The underlying $\mathcal{O}_K$-module of $\overline{\mathcal{O}}_K(\lambda)$ is $\mathcal{O}_K$; the norm of $1 \in \mathcal{O}_K$ at each $\sigma \in \Sigma$ is given by $|1|_\sigma \stackrel{\text{def}}{=} \lambda^{-1}$. Thus, if λ is much greater than 1, $\overline{\mathcal{O}}_K(\lambda)$ will have lots of global sections (i.e., its h^0 will be large). Observe further that if $\lambda \geq 1$, then one has a natural inclusion

$$\overline{\mathcal{O}}_K = \overline{\mathcal{O}}_K(1) \hookrightarrow \overline{\mathcal{O}}_K(\lambda)$$

(which induces the identity map $\mathcal{O}_K \to \mathcal{O}_K$ on the underlying $\mathcal{O}_K$-modules). This inclusion is *integral* at all the primes (finite and infinite) of K.

If $\overline{E}$ is an arithmetic vector bundle on K, let us write $\overline{E}(\lambda) \stackrel{\text{def}}{=} \overline{E} \otimes_{\overline{\mathcal{O}}_K} \overline{\mathcal{O}}_K(\lambda)$. Note that

$$\deg(\overline{E}(\lambda)) = \deg(\overline{E}) + rd \cdot \log(\lambda)$$

If $\lambda \geq 1$, we have a natural inclusion

$$\overline{E}(\lambda^{-1}) \hookrightarrow \overline{E}$$

(given by tensoring the inclusion of the preceding paragraph with $\overline{E}(\lambda^{-1})$). Note that by "pushing forward" (or, equivalently, executing a "change of structure group") via this inclusion, any $\overline{E}(\lambda^{-1})$-torsor induces an $\overline{E}$-torsor. Moreover, it follows easily from the definitions that every $\overline{E}$-torsor arises in this fashion from some (not necessarily unique) $\overline{E}(\lambda^{-1})$-torsor.

Now let $\lambda \in \mathbf{R}$ be ≥ 1. Let us assume that we are given an $\overline{E}$-torsor, together with an $\overline{E}(\lambda^{-1})$-torsor from which it arises. This $\overline{E}(\lambda^{-1})$-torsor may be thought of as corresponding to an exact sequence (cf. Example 2.3)

$$0 \to \overline{E}(\lambda^{-1}) \to \overline{M} \to \overline{\mathcal{O}}_K \to 0$$

If we tensor this exact sequence with $\overline{\mathcal{O}}_K(\lambda)$, then we obtain an exact sequence

$$0 \to \overline{E} \to \overline{M}(\lambda) \to \overline{\mathcal{O}}_K(\lambda) \to 0$$

Now we would like to compute (cf. the proof of Theorem 2.4) the various h^0's of this exact sequence. Let us write C' (> 0) for the universal constant of Theorem 1.5. Then Theorem 1.5 implies that

$$\begin{aligned}
\mathrm{h}^0(\overline{M}(\lambda)) &\geq \mathrm{h}^0((\overline{M}(\lambda))^*) - (r+1)C'd(\log(d) + \log(r+1) + 1) + \chi(\overline{M}(\lambda)) \\
&\geq \deg(\overline{M}(\lambda)) - 2C'rd(\log(rd) + 2) + (r+1)\chi(\overline{\mathcal{O}}_K) \\
&= \deg(\overline{E}) + d \cdot \log(\lambda) - 2C'rd(\log(rd) + 2) + (r+1)\chi(\overline{\mathcal{O}}_K)
\end{aligned}$$

Next, let us observe that for $i = 0, \ldots, r$, we have

$$\deg(\overline{L}_i^*) = \deg(\overline{\omega}_K) - \deg(\overline{L}_i) \leq \log(D_K) - \deg(\overline{L}_i) < 0$$

(by Proposition 1.4 and the assumptions on $\overline{E}$). Thus, in particular, we have (for $i = 0, \ldots, r$) $\mathrm{h}^0(\overline{L}_i^*) = 0$. Since the given filtration on $\overline{E}$ induces a filtration on $\overline{E}^*$ with subquotients equal to $\overline{L}_i^*$ (for $i = 0, \ldots, r$), we thus obtain:

$$\mathrm{h}^0(\overline{E}^*) = 0$$

In particular, Theorem 1.5 implies that

$$\begin{aligned}\mathrm{h}^0(\overline{E}) &\leq \chi(\overline{E}) + C'rd(\log(rd) + 1) \\ &= \deg(\overline{E}) + r \cdot \chi(\overline{\mathcal{O}}_K) + C'rd(\log(rd) + 1)\end{aligned}$$

Putting everything together, we see that

$$\begin{aligned}h^0(\overline{M}(\lambda)) - h^0(\overline{E}) &\geq \chi(\overline{\mathcal{O}}_K) + d \cdot \log(\lambda) - 3C'rd(\log(rd) + 2) \\ &\geq d \cdot \log(\lambda) - \frac{1}{2}\log(D_K) - 3C'rd(\log(rd) + 2)\end{aligned}$$

Now let us set

$$\lambda \stackrel{\text{def}}{=} \frac{1}{2d}\log(D_K) + C \cdot r(\log(rd) + 1)$$

where $C \stackrel{\text{def}}{=} 7C'$. Then we obtain that $h^0(\overline{M}(\lambda)) - h^0(\overline{E}) > 0$, i.e., there exists some globally integral $m \in \overline{M}(\lambda)$ whose image m' in $\overline{\mathcal{O}}_K(\lambda)$ is *nonzero*. Moreover, the zero locus of $m' \in \overline{\mathcal{O}}_K(\lambda)$ has logarithmic degree $= \deg(\overline{\mathcal{O}}_K(\lambda)) = d \cdot \log(\lambda)$. Thus, sorting through the definitions, we conclude that $\frac{m}{m'}$ defines a splitting of the torsor $\overline{T}$ with denominators of size

$$\leq d \cdot \lambda = \frac{1}{2}\log(D_K) + C \cdot rd(\log(rd) + 1)$$

as desired. $\bigcirc$

§4. Examples from Geometry

In this §, we survey various examples of *Arakelov-theoretic torsors that arise naturally in arithmetic geometry.* For a more detailed discussion of the ideas surrounding this topic, we refer to the Introductions of [Mzk1,2] (especially, §0.8, 2.3, of the Introduction to [Mzk2]). In particular, we explain the relationship of the general notion of a torsor in Arakelov theory to the main topic of this paper, which is, effectively, *the study of a specific Arakelov-theoretic torsor canonically associated to the universal extension of an elliptic curve.*

Let K be a finite extension of $\mathbf{Q}$, whose ring of integers we denote $\mathcal{O}_K$. Let X be a *smooth scheme over* $\mathcal{O}_K$. Let L be a *line bundle on* X. Then L admits a *connection* Zariski locally on X (since Zariski locally it is isomorphic to the trivial line bundle), hence naturally defines a cohomology class

$$c_1(L) \in H^1(X, \Omega_{X/\mathcal{O}_K})$$

which is the *Hodge-theoretic first Chern class* of the line bundle L. More geometrically, one may think of the space of connections on L as a *torsor*

$$T_L \to X$$

over the vector bundle $\Omega_{X/\mathcal{O}_K}$. That is to say, $\Omega_{X/\mathcal{O}_K}$ acts on T_L (by adding a differential ω to a given connection ∇ to form a new connection $\nabla+\omega$), and moreover, any two sections of T_L differ by a section of $\Omega_{X/\mathcal{O}_K}$.

Typically, this torsor will not admit any *algebraic* sections $X \to T_L$ over X (cf., e.g., [Mzk1], Chapter I, §3, Theorem 3.4). In fact, if the topological first Chern class of L (with coefficients in $\mathbf{Q}$) is nonzero, then it follows that such sections do not exist. For instance, if L is *ample* and X is proper (of dimension ≥ 1) over $\mathcal{O}_K$, then its first Chern class will be nonzero, so $T_L \to X$ will not admit any global sections.

Write

$$\mathcal{X} \stackrel{\mathrm{def}}{=} X(\mathbf{C})$$

for the complex manifold associated to X over $\mathbf{C}$. More generally, in the following discussion, we shall use ordinary Roman letters for algebraic objects over $\mathbf{Z}$, and calligraphic letters for the associated complex analytic objects. (The sole exception to this rule will be the notation "$\mathcal{O}$" for the structure sheaf.) Thus, we write $\mathcal{T}_{\mathcal{L}} \stackrel{\mathrm{def}}{=} T_L(\mathbf{C})$. Then one may consider *real analytic sections*

$$s : \mathcal{X} \to \mathcal{T}_{\mathcal{L}}$$

of the torsor $\mathcal{T}_{\mathcal{L}} \to \mathcal{X}$. Typically, *holomorphic sections* will not exist (for instance, if X is proper, then holomorphic sections are equivalent to algebraic ones). However, if one works instead in the real analytic category, then it is not difficult to show that real analytic sections always exist. For instance, if $| - |_{\mathcal{L}}$ is a C^∞ metric on the line bundle $\mathcal{L}$, then it is a well-known fact from complex geometry (cf., e.g., [Wells], Chapter III, §2) that $| - |_{\mathcal{L}}$ defines a natural C^∞ connection ∇ on $\mathcal{L}$. Thus, (by taking the "(1,0)-part" of) the connection ∇ defines a C^∞ section of the torsor $\mathcal{T}_{\mathcal{L}} \to \mathcal{X}$. If $| - |_{\mathcal{L}}$ is *real analytic* (and, in fact, it is not difficult to show that every C^∞ metric may be approximated by a real analytic one), then the corresponding section of $\mathcal{T}_{\mathcal{L}} \to \mathcal{X}$ will be real analytic. In summary,

Real analytic metrics on $\mathcal{L}$ define real analytic sections of $\mathcal{T}_{\mathcal{L}} \to \mathcal{X}$.

Moreover, if a real analytic $| - |_{\mathcal{L}}$ defines the section $s : \mathcal{T}_{\mathcal{L}} \to \mathcal{X}$, then the real analytic $(1,1)$-form

$$\overline{\partial}(s) \in \Gamma^{\text{real analytic}}(\mathcal{X}, \ \wedge^{1,1} \Omega_{\mathcal{X}})$$

on $\mathcal{X}$ is (up to a universal constant factor) the *curvature* of the metric $| - |_{\mathcal{L}}$.

Next, we list some typical *examples* of this sort of situation:

Example 4.1. $X =$ *an abelian variety over an open subset of* $\mathrm{Spec}(\mathcal{O}_K)$. In this case, there is a unique (up to constant multiple) real analytic metric on $\mathcal{L}$ whose curvature is invariant with respect to translation (cf. [Mumf3], §12). Thus, we have *canonical metrics* on $\mathcal{L}$. For instance, in the case where X is an *elliptic curve* E, and $L = \mathcal{O}_E(e_E)$ (where e_E is the origin of E), T_L has a natural *abelian group scheme structure* (cf. Chapter III, §1, 4). This abelian group scheme is called the *universal extension of* E and is denoted by $E^\dagger$ (cf. Chapter III for a more detailed discussion of the universal extension). Moreover, in this case, the real analytic section $\mathcal{E} \to \mathcal{E}^\dagger$ defined by the canonical metric on $\mathcal{L}$ is simply the section defined by the real analytic subvariety

$$\mathcal{E}^\dagger_{\mathbf{R}} \subseteq \mathcal{E}^\dagger$$

which is the *closure of the torsion points of* $\mathcal{E}^\dagger$. We refer to the discussion of the "Real Analytic Splitting" in Chapter III, §3, for more details. *This splitting in the case of elliptic curves will play a fundamental role in this paper.*

Example 4.2. $X = A_g$, *the moduli stack of principally polarized abelian varieties of dimension g over* $\mathbf{Z}$. In this case, the Siegel upper half-plane uniformization of the analytic stack $\mathcal{A}_g$ carries a natural Kähler metric which is the unique metric on the Siegel upper half-plane that is invariant with respect to the natural action of $Sp_{2g}(\mathbf{R})$ on the Siegel upper half-plane. For instance, if $g = 1$, then this metric is the well-known *Poincaré metric*

$$\frac{dx \wedge dy}{y^2}$$

on the upper half-plane. This metric induces a real analytic metric on the canonical bundle of $\mathcal{A}_g$ (which is well-known to be ample).

Example 4.3. *X is a hyperbolic curve over an open subset of* $\mathrm{Spec}(\mathcal{O}_K)$. (A "hyperbolic curve" is a family of curves obtained by removing a divisor of degree r which is étale over the base from a family of smooth, proper, connected genus g curves such that $2g-2+r > 0$.) In this case, it follows from a well-known theorem ("Köbe's uniformization theorem") of complex analysis that the univeral cover of the associated analytic object $\mathcal{X}$ is (a disjoint union of copies of) the upper half-plane. Thus, the Poincaré metric on the upper half-plane descends to define a real analytic Kähler metric on $\mathcal{X}$. Moreover, this real analytic Kähler metric defines a real analytic metric on the canonical bundle of $\mathcal{X}$ (cf. the Introductions to [Mzk1,2]).

Example 4.4. *$X = M_{g,r}$, the moduli stack of hyperbolic curves of type (g, r) over* $\mathbf{Z}$. In this case, there is a canonical real analytic Kähler metric on the associated analytic object $\mathcal{M}_{g,r}$ called the *Weil-Petersson metric*. This real analytic Kähler metric defines a real analytic metric on the canonical bundle of $\mathcal{M}_{g,r}$ (cf. the Introductions to [Mzk1,2]).

We also note that the case of open X, or of X which have semi-stable reduction over $\mathcal{O}_K$, may be handled in a parallel fashion to the smooth case, using *log structures*. Since we do not need to use log structures in any rigorous sense in the present survey, we leave this generalization to the reader.

Thus, to summarize, we see that *canonical line bundles L on smooth $\mathcal{O}_K$-schemes X naturally define torsors T_L (over the sheaf of differentials $\Omega_{X/\mathcal{O}_K}$ on X) which typically have natural real analytic sections at the infinite primes of K arising from canonical real analytic Kähler metrics on $X(\mathbf{C})$*. In particular, if one is given an $\mathcal{O}_K$-rational point

$$x \in X(\mathcal{O}_K)$$

then, by restriction, one obtains a torsor $T_L|_x$ over the $\mathcal{O}_K$-vector bundle $\Omega_{X/\mathcal{O}_K}|_x$, equipped with *trivializations at the infinite primes of K* (obtained by restricting the natural real analytic section of $\mathcal{T}_\mathcal{L} \to \mathcal{X}$). Since $\Omega_{X/\mathcal{O}_K}|_x$ also gets a metric (obtained by restricting the Kähler metric on $\mathcal{X} = X(\mathbf{C})$), we thus see that we get precisely the data of Definition 2.1, i.e., *for each rational point x, we obtain an arithmetic torsor (as in §2) naturally associated to x*. This shows how arithmetic torsors arise naturally in arithmetic geometry.

In the present paper, we will especially be interested in the cases where X is either (i) an elliptic curve E, or (ii) the moduli stack of elliptic curves over $\mathbf{Z}$. In fact, the torsor arising from Examples 4.2, 4.3, above in the case of the moduli stack of elliptic curves may be thought of as precisely the torsor obtained from the tautological elliptic curve by looking at tangent space to the origin of the universal extension of the tautological elliptic curve (cf. Chapter III, Proposition 1.3; Chapter III, §3, "The Real Analytic Splitting"; [Mzk2], Introduction, §0.7, 0.8). Thus, in summary, the torsors that we are interested in in the present paper all arise as the *restriction to some sort of rational point* (where (ii) above corresponds effectively to the case where the rational point is valued in some sort of ring of "dual numbers") of the *arithmetic torsor*

$$(E^{\dagger} \to E, \ \mathcal{E}^{\dagger}_{\mathbf{R}} \subseteq \mathcal{E}^{\dagger})$$

given by the *universal extension of an elliptic curve.* Put another way,

> *The main goal of the present paper is to understand the arithmetic torsor* $(E^{\dagger} \to E, \mathcal{E}^{\dagger}_{\mathbf{R}} \subseteq \mathcal{E}^{\dagger})$ *given by the universal extension of an elliptic curve.*

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Chapter II: The Galois Action on Torsion Points

§0. Introduction

In some sense, the main goal of the present paper is to prove a *comparison theorem* that relates the natural *Arakelov-theoretic torsors* arising from an elliptic curve (cf. Chapter I, especially, Chapter I, §4) to the *Galois action on the torsion points* of the elliptic curve. In Chapter I, we introduced the "torsor side" of this comparison; thus, in the present Chapter, we wish to discuss the "torsion point side" of the comparison. More precisely, in this Chapter, we show that, in certain situations, the Galois action on the torsion points of an elliptic curve is as transitive as possible, i.e., that (cf. Theorem 4.4):

> *If we consider all elliptic curves over number fields of bounded degree which have semi-stable reduction everywhere and at least one prime of bad reduction, then for any prime number l of the order of the height h_E of the elliptic curve, the image of the associated Galois representation in $GL_2(\mathbf{Z}_l)$ contains $SL_2(\mathbf{Z}_l)$.*

In particular, the elliptic curve will not have any rational torsion points of order l. Thus, this result may be regarded as a sort of "poor man's uniform boundedness conjecture" (now Merel's theorem ([Mer]) – although, in fact, it is not strictly implied by Merel's theorem). Alternatively, it may regarded as an effective version of a theorem of Serre ([Ser], Chapter IV, Theorem 3.2).

The proof is similar to that of Faltings' proof of the Tate Conjecture ([Falt]), only technically much simpler. That is to say, the main technique is essentially the standard one (going back to Tate) for proving "Tate conjecture-type results."

§1. Some Elementary Group Theory

Let $l \geq 5$ be a prime number. In this §, we review some well-known results concerning the group theory of $SL_2(\mathbf{Z}_l)$.

Lemma 1.1. *Let $G \subseteq SL_2(\mathbf{F}_l)$ be the subgroup generated by the matrices $\alpha \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\beta \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Then $G = SL_2(\mathbf{F}_l)$.*

Proof. Note that if $\mu, \lambda \in \mathbf{F}_l$, then $\beta^\mu \cdot \alpha^\lambda$ (where this expression makes sense since both α^l and β^l are equal to the identity matrix) takes the vector $v \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} \lambda \\ \mu\cdot\lambda+1 \end{smallmatrix}\right)$. In particular, if we let $\gamma \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then for any $\lambda \in \mathbf{F}_l^\times$, we have $\gamma \cdot v, \lambda \cdot \gamma \cdot v \in G \cdot v$. We thus obtain that $\lambda \cdot v \in G \cdot v$. Thus, in summary, we have proven that $\mathbf{F}_l^2 - \{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\} \subseteq G \cdot v$.

Now let us prove that an arbitrary element $\delta \in SL_2(\mathbf{F}_l)$ is contained in G. By the conclusion of the preceding paragraph, we may assume that $\delta \cdot v = v$. But this implies that δ is an upper triangular matrix all of whose diagonal entries are "1." Thus, δ is a power of α, hence contained in G, as desired. $\bigcirc$

Corollary 1.2. *The finite group $SL_2(\mathbf{F}_l)$ is simple.*

Proof. Let $\lambda \in \mathbf{F}_l^\times$ be such that $\lambda^2 \neq 1$. (Note that such λ exists since $l \geq 5$.) Let $\epsilon \stackrel{\text{def}}{=} \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$. Then $\epsilon \cdot \alpha \cdot \epsilon^{-1} \cdot \alpha^{-1} = \alpha^{\lambda^2 - 1}$. Thus, α (and, similarly, β) is contained in the commutator subgroup of $SL_2(\mathbf{F}_l)$, so Corollary 1.2 follows from Lemma 1.1. $\bigcirc$

Lemma 1.3. *Let $G \subseteq GL_2(\mathbf{F}_l)$ be a subgroup that contains the matrix $\alpha \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, as well as at least one matrix that is not upper triangular. Then $SL_2(\mathbf{F}_l) \subseteq G$.*

Proof. Note that α generates an l-Sylow subgroup S of $GL_2(\mathbf{F}_l)$, and that the number of l-Sylow subgroups of $GL_2(\mathbf{F}_l)$ is precisely $l + 1$. Since the normalizer of S in $GL_2(\mathbf{F}_l)$ is the set of upper triangular matrices, and we have assumed that G contains at least one such matrix, it follows that the number n_G of l-Sylow subgroups of G is ≥ 2. On the other hand, by the general theory of Sylow subgroups, it follows that n_G is congruent to 1 modulo l. Since $2 \leq n_G \leq l + 1$, we thus obtain that $n_G = l + 1$. In particular, in the notation of Lemma 1.1, we conclude that $\alpha, \beta \in G$. Thus, by Lemma 1.1, we have $SL_2(\mathbf{F}_l) \subseteq G$, as desired. $\bigcirc$

Corollary 1.4. *Let l be a prime number ≥ 5. Let $G \subseteq GL_2(\mathbf{Z}_l)$ be a closed subgroup whose image H in $GL_2(\mathbf{F}_l)$ contains the matrix $\alpha \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, as well as a matrix which is not upper triangular. Then $SL_2(\mathbf{Z}_l) \subseteq G$.*

Proof. By Lemma 1.3, we have that $SL_2(\mathbf{F}_l) \subseteq H$. Let $G' \subseteq SL_2(\mathbf{Z}_l)$ be the commutator subgroup of G. Thus, by Corollary 1.2, G' surjects onto $SL_2(\mathbf{F}_l)$. Now by [Ser], Chapter IV, §3.4, Lemma 3, this implies that $SL_2(\mathbf{Z}_l) = G' \subseteq G$, as desired. $\bigcirc$

§2. The Height of an Elliptic Curve

Let K be a *number field*, of degree d over $\mathbf{Q}$. Let $E \to \mathrm{Spec}(\mathcal{O}_K)$ be a semi-abelian variety of dimension 1 whose generic fiber E_K is proper. Thus, E_K is an elliptic curve over K. Let us write

$$\omega_E$$

for the finite, flat $\mathcal{O}_K$-module of rank one consisting of the invariant differentials on E.

If $\sigma : K \hookrightarrow \mathbf{C}$ is a complex embedding, then we get a natural metric on $(\omega_E)_\sigma \stackrel{\mathrm{def}}{=} (\omega_E) \otimes_{K,\sigma} \mathbf{C}$ by integration: if $\alpha \in (\omega_E)_\sigma$, then

$$|\alpha|^2 \stackrel{\mathrm{def}}{=} \int_{E_\sigma} \alpha \wedge \overline{\alpha}$$

(where $E_\sigma \stackrel{\mathrm{def}}{=} E \otimes_{K,\sigma} \mathbf{C}$, and $\overline{\alpha}$ is the complex conjugate of α). Thus, by equipping ω_E which this metric at the archimedean places of K, we obtain an *arithmetic line bundle* $\overline{\omega}_E$ (cf. Chapter I, Definition 1.1). Let us write

$$h_E \stackrel{\mathrm{def}}{=} \deg(\overline{\omega}_E) \in \mathbf{R}$$

This number is often referred to as the *Faltings height of the elliptic curve* E.

Next, let us observe that $E \to \mathrm{Spec}(\mathcal{O}_K)$ defines a classifying morphism

$$\phi : \mathrm{Spec}(\mathcal{O}_K) \to \overline{\mathcal{M}}_{1,0}$$

where $\overline{\mathcal{M}}_{1,0}$ is the moduli stack of semi-abelian varieties of dimension one over $\mathbf{Z}$. As is well-known, this stack has a "divisor at infinity" $\infty_{\mathcal{M}} \subseteq \overline{\mathcal{M}}_{1,0}$, whose complement $\mathcal{M}_{1,0} \subseteq \overline{\mathcal{M}}_{1,0}$ is the moduli stack of elliptic curves over $\mathbf{Z}$. Thus, we may consider the divisor

$$\infty_E \stackrel{\mathrm{def}}{=} \phi^{-1}(\infty_{\mathcal{M}}) \subseteq \mathrm{Spec}(\mathcal{O}_K)$$

Moreover, we let

$$\deg(\infty_E) \stackrel{\mathrm{def}}{=} \log(\#(\mathcal{O}_{\infty_E})) \in \mathbf{R}$$

Now we have the following well-known result:

Proposition 2.1. *There exists a universal constant C (independent of K, d, and E) such that:* $\frac{1}{12} \cdot \deg(\infty_E) \leq h_E + d \cdot C$.

Proof. This follows immediately from the formula of Proposition 1.1 of [Silv2], together with the fact that the archimedean term on the right-hand side of this formula is universally bounded below by $d \cdot C$, where C is a universal constant independent of K, d, and E. ○

Proposition 2.2. *Let $\alpha \in \mathbf{R}$. Then the number of isomorphism classes of elliptic curves E_K over number fields K of degree $\leq d$ such that $h_E \leq \alpha$ is finite.*

Proof. This follows from [Silv1], Proposition 8.2 and [Silv2], Proposition 1.1. ◯

§3. The Galois Action on the Torsion of a Tate Curve

Let K be a finite extension of $\mathbf{Q}_p$, where p is a prime number. Let us denote its residue field by k; the maximal ideal of its ring of integers by $\mathfrak{m}_K$; and its associated valuation map by $v_K : K^\times \to \mathbf{Z}$ (normalized so that v_K is surjective). Also, let us write $\overline{K}$ for an algebraic closure of K, and $\Gamma_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K)$. Let $E \to \mathrm{Spec}(\mathcal{O}_K)$ be a semi-abelian scheme of dimension one over $\mathcal{O}_K$. Let us assume, moreover, that the generic fiber E_K of E is proper, while the special fiber E_k of E is equal to $(\mathbf{G}_{\mathrm{m}})_k$, the multiplicative group over k.

Let l be a prime number (possibly equal to p). Let us write

$$M_l(E) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbf{Z}/l \cdot \mathbf{Z}, E(\overline{K}))$$

for the *"mod l" Tate module* of E_K. Thus, $M_l(E)$ is (noncanonically) isomorphic as an $\mathbf{F}_l$-module to $\mathbf{F}_l^2$, and is, moreover, equipped with a continuous action by Γ_K (induced by the natural action of Γ_K on $\overline{K}$). Also, it is well-known (see, e.g., [FC], Chapter III, Corollary 7.3) that $M_l(E)$ fits into a natural exact sequence of Γ_K-modules

$$0 \to \mathbf{F}_l(1) \to M_l(E) \to \mathbf{F}_l \to 0$$

where the "(1)" is a Tate twist, and "$\mathbf{F}_l$" is equipped with the trivial Galois action. Moreover, the extension class associated to this exact sequence is precisely that obtained by extracting an l^{th} root of the *Tate parameter* $q_E \in \mathfrak{m}_K$. The Tate parameter is an element of $\mathfrak{m}_K$ that is naturally associated to E and has the property that the subscheme $\mathcal{O}_K/(q_E)$ is equal to the pull-back via the classifying morphism $\mathrm{Spec}(\mathcal{O}_K) \to \overline{\mathcal{M}}_{1,0}$ of the divisor $\infty_{\mathcal{M}} \subseteq \overline{\mathcal{M}}_{1,0}$. Thus, the above exact sequence splits if and only if q_E has an l^{th} root in K. Note that in order for this to happen, it must be the case that $v_K(q_E)$ is divisible by l. In particular, we have the following result:

Lemma 3.1. *Suppose that $N \subseteq M_l(E)$ is a one-dimensional $\mathbf{F}_l$-subspace which is stabilized by Γ_K. Then* either $v_K(q_E) \in l \cdot \mathbf{Z}$, or N *is equal to the subspace* $\mathbf{F}_l(1) \subseteq M_l(E)$ *of the above exact sequence.*

Definition 3.2. We shall refer to $v_K(q_E) \in \mathbf{Z}_{>0}$ as the *local height of* E.

Next, let us recall that the submodule $\mathbf{F}_l(1) \subseteq M_l(E)$ defines a finite, flat subgroup scheme

$$\mu_l \subseteq E$$

over $\mathcal{O}_K$. Thus, we may form the quotient

$$E' \stackrel{\text{def}}{=} E/\mu_l$$

Then $E' \to \text{Spec}(\mathcal{O}_K)$ is a one-dimensional semi-abelian scheme over $\mathcal{O}_K$ whose special fiber is $(\mathbf{G}_{\text{m}})_k$ and whose generic fiber is proper. Moreover, its Tate parameter $q_{E'}$ satisfies

$$q_{E'} = q_E^l$$

Let us write

$$\deg(\infty_E) \stackrel{\text{def}}{=} \log(\#(\mathcal{O}_K/(q_E))) \in \mathbf{R}$$

Then we have

$$\deg(\infty_{E'}) = l \cdot \deg(\infty_E)$$

§4. An Effective Estimate of the Image of Galois

Let us write $\overline{\mathbf{Q}}$ for an algebraic closure of $\mathbf{Q}$. Let K be a subfield of $\overline{\mathbf{Q}}$ of degree $d < \infty$ over $\mathbf{Q}$, and let $E \to \text{Spec}(\mathcal{O}_K)$ be as in §2.

Let l be a prime number. Suppose that we are given a subgroup scheme $G_K \subseteq E_K$ (over K) such that over $\overline{\mathbf{Q}}$, G_K becomes isomorphic to $\mathbf{Z}/l \cdot \mathbf{Z}$. We shall call such G_K *l-cyclic*. Write $(E_G)_K$ for the quotient of E_K by G_K. Note that since $(E_G)_K$ is isogenous to E_K, it has semi-stable reduction at all the primes of K, hence extends to a one-dimensional semi-abelian scheme $E_G \to \text{Spec}(\mathcal{O}_K)$.

Lemma 4.1. *Suppose that $G_K \subseteq E_K$ is l-cyclic, and that l is greater than the local heights of E at all of its primes of bad (multiplicative) reduction. Then there exists a universal constant C (independent of K, d, and E) such that*

$$\frac{1}{12} l \cdot \deg(\infty_E) \leq h_E + d \cdot C + 2d \cdot \log(l)$$

Proof. Note that the assumption on l implies (by Lemma 3.1) that at all the primes of bad reduction, G_K corresponds to the subspace $\mathbf{F}_l(1)$ of Lemma 3.1. Thus, at primes of bad reduction, E_G may be identified with the elliptic curve E' discussed at the end of §3. In particular,

$$\deg(\infty_{E_G}) = l \cdot \deg(\infty_E)$$

On the other hand, the degree l covering morphism $(E_G)_K \to E_K$ extends (cf., e.g., [FC], Chapter I, Proposition 2.7) to a morphism $E_G \to E$. Thus, we have a natural inclusion $\omega_E \subseteq \omega_{E_G}$ whose cokernel is annihilated by l. Moreover, since integrating a $(1,1)$-form over E_σ differs from integrating over $(E/G)_\sigma$ by a factor of l, it follows that

$$\deg(\overline{\omega}_{E_G}) \leq \deg(\overline{\omega}_E) + 2d \cdot \log(l)$$

Thus, Lemma 4.1 follows from Proposition 2.1. ○

Lemma 4.2. *There exists a universal (positive) constant C (independent of d) such that for each positive integer d, there exists a finite subset $\mathcal{E}_d \subseteq \mathcal{M}_{1,0}(\overline{\mathbf{Q}})$ with the following property: Suppose that there exists an l-cyclic $G_K \subseteq E_K$, where $[K : \mathbf{Q}] = d$; E_K is an elliptic curve over K with semi-stable reduction at all primes and at least one prime of bad reduction; and $l \geq 100 \cdot (h_E + C \cdot d^2)$. Then the point of $[E_K] \in \mathcal{M}_{1,0}(\overline{\mathbf{Q}})$ defined by E_K belongs to $\mathcal{E}_d$.*

Proof. First, observe that if v is any local height of E_K, then $\deg(\infty_E) \geq v \cdot \log(2)$. Thus, Proposition 2.1 implies that by choosing C appropriately, we may assume that $h_E + C \cdot d^2 \geq h_E + C \cdot d \geq \frac{1}{12} \cdot \deg(\infty_E)$. Thus, we obtain that

$$\begin{aligned} l &\geq \frac{100}{12} \cdot \deg(\infty_E) \\ &\geq (\frac{100 \cdot \log(2)}{12}) \cdot v \\ &> v \end{aligned}$$

which shows that the hypotheses of Lemma 4.1 are satisfied. Thus, we conclude (from Lemma 4.1) that if we choose C so that $l \geq \frac{48d \cdot \log(l)}{\log(2)}$ (cf. Lemma 4.3 below) then

$$\begin{aligned} \frac{1}{12} l \cdot \deg(\infty_E) &\leq h_E + d \cdot C' + 2d \cdot \log(l) \\ &\leq h_E + d \cdot C' + \frac{l \cdot \log(2)}{24} \end{aligned}$$

for some universal constant C'. Next, observe that since E_K *has at least one prime of bad reduction*, it follows that $\log(2) \leq \deg(\infty_E)$. Thus, substituting into the above inequality, we obtain that

$$\frac{l \cdot \log(2)}{24} \leq h_E + d \cdot C'$$

On the other hand, $\frac{\log(2)}{24} \geq \frac{2}{100}$, and, by assumption, $l \geq 100 \cdot h_E$, so (substituting) we obtain that $2h_E \leq h_E + d \cdot C'$, i.e., $h_E \leq d \cdot C'$. But this implies, by Proposition 2.2, that $[E_K]$ belongs to some finite exceptional set $\mathcal{E}_d$, as desired. ○

Lemma 4.3. *Let x and y be real numbers such that $x, y \geq 2$, and $x \geq 2y^2$. Then $x \geq y \cdot \log(x)$.*

Proof. First observe that $y \geq \frac{1}{2} \cdot \log(2) + \log(y)$ for $y \geq 2$. Indeed, this is true for $y = 1$ (since $4 \geq 3 \geq 3 \cdot \log(2)$), and the function $\phi(y) = y - \log(y)$ has derivative $1 - \frac{1}{y}$, which is ≥ 0 for $y \geq 1$. Thus when $x = 2y^2$, we have $x = 2y^2 \geq 2y \cdot \log(2^{\frac{1}{2}} \cdot y) = y \cdot \log(x)$, as desired. Moreover, the function $\psi(x) = x - y \cdot \log(x)$ has derivative $1 - \frac{y}{x}$, which is ≥ 0 for $x \geq 2y^2 \geq 2y \geq y$ (since $y \geq 1$). Thus, $\psi(x)$ is increasing for $x \geq 2y^2$, so $\psi(x) \geq 0$ for $x \geq 2y^2$, as desired. ○

Theorem 4.4. *There exist:*

(1) *a universal (positive) constant C (independent of d); and*

(2) *for each positive integer d, a finite subset $\mathcal{E}_d \subseteq \mathcal{M}_{1,0}(\overline{\mathbf{Q}})$*

with the following property: Suppose that

(i.) $K \subseteq \overline{\mathbf{Q}}$ is a subfield of degree $d < \infty$;

(ii.) E_K is an elliptic curve over K with semi-stable reduction at all primes and at least one prime of bad reduction;

(iii.) the isomorphism class $[E_K] \in \mathcal{M}_{1,0}(\overline{\mathbf{Q}})$ does not belong to $\mathcal{E}_d$;

(iv.) l is a prime number $\geq 100 \cdot (h_E + C \cdot d^2)$.

Then the image of the Galois representation $\mathrm{Gal}(\overline{\mathbf{Q}}/K) \to GL_2(\mathbf{Z}_l)$ associated to E_K contains $SL_2(\mathbf{Z}_l)$.

Proof. As we saw in the proof of Lemma 4.2, in the situation under consideration, the local height of E_K at a finite prime of E_K of bad reduction can never be divisible by l. Since, by hypothesis, at least one such finite prime exists, it follows that the image of Galois in $GL_2(\mathbf{F}_l)$ contains the element "α" (cf. Corollary 1.4), i.e., $\binom{1\ 1}{0\ 1}$. Moreover, by Lemma 4.2, it follows that the image of Galois in $GL_2(\mathbf{F}_l)$ contains at least one matrix which is not upper triangular. Thus, we conclude from Corollary 1.4 that the image of Galois in $GL_2(\mathbf{Z}_l)$ contains $SL_2(\mathbf{Z}_l)$, as desired. ○

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Chapter III: The Universal Extension of a Log Elliptic Curve

§0. Introduction

In this Chapter, we review various well-known facts concerning the universal extension of an elliptic curve. The main ideas that we discuss are as follows: Given a *log elliptic curve* (cf. Definition 1.1) $E \to S$, we define its *universal extension* $E^\dagger \to E$ (cf. Definition 1.2) as a certain group scheme of line bundles on E of degree 0 equipped with a connection. For degenerating log elliptic curves (i.e., "Tate curves"), this universal extension admits a *canonical splitting* (cf. Theorem 2.1) which is essentially uniquely characterized by the fact that it is a group homomorphism. The universal extension $E^\dagger \to E$ also has a *Hodge-theoretic interpretation* (cf. Theorem 4.2) as the Hodge-theoretic first Chern class of the divisor defined by the origin of E. Moreover, this Hodge-theoretic interpretation of the universal extension allows one to *analytically continue* (cf. Theorem 5.6) the canonical splitting, relative to the function-theoretic context of [Mumf]. We construct this analytic continuation by studying the *Schottky-Weierstrass ζ-function* (cf. §5), which is a certain analogue of the classical Weierstrass ζ-function, but for the Schottky uniformization (as opposed to the full uniformization by the complex plane, as in the classical case) of an elliptic curve. This analytic continuation allows one to give explicit coordinates for the *torsion points of the universal extension* (cf. Corollary 5.9), which play a fundamental role in this paper. During our analysis of the universal extension, we pause, in §3 (cf. also Remark 2 in §6), to review the complex analogues of the ideas discussed in this Chapter. Finally, in §6, 7, we discuss certain *"higher" analogues of the Schottky-Weierstrass ζ-function.*

§1. Definition of the Universal Extension

In this §, we would like to define a certain canonical extension of a "log elliptic curve" – called the *universal extension of the log elliptic curve* – which will play a *key role* in the present paper. One useful reference for this universal extension (in the non-logarithmic case) is Appendix C of [Katz].

Let

$$\overline{\mathcal{M}}_{1,1}$$

be the *moduli stack of one-pointed stable curves of genus one over* $\mathbf{Z}$ (see, e.g., [Knud] for more details). Note that $\overline{\mathcal{M}}_{1,1}$ is equipped with a natural log structure (in the sense of [Kato]) defined by its divisor at infinity. We denote the resulting *log stack* by $\overline{\mathcal{M}}_{1,1}^{\log}$. Let

$$(\mathcal{C} \to \overline{\mathcal{M}}_{1,1}, \epsilon : \overline{\mathcal{M}}_{1,1} \to \mathcal{C})$$

denote the *tautological one-pointed stable curve of genus one*. Now observe that the inverse image on $\mathcal{C}$ of the divisor at infinity of $\overline{\mathcal{M}}_{1,1}$ is a divisor with normal crossings on $\mathcal{C}$, hence defines in a natural way a log structure on $\mathcal{C}$. Denote the resulting log stack by $\mathcal{C}^{\log}$. Let $\overline{\mathcal{M}}_{1,0}^{\log} \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{1,1}^{\log}$. Thus, we have a natural *log smooth morphism*

$$\mathcal{C}^{\log} \to \overline{\mathcal{M}}_{1,0}^{\log}$$

together with a section $\epsilon : \overline{\mathcal{M}}_{1,0} \to \mathcal{C}$. Let $\mathcal{E} \subseteq \mathcal{C}$ be the open substack of $\mathcal{C}$ of points that are smooth over $\overline{\mathcal{M}}_{1,0}$. Since ϵ maps into $\mathcal{E}$, we have (by abuse of notation) a section $\epsilon : \overline{\mathcal{M}}_{1,0} \to \mathcal{E}$. Moreover,

$$(\mathcal{E} \to \overline{\mathcal{M}}_{1,0}, \epsilon : \overline{\mathcal{M}}_{1,0} \to \mathcal{E})$$

forms a *semi-abelian scheme* (see, e.g., [FC], Definition 2.3 of Chapter I, for more details) of dimension 1 over $\overline{\mathcal{M}}_{1,0}$.

Definition 1.1. We shall refer to $(\mathcal{C}^{\log} \to \overline{\mathcal{M}}_{1,0}^{\log}, \epsilon : \overline{\mathcal{M}}_{1,0} \to \mathcal{E})$ as the *universal log elliptic curve.* If $S^{\log}$ is a fine noetherian log scheme (see [Kato] for more details), then we shall refer to the datum of a morphism $\alpha : S^{\log} \to \overline{\mathcal{M}}_{1,0}^{\log}$ as a *log elliptic curve.*

This definition prompts the following basic remarks:

(1) The point of distinguishing $\overline{\mathcal{M}}_{1,0}^{\log}$ from $\overline{\mathcal{M}}_{1,1}^{\log}$ (even though both notations denote the same log stack) is that *the natural log structures of the tautological objects that they parametrize differ.* This leads for instance to different canonical p-adic uniformization theories (as in [Mzk1]).

(2) Let $\alpha : S^{\log} \to \overline{\mathcal{M}}_{1,0}^{\log}$ be a *log elliptic curve.* Then pulling back $\mathcal{C}^{\log} \to \overline{\mathcal{M}}_{1,0}^{\log}$, $\mathcal{E}$, and ϵ gives rise to $C^{\log} \to S^{\log}$, $E \to S$, $e : E \to S$. Often, by abuse of terminology, we shall say "let $C^{\log} \to S^{\log}$ be a log elliptic curve." This means that implicitly we assume that some α has been given and that $C^{\log}$ has been constructed from α in the fashion just described.

This completes the discussion of the definition of a log elliptic curve.

Let

$$f^{\log} : C^{\log} \to S^{\log}$$

be a *log elliptic curve*. Let

$$\omega_E \stackrel{\text{def}}{=} e^* \Omega_{E/S}$$

Thus, ω_E is a line bundle on S. Let $W \to S$ denote the affine group scheme defined by ω_E. Thus, W is the spectrum of the symmetric algebra of $\tau_E \stackrel{\text{def}}{=} \omega_E^\vee$ (the dual bundle to ω_E) over $\mathcal{O}_S$. Now we would like to define a natural extension of group schemes

$$0 \to W \to E^\dagger \to E \to 0$$

(cf. Appendix C, §C1, of [Katz]), as follows. First, let T be an S-scheme. Since $E^\dagger$ will be defined as a *functor over* E, points of $E^\dagger(T)$ will consist of points of $E(T)$ plus some extra information:

> We define $E^\dagger(T)$ to be the set of isomorphism classes of pairs (x, ∇_x), where $x \in E(T)$, and ∇_x is a *logarithmic connection* (with respect to the morphism $C^{\log} \to S^{\log}$) *on the line bundle* $\mathcal{L}_x \stackrel{\text{def}}{=} \mathcal{O}_{C_T}(x - e)$ (where $C_T = C \times_S T$).

(Here in the notation "$\mathcal{O}_{C_T}(x - e)$," we use "e" (respectively, "x") to denote the (Cartier) divisor defined by the image of the section e (respectively, x) in C_T.) It is immediately clear that $E^\dagger$ is a torsor over E under the group scheme W (since ω_E may be identified with the push-forward of $\Omega_{C^{\log}/S^{\log}}$ via $f^{\log} : C^{\log} \to S^{\log}$). Moreover, $E^\dagger$ has a natural abelian group law of its own given as follows: The sum of (x, ∇_x) and (y, ∇_y) is the line bundle $\mathcal{L}_{x+y} \cong \mathcal{L}_x \otimes_{\mathcal{O}_{C_T}} \mathcal{L}_y$ equipped with the connection $\nabla_x \otimes \nabla_y$.

Definition 1.2. We shall refer to the exact sequence

$$0 \to W \to E^\dagger \to E \to 0$$

of smooth, abelian group schemes over S as the *universal extension of* $E \to S$ (or $C^{\log} \to S^{\log}$).

Let

$$\mathcal{H} \stackrel{\text{def}}{=} \mathbf{R}^1 f^{\log}_{\text{DR},*} \mathcal{O}_C$$

be the *first (logarithmic) de Rham cohomology module of* $f^{\log} : C^{\log} \to S^{\log}$ (see, e.g., [Kato]). Then $\mathcal{H}$ is a rank two vector bundle on S equipped with a filtration called the *Hodge filtration.* This filtration gives rise to an exact sequence

$$0 \to \omega_E \to \mathcal{H} \to \tau_E \to 0$$

Proposition 1.3. *There is a canonical isomorphism between $\mathcal{H}$ and the tangent space to the group scheme $E^\dagger$ at the origin. Moreover, the exact sequence above arising from the Hodge filtration on $\mathcal{H}$ corresponds under this isomorphism to the filtration induced on the tangent space at the origin to $E^\dagger$ by the exact sequence* $0 \to W \to E^\dagger \to E \to 0$.

Proof. This follows immediately by working over the dual numbers $S[\varepsilon]/(\varepsilon^2)$ and thinking about the functorial definitions of $E^\dagger$ and de Rham cohomology. ○

§2. Canonical Splitting at Infinity

We maintain the notations of §1. In this §, we would like to discuss certain *canonical splittings of the universal extension $E^\dagger \to E$ of E.*

In this §, we assume that the inverse image (via the implicit *classifying morphism* $\alpha : S^{\log} \to \overline{\mathcal{M}}_{1,0}^{\log}$ for the given log elliptic curve) of the divisor at infinity of $\overline{\mathcal{M}}_{1,0}$ is a (Cartier) divisor $D \subseteq S$. Write $\mathcal{J}$ for the ideal defining D. Thus, $\mathcal{J}$ is a line bundle on S. Moreover, we have a natural *identification*:

$$E|_D = (\mathbf{G}_{\mathrm{m}})_D$$

Now let $\widehat{S}$ be the *completion* of S at D. Thus, $\widehat{S}$ is a formal scheme. Let us write $E_{\widehat{S}}$ for the *formal object* obtained by pulling back $E \to S$ to $\widehat{S}$. Then one knows that

$$E_{\widehat{S}} \cong (\mathbf{G}_{\mathrm{m}})_{\widehat{S}}$$

(where the isomorphism is an isomorphism of $\mathcal{J}$-adic formal group objects over the formal scheme $\widehat{S}$). Indeed, this is a relative simple special case (the case already known to Tate) of the theory of [FC], Chapter III. In fact, this isomorphism is the *unique such isomorphism that reduces to the identity over D* (by [FC], Chapter I, Theorem 2.2).

Now let us pull-back the universal extension $E^\dagger \to E$ to a morphism

$$E^\dagger_{\widehat{S}} \to E_{\widehat{S}}$$

over $\widehat{S}$. We would like to construct a certain canonical section of this morphism. This canonical section will be a *homomorphism of group objects*. Thus, by [FC], Chapter I, Theorem 2.2, it will exist and be unique once it is constructed over D.

The *definition of this canonical section over* D is given as follows: As discussed above, we identify E_D with $(\mathbf{G}_{\mathrm{m}})_D$. Let us think of the affine ring of $(\mathbf{G}_{\mathrm{m}})_D$ as the ring of Laurent polynomials in an indeterminate u. Now let x be a D-valued point of $(\mathbf{G}_{\mathrm{m}})_D$. Then (as in Appendix C, §C1, of [Katz]) points of $E^{\dagger}$ lying over x may be identified with sections of $\omega_{C^{\log}/S^{\log}} \otimes_{\mathcal{O}_S} \mathcal{O}_D$ that are regular away from x and e, but have a simple pole with residue 1 (respectively, -1) at x (respectively, e). For instance, if x corresponds to the section u_x of $\mathcal{O}_D^{\times}$, then

$$\omega_x \stackrel{\text{def}}{=} \frac{du}{(u-u_x)} - \frac{du}{(u-1)}$$

is such a differential. Since the correspondence $x \mapsto \omega_x$ is clearly functorial, we thus obtain a *scheme-theoretic section*

$$E_D \to E_D^{\dagger}$$

It remains to check that this is a *homomorphism of group schemes*. To see this, suppose that $y \in E(D)$, $z \stackrel{\text{def}}{=} x \cdot y$. Then the isomorphism between $\mathcal{O}_{C_D}(x-e) \otimes \mathcal{O}_{C_D}(y-e)$ and $\mathcal{O}_{C_D}(z-e)$ is given by division by the rational function

$$\frac{(u-1)(u-u_z)}{(u-u_x)(u-u_y)}$$

Moreover, the logarithmic derivative of this rational function is clearly equal to $\omega_z - \omega_x - \omega_y$. This observation implies that the section constructed is a homomorphism of group schemes, as desired.

Thus, in summary, we see that we have constructed a *homomorphism of group objects*

$$\kappa : E_{\widehat{S}} \to E_{\widehat{S}}^{\dagger}$$

in the formal category over $\widehat{S}$ which forms a section of $E_{\widehat{S}}^{\dagger} \to E_{\widehat{S}}$. Moreover, since there are no nontrivial homomorphisms from $\mathbf{G}_{\mathrm{m}}$ into W (note: W is Zariski locally isomorphic to $\mathbf{G}_a$), it follows that κ is the unique such section. We state this as a theorem:

Theorem 2.1. *The extension*

$$0 \to W_{\widehat{S}} \to E_{\widehat{S}} \to E_{\widehat{S}}^{\dagger} \to 0$$

of group objects over the formal scheme $\widehat{S}$ admits a unique splitting

$$\kappa : E_{\widehat{S}} \to E^{\dagger}_{\widehat{S}}$$

– which we shall refer to as the canonical splitting *of this extension. In particular, it follows that there exists a natural isomorphism*

$$E^{\dagger}_{\widehat{S}} \cong (\mathbf{G}_{\mathrm{m}})_{\widehat{S}} \times W_{\widehat{S}}$$

of group objects over $\widehat{S}$.

Now recall that $E^{\dagger}$ may be thought of as an ω_E-torsor over E. Thus, in particular, we may write this torsor as an extension of coherent $\mathcal{O}_E$-modules

$$0 \to \mathcal{O}_E \to \mathcal{T} \to \tau_E|_E \to 0$$

Let $\mathcal{R}_{E^{\dagger}}$ be the sheaf of quasi-coherent $\mathcal{O}_E$-algebras whose spectrum is $E^{\dagger}$:

$$E^{\dagger} = \mathit{Spec}(\mathcal{R}_{E^{\dagger}})$$

Thus, Zariski locally on E, $\mathcal{R}_{E^{\dagger}}$ is the symmetric algebra over $\mathcal{O}_E$ of $\tau_E|_E$. Moreover, there is a natural inclusion

$$\mathcal{T} \subseteq \mathcal{R}_{E^{\dagger}}$$

and $\mathcal{T}$ generates $\mathcal{R}_{E^{\dagger}}$ as an $\mathcal{O}_E$-algebra. If i is a positive integer, let us denote by

$$\mathcal{R}_{E^{\dagger}}[i] \subseteq \mathcal{R}_{E^{\dagger}}$$

the $\mathcal{O}_E$-submodule of $\mathcal{R}_{E^{\dagger}}$ generated by $\mathcal{O}_E$-linear combinations of products of i sections of $\mathcal{T}$. (Also, let $\mathcal{R}_{E^{\dagger}}[0] \stackrel{\mathrm{def}}{=} \mathcal{O}_E$.)

Definition 2.2. We shall refer to sections of $\mathcal{O}_{E^{\dagger}}$ that lie inside $\mathcal{R}_{E^{\dagger}}[i]$ as being *of torsorial degree* $\leq i$. More generally, if $\mathcal{L}$ is a line bundle on E, then we shall refer to sections of $\mathcal{L} \otimes_{\mathcal{O}_E} \mathcal{O}_{E^{\dagger}}$ that lie inside $\mathcal{L} \otimes_{\mathcal{O}_E} \mathcal{R}_{E^{\dagger}}[i]$ as being *of torsorial degree* $\leq i$.

Now by Theorem 2.1, we have a natural splitting

$$E^{\dagger}_{\widehat{S}} \cong E_{\widehat{S}} \times W_{\widehat{S}}$$

This natural splitting then induces an isomorphism

$$(\mathcal{R}_{E^{\dagger}})_{\widehat{S}} \cong \mathcal{O}_{E_{\widehat{S}}}[\tau_E|_{E_{\widehat{S}}}]$$

(where "$[-]$" means "the symmetric algebra of"). That is to say, whereas $\mathcal{R}_{E^{\dagger}}$ is, in general, only Zariski locally isomorphic to a polynomial algebra, *over* $\widehat{S}$, it becomes naturally isomorphic to a polynomial algebra (i.e., the symmetric algebra of $\tau_E|_{E_{\widehat{S}}}$). Thus, if f is a section of $\mathcal{L} \otimes \mathcal{O}_{E^{\dagger}_{\widehat{S}}}$, one can make the following definition:

Definition 2.3. We shall refer to the degree i term of f (relative to the above isomorphism of $(\mathcal{R}_{E^{\dagger}})_{\widehat{S}}$ with a polynomial algebra) as the *degree i component*

$$\mathrm{Comp}_i(f) \in \mathcal{L} \otimes_{\mathcal{O}_S} \tau_E^{\otimes i}|_{E_{\widehat{S}}}$$

of f.

§3. Canonical Splittings in the Complex Case

Before proceeding, we pause to review the basic theory of the universal extension and its canonical splittings over the *complex numbers* (cf., e.g., [Katz], Appendix C). Thus, let E be an *elliptic curve over* $\mathbf{C}$. Then its universal extension $E^{\dagger}$ may be *constructed analytically* as follows: First, recall the *de Rham isomorphism*

$$H^1_{\mathrm{DR}}(E, \mathcal{O}_E) \cong H^1_{\mathrm{sing}}(E, \mathbf{C})$$

between the de Rham cohomology (algebraic or holomorphic) of E and the singular cohomology of E with complex coefficients. Let

$$\Lambda \subseteq \mathcal{H} \stackrel{\mathrm{def}}{=} H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$$

be the subgroup defined (using the de Rham isomorphism) by

$$H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbf{Z}) \subseteq H^1_{\mathrm{sing}}(E, \mathbf{C})$$

Thus, Λ is a free $\mathbf{Z}$-module of rank 2, and $\mathbf{C} \otimes_{\mathbf{Z}} \Lambda \cong \mathcal{H}$.

Now by Proposition 1.3 and the well-known theory of the (complex analytic) exponential map of a complex Lie group, we have a natural uniformizing map

$$\mathcal{H} \to E^{\dagger}$$

compatible with the additive group structures of $\mathcal{H}$ and $E^{\dagger}$. Moreover, the kernel of this homomorphism (cf. the discussion at the end of [Katz], Appendix C, §C5) is

$$\Lambda \subseteq \mathcal{H}$$

Thus, we have the following:

Proposition 3.1. *If E is an elliptic curve over $\mathbf{C}$, then (the complex Lie group defined by) $E^{\dagger}$ is naturally isomorphic to the quotient*

$$\mathcal{H}/\Lambda$$

where $\mathcal{H}$ is the first de Rham cohomology module of E, and Λ is the $\mathbf{Z}$-submodule arising from the singular cohomology of E with coefficients in $2\pi i \cdot \mathbf{Z}$.

Next, we would like to discuss certain *canonical splittings* that exist in the complex analytic case. In fact, *in the complex analytic case, there are three different types of canonical splitting*, all of which are well-known and classical, and all of which will be relevant to the theory of the present paper (but in different ways). The three canonical splittings that we will consider are the following:

The Real Analytic Splitting: Write $\Lambda_{\mathbf{R}}$ (respectively, $\Lambda_{\mathbf{C}}$) for $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$ (respectively, $\Lambda \otimes_{\mathbf{Z}} \mathbf{C}$). Define

$$E_{\mathbf{R}} \stackrel{\text{def}}{=} \Lambda_{\mathbf{R}}/\Lambda \subseteq \Lambda_{\mathbf{C}}/\Lambda = E^{\dagger}$$

Thus, $E_{\mathbf{R}}$ is a *real analytic* submanifold of $E^{\dagger}$. Moreover, it is easy to see that $E_{\mathbf{R}}$ maps bijectively onto E, i.e., we have an *isomorphism of real analytic Lie groups*

$$E_{\mathbf{R}} \cong E$$

Unlike E, however, $E_{\mathbf{R}}$ is only a "real analytic torus," *not* a complex analytic torus inside $E^{\dagger}$. Put another way, $E_{\mathbf{R}}$ defines a real analytic section

$$\kappa_{\mathbf{R}} : E \to E^{\dagger}$$

of $E^{\dagger} \to E$ which is a homomorphism in the category of real analytic Lie groups.

Definition 3.2. We shall refer to $\kappa_{\mathbf{R}} : E \to E^{\dagger}$ as the *canonical real analytic section of* $E^{\dagger} \to E$.

Remark 1. Note in particular that since $\kappa_{\mathbf{R}}$ is a group homomorphism, it maps *torsion points of* E *to torsion points of* $E^{\dagger}$. Thus, although $\kappa_{\mathbf{R}}$ is only defined in the real analytic category, *its restriction to torsion points can be defined entirely algebraically.* For instance, if E is defined over a number field, then the restriction of $\kappa_{\mathbf{R}}$ to torsion points of E is also defined over a number field. This fact will be of *fundamental importance* in this paper.

Remark 2. Although the following discussion will not be very relevant to the present paper, the reader might wonder (in view of the choice of notation) whether $\kappa_{\mathbf{R}}$ is, in some sense, "analogous" to the formal analytic κ of Theorem 2.1. In fact, there is an analogy: Namely, $E_{\mathbf{R}} \subseteq E^{\dagger}$ may be obtained as the invariant subset of a certain natural complex conjugation morphism on $E^{\dagger}$. Moreover, frequently in discussions of global motives, it is natural to think of complex conjugation as "Frobenius at the infinite prime." On the other hand, the splitting κ may also be constructed p-adically as follows: $E^{\dagger}$ may be thought of as a certain crystal in group schemes on the log crystalline site of $S^{\log}$. Put another way, $E^{\dagger}$ is, in essense, the log crystalline cohomology of $C^{\log} \to S^{\log}$ with coefficients in $\mathcal{O}^{\times}$. Moreover, as a log crystalline cohomology object, $E^{\dagger}$ is equipped with a natural Frobenius action. The section κ may then be constructed p-adically as the unique *Frobenius invariant* section of $E^{\dagger} \to E$. That is to say, both κ and $\kappa_{\mathbf{R}}$ may be constructed as invariant subsets of Frobenius actions at finite and infinite primes, respectively. We refer to the discussion of [Mzk2], Introduction, §1, for more details.

The $\mathbf{G}_{\mathrm{m}}$-Splitting: To define this splitting, fix a rank one $\mathbf{Z}$-submodule:

$$\Lambda_1 \subseteq \Lambda$$

This splitting defines an *intermediate covering* of the covering defined by the exponential map of E:

$$\tau_E \to \tau_E/\Lambda_1 \to \tau_E/\Lambda = E$$

(where by abuse of notation, we also denote by Λ the image of $\Lambda \subseteq \mathcal{H}$ under the projection $\mathcal{H} \to \tau_E$ arising from the Hodge filtration). Note that τ_E/Λ_1 is isomorphic (by an isomorphism which is unique up to composition with the inversion map) to $\mathbf{G}_{\mathrm{m}}$ (i.e., the multiplicative group $\mathbf{G}_{\mathrm{m}}$ over $\mathbf{C}$, regarded as a complex analytic Lie group). Thus, in the following discussion, we shall make the following identification:

$$\tau_E/\Lambda_1 = \mathbf{G}_{\mathrm{m}}$$

In other words, the choice of Λ_1 defines a uniformization of E by $\mathbf{G}_{\mathrm{m}}$. In particular, we may think of this uniformization as giving a presentation of E as follows:

$$E = \mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$$

where $q \in \mathbf{G}_{\mathrm{m}} = \mathbf{C}^{\times}$. This uniformization of E is referred to as the *Schottky uniformization of* E (and depends on the choice of $\Lambda_1 \subseteq \Lambda$).

Now let us consider the *universal extension* $E^{\dagger}$ of E. Now we have a commutative diagram:

$$\begin{array}{ccccccc}
\Lambda_1 & & = & \Lambda_1 & = & \Lambda_1 \\
\bigcap & & & \bigcap & & \bigcap \\
(\Lambda_1)_{\mathbf{C}} \stackrel{\mathrm{def}}{=} \Lambda_1 \otimes_{\mathbf{Z}} \mathbf{C} & & \subseteq & \Lambda_{\mathbf{C}} = \mathcal{H} & \longrightarrow & \tau_E
\end{array}$$

If we form the quotient of the second line by the first, we then obtain two morphisms

$$(\Lambda_1)_{\mathbf{C}}/\Lambda_1 \longrightarrow E^{\dagger}|_{\mathbf{G}_{\mathrm{m}}} \longrightarrow \mathbf{G}_{\mathrm{m}}$$

whose composite is an isomorphism. Put another way, we have constructed a *holomorphic section*

$$\kappa_{\Lambda_1} : \mathbf{G}_{\mathrm{m}} \to E^{\dagger}|_{\mathbf{G}_{\mathrm{m}}}$$

of the pull-back of $E^{\dagger} \to E$ by $\mathbf{G}_{\mathrm{m}} \to E$.

Definition 3.3. We shall refer to κ_{Λ_1} as the $\mathbf{G}_{\mathrm{m}}$-*splitting associated to* Λ_1.

Remark. This splitting is the *complex analogue of the splitting of Theorem 2.1*. Indeed, it is not difficult to check that if, in the context of Theorem 2.1, i.e., of a *degenerating elliptic curve*, one takes for $\Lambda_1 \subseteq H^1_{\mathrm{sing}}(E, 2\pi i\mathbf{Z}) = H_1^{\mathrm{sing}}(E,\mathbf{Z})^{\vee}$ (where "$\vee$" denotes the dual $\mathbf{Z}$-module) the $\mathbf{Z}$-submodule which is the annihilator of the submodule

$$\mathrm{Ker}\{H_1^{\mathrm{sing}}(E,\mathbf{Z}) \to H_1^{\mathrm{sing}}(\text{the degenerate elliptic curve}, \mathbf{Z}) = \mathbf{Z}\}$$

i.e., the annihilator of the *vanishing cycle*, then for this choice of Λ_1, the splitting κ_{Λ_1} just constructed coincides with the complex analytic splitting of the universal extension defined by the κ of Theorem 2.1 in a (complex analytic) neighborhood of the point at infinity $\infty \in \overline{\mathcal{M}}_{1,0}(\mathbf{C})$. Indeed, the fact that these two splittings coincide follows from the fact that there do not exist any nontrivial homomorphisms of complex Lie groups $\mathbf{C}^\times \to \mathbf{C}$.

The η-Splitting: Here, we follow the treatment of [Katz], Appendix C, §C5. Fix a $\mathbf{C}$-linear isomorphism $\mathbf{C} \cong \omega_E$. This isomorphism makes it easier to describe the splitting in question. In fact, however, *the splitting will be entirely independent of the choice of isomorphism* $\mathbf{C} \cong \omega_E$. Once this isomorphism is fixed, we get a choice of invariant differential $\omega = dz$ (where z is the standard coordinate on $\mathbf{C}$), and so we can write

$$E = \mathbf{C}/\Lambda$$

(where by abuse of notation, we also denote by Λ the image of $\Lambda \subseteq \mathcal{H}$ under the projection $\mathcal{H} \to \tau_E$ arising from the Hodge filtration). Moreover, one has the Weierstrass $\wp$-function (see, e.g., [Ahlf], p. 272, for a treatment)

$$\wp(z) \stackrel{\text{def}}{=} \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

Let us write $x \stackrel{\text{def}}{=} \wp(z)$, and $y \stackrel{\text{def}}{=} \wp'(z)$ (the derivative with respect to z). Then, as stated in [Katz], Appendix C, §C5, the differentials

$$\omega = \frac{dx}{y} \qquad \text{and} \qquad \eta \stackrel{\text{def}}{=} \frac{x\ dx}{y}$$

define by integration

$$\omega(\lambda) \stackrel{\text{def}}{=} \int_\lambda \omega; \quad \eta(\lambda) \stackrel{\text{def}}{=} \int_\lambda \eta$$

(for $\lambda \in \Lambda$) a basis of $\mathcal{F} = \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{C}) = \Lambda_{\mathbf{C}}$ (where we use the fact that the cup pairing on cohomology defines a natural isomorphism of Λ with its dual). Thus, in particular, we can write

$$\mathcal{H} \cong (\mathbf{C} \cdot \omega) \oplus (\mathbf{C} \cdot \eta)$$

Moreover, *although ω and η depend on the choice of isomorphism $\omega_E \cong \mathbf{C}$, this decomposition does not.* Thus, we get a natural splitting

$$\kappa_\eta : \tau_E \to \mathcal{H}$$

of the exact sequence $0 \to \omega_E \to \mathcal{H} \to \tau_E \to 0$ (cf. the discussion preceding Proposition 1.3). Here, we shall often think of τ_E and $\mathcal{H}$ as the universal covering spaces of E and $E^\dagger$, respectively.

Definition 3.4. We shall refer to κ_η as the *η-splitting of $\mathcal{H} \to \tau_E$.*

Remark. The relationship of this splitting with the theory of this paper is as follows: In [Katz], Appendix C, §C2, a certain natural *algebraic rational section* of the universal extension is constructed. This section will be discussed from the point of view of the present paper in §4 below. It is then shown in [Katz], Appendix C, §C6, that the *Weierstrass ζ-function may be thought of as the difference between this algebraic rational section and the splitting of Definition 3.4.* On the other hand, in the present paper, we will consider (cf. §5 below) the difference between this algebraic rational section and the splitting of Theorem 2.1. This difference will play an important role in the theory of the present Chapter. Moreover, we would like to think of this difference – which we will refer to as the *Schottky-Weierstrass ζ-function* – as being *analogous* to the classical Weierstrass ζ-function. In fact, we shall see below in §5 that many of the important properties of the Schottky-Weierstrass ζ-function are proven by arguments exactly analogous to those used to prove the basic properties of the classical Weierstrass ζ-function. The reason for the inclusion of the word "Schottky" here is that this "Schottky-Weierstrass ζ-function" is a sort of Weierstrass ζ-function with respect to the *Schottky uniformization* $\mathbf{G}_{\mathrm{m}} \to E$ of E (cf. the discussion surrounding Definition 3.3 above).

§4. Hodge-Theoretic Interpretation of the Universal Extension

In this §, we show how to interpret the universal extension as the *Hodge-theoretic first Chern class of a certain divisor.* The material of this § follows immediately from (the obvious logarithmic generalization of) [Falt], Lemma IV.4, but, for the convenience of the reader, we give a self-contained treatment here. In particular, the theory of the present § will allow us to give a *functorial definition of the splitting of [Katz], Lemma C2.1,* which will play an important role in §5 below. In [Katz], by contrast, only an explicit definition of this splitting in terms of certain "special functions" was given, without any explanation of its functorial definition. In fact, for proving the results of §5 below, it will be important to know the definition of this splitting in terms of special functions, but nonetheless it is interesting to know that this splitting can be defined by means of abstract nonsense.

Let $C^{\log} \to S^{\log}$ be a log elliptic curve. Let

$$\pi_1, \pi_2 : C \times_S E \to E$$

be the projections to the first and second factors, and denote by $E = \Delta_E \subseteq E \times_S E \subseteq C \times_S E$ the diagonal section. (Thus, Δ_E is a *(closed) divisor* in $C \times_S E$.) Let

$$Z \to C \times_S E$$

be the *torsor of relative logarithmic differentials* (i.e., differentials with respect to the logarithmic morphism $\pi_2^{\log} : C^{\log} \times_{S^{\log}} E^{\log} \to E^{\log}$) *that are regular everywhere, except at the divisor* Δ_E*, where they have a simple pole with residue* 1. Thus, Z is an $\omega_E|_{C \times_S E}$-torsor (where we think of the differentials "ω_E" as arising from the first factor of $C \times_S E$) on $C \times_S E$. Another way to think of Z is that it is the torsor defined by the *Hodge-theoretic first Chern class* in

$$\mathbf{R}^1(\pi_2)_*(C \times_S E, \omega_E|_{C \times_S E})$$

of the divisor Δ_E.

Similarly, let

$$Y \to C \times_S E$$

be the *torsor of relative logarithmic differentials* (i.e., differentials with respect to the logarithmic morphism $\pi_2^{\log} : C^{\log} \times_{S^{\log}} E^{\log} \to E^{\log}$) *that are regular everywhere, except at the divisor* $\{e\} \times E \subseteq C \times_S E$*, where they have a simple pole with residue* 1. (Here, "e" denotes the origin of E.) Thus, Y is an $\omega_E|_{C \times_S E}$-torsor on $C \times_S E$. Another way to think of Z is that it is the torsor defined by the *Hodge-theoretic first Chern class* in

$$\mathbf{R}^1(\pi_2)_*(C \times_S E, \omega_E|_{C \times_S E})$$

of the divisor $\{e\} \times E \subseteq C \times_S E$. Note that in fact, Y is the pull-back to $C \times_S E$ via π_1 of the torsor

$$T_{\mathrm{or}} \to C$$

which is the Hodge-theoretic first Chern class of the divisor $\{e\}$ in C.

Let us denote by

$$Z_\Delta \to E; \quad Y_\Delta \to E$$

the restrictions of the torsors Z and Y to $\Delta_E \subseteq C \times_S E$. Thus, Z_Δ and Y_Δ are $\omega_E|_E$-torsors on E. Since the composite of $E = \Delta_E \hookrightarrow C \times_S E$ with π_1 is the natural inclusion $E \hookrightarrow C$, it follows that

$$Y_\Delta = T_{\mathrm{or}}|_E$$

i.e., the torsor associated to the Hodge-theoretic first Chern class of the divisor $\{e\}$ in C.

On the other hand, we propose to show in the following that *when* 2 *is invertible on* S, the torsor Z_Δ is trivial, and, moreover that one can construct an explicit trivialization of Z_Δ: Let $\mathcal{J}$ be the sheaf of ideals on $C \times_S E$ that defines Δ_E. Then we have an exact sequence

$$0 \to \mathcal{J}^2/\mathcal{J}^3 \to \mathcal{J}/\mathcal{J}^3 \to \mathcal{J}/\mathcal{J}^2 \to 0$$

We will regard this exact sequence as an exact sequence of $\mathcal{O}_E$*-modules* via the action of $\mathcal{O}_E$ arising from the second factor of $C \times_S E$. We would first like to show that this exact sequence (of $\mathcal{O}_E$-modules) admits a *natural splitting* when 2 is invertible on S.

We construct this splitting as follows: First, observe that the isomorphism

$$C \times_S E \cong C \times_S E$$

given by $(\alpha, \beta) \mapsto (\alpha - \beta, \beta)$ (where α is a point of C, β a point of E, and we use the fact that the group scheme E acts on C) maps $\Delta_E \subseteq C \times_S E$ isomorphically onto $\{e\} \times E \subseteq C \times_S E$. Thus, if $\mathcal{I}_e \subseteq C$ is the sheaf of ideals defining $\{e\} \subseteq C$, then we have an exact sequence of sheaves

$$0 \to \mathcal{I}_e^2/\mathcal{I}_e^3 \to \mathcal{I}_e/\mathcal{I}_e^3 \to \mathcal{I}_e/\mathcal{I}_e^2 \to 0$$

of $\mathcal{O}_S$-modules such that the exact sequence of the preceding paragraph is the pull-back via $\pi_1 : C \times_S E \to C$ of this exact sequence. Thus, it suffices to construct a splitting of this exact sequence. To do this, we consider the automorphism "-1" on E, which extends to an automorphism of C. This automorphism induces an automorphism α of the above exact sequence (which covers the automorphism "-1" of C). Moreover, the automorphism α is of order 2, and (as one sees easily by identifying $\mathcal{I}_e/\mathcal{I}_e^2$ with the cotangent space ω_E to C at e) induces multiplication by -1 (respectively, 1) on $\mathcal{I}_e/\mathcal{I}_e^2$ (respectively, $\mathcal{I}_e^2/\mathcal{I}_e^3$). Thus, the endomorphism $\frac{1}{2} \cdot (1 - \alpha)$ of the above exact sequence induces a splitting of the above exact sequence, as desired.

Thus, we obtain a splitting

$$\mathcal{S}_{\mathcal{J}} : \mathcal{J}/\mathcal{J}^2 \to \mathcal{J}/\mathcal{J}^3$$

of the exact sequence

$$0 \to \mathcal{J}^2/\mathcal{J}^3 \to \mathcal{J}/\mathcal{J}^3 \to \mathcal{J}/\mathcal{J}^2 \to 0$$

Now let s be a local section of $\mathcal{J}/\mathcal{J}^2$ in a (Zariski) neighborhood of Δ_E. Then $\mathcal{S}_{\mathcal{J}}(s)$ lifts (noncanonically) to a local section t of $\mathcal{J}$ with a zero of order 1 at Δ_E. Thus, if we form the logarithmic derivative $\frac{dt}{t}$ of t (where we differentiate with respect to the *first factor* of $C \times_S E$), we get a differential which is regular in a neighborhood of Δ_E, except that it has a simple pole at Δ_E with a residue of order 1. Thus, $\frac{dt}{t}$ defines a section of Z over this neighborhood. Moreover, one checks immediately that the restriction of this section to Δ_E is independent of the choice of s and t. Thus, we get a section

$$\mathcal{S}_\Delta : E \to Z_\Delta$$

of Z_Δ which is globally defined over E. That is to say, we have proven the following:

Proposition 4.1. *If 2 is invertible on S, then the torsor $Z_\Delta \to E$ admits a canonical section $\mathcal{S}_\Delta : E \to Z_\Delta$ (as defined above).*

Next, let us observe that it follows from the definition of the universal extension $E^\dagger$ in §1 that the torsor

$$Z - Y$$

formed by taking the difference of the torsors Y and Z on $C \times_S E$ is trivial on the fibers of $\pi_2 : C \times_S E \to E$, and, moreover, that the torsor $Z - Y$ may thus be "pushed forward" via π_2 to form an ω_E-torsor on E which is (by definition) equal to $E^\dagger \to E$. In fact, because $Z - Y$ is trivial on the fibers of π_2, it is in fact equal to the pull-back by π_2 of the torsor $E^\dagger \to E$. In particular, since the composite of $E = \Delta_E \hookrightarrow C \times_S E$ with π_2 is the identity, it follows that we have a natural *equality of ω_E-torsors on E*:

$$Z_\Delta - Y_\Delta = (Z - Y)|_{\Delta_E} = E^\dagger$$

Thus, by Proposition 4.1, we obtain that:

Theorem 4.2. *If 2 is invertible on S, then there is a canonical isomorphism of ω_E-torsors on E*

$$E^\dagger \cong -T_{\mathrm{or}}|_E$$

between $E^\dagger \to E$ and -1 times the torsor associated to the Hodge-theoretic first Chern class of the origin $\{e\} \subseteq E$.

Next, we would like to observe a certain consequence of Proposition 4.2:

Corollary 4.3. Even if 2 is not invertible on S, *the* $\omega_E|_E$*-torsor* $E^\dagger \to E$ *extends naturally to an* $\omega_E|_C$*-torsor* $E^\dagger_C \to C$*. If* $S^{\log} = (\overline{\mathcal{M}}^{\log}_{1,0})_{\mathbf{Z}}$*, then this extension is unique.*

Proof. We begin with the case where 2 is *invertible*. Then Theorem 4.2 implies that $E^\dagger \to E$ is isomorphic $-T_{\mathrm{or}}|_E$, which is just the restriction to $E \subseteq C$ of the torsor $-T_{\mathrm{or}} \to C$. This proves *extendability*.

Next, we prove *uniqueness* in the case where $S^{\log} = (\overline{\mathcal{M}}^{\log}_{1,0})_{\mathbf{Z}}$. Write $D_S \subseteq S$ for the divisor at infinity, and $D_C \subseteq C$ for the complement of E in C with the "reduced induced" scheme structure. Thus, D_C maps isomorphically down to D_S, and $D_C \cong D_S \cong \mathrm{Spec}(\mathbf{Z})$. *Uniqueness* then follows from the fact that D_C has *codimension* 2 in C (which is a *regular* algebraic stack).

Thus, it remains to prove *extendability* in the case where $S^{\log} = (\overline{\mathcal{M}}^{\log}_{1,0})_{\mathbf{Z}}$. Note that (relative to the notation introduced in the preceding paragraph) there is a natural exact sequence of "local cohomology groups":

$$\ldots \to \mathrm{H}^1(C, \omega_E|_C) \to \mathrm{H}^1(E, \omega_E|_E) \to \mathrm{H}^2_{D_C}(C, \omega_E|_C) \to \ldots$$

where $\mathrm{H}^2_{D_C}$ denotes "cohomology with supports in D_C." On the other hand, it is well-known that since the subscheme $D_C \subseteq C$ is a *local complete intersection of codimension* 2, the cohomology module $\mathrm{H}^2_{D_C}$ is *flat* over $\mathbf{Z}$. (Indeed, this amounts to a well-known computation involving the *Koszul complex* (cf., e.g., [Mats]): If t_1 and t_2 are local parameters on S at D_C, then this cohomology is a direct limit of $\mathbf{Z}$-modules of the form $\mathbf{Z}[t_1, t_2]/(t_1^n, t_2^n)$, as $n \to \infty$.) Thus, it follows that the vanishing of the obstruction to extending $E^\dagger$ to C may be checked after restriction to $\mathrm{Spec}(\mathbf{Z}[\frac{1}{2}])$. This completes the proof of the corollary. ○

Remark. The reader may find it strange that despite the fact that we have shown the torsors $E^\dagger_C \to C$ and $-T_{\mathrm{or}} \to C$ to be isomorphic after one inverts 2, we may not conclude immediately that they are isomorphic even without inverting 2. The reason for this is the possibility of the existence of 2-torsion in $H^1(S, \omega_E)$. In fact, we shall see later that the isomorphism between $E^\dagger_C \to C$ and $-T_{\mathrm{or}} \to C$ does, in fact, fail to be integral at the prime 2 (cf. the Remark following Corollary 5.8).

Note that Corollary 4.3 thus implies that the vector bundle $\mathcal{T}$ considered in the discussion preceding Definition 2.3 extends naturally to a vector bundle $\mathcal{T}_C$ on C that fits into an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{T}_C \to \tau_E|_C \to 0$$

extending the exact sequence of *loc. cit.*

Finally, we make the connection with the *splitting of [Katz], Lemma C2.1.* The point is the following: The torsor $-T_{\mathrm{or}} \to C$ is the torsor of differentials that are regular everywhere except for a simple pole at the origin of residue -1. Thus, over

$$U \stackrel{\mathrm{def}}{=} C - \{e\}$$

this torsor is just the torsor of global regular differentials. In particular, over U, it admits a section given by the "zero differential." Moreover, in a neighborhood of e, it is easy to see that the zero differential, *regarded as the sort of differential parametrized by* $-T_{\mathrm{or}}$, has a pole of order 1 at e. Thus, we get a section

$$\mathcal{S}_{\mathrm{or}} : U \to -T_{\mathrm{or}}$$

with a pole of order 1 at the origin.

Now let us assume that 6 *is invertible on* S. Then one obtains two sections

$$\mathcal{S}_1, \mathcal{S}_2 : U \to E_C^{\dagger}$$

of the universal extension over U: the first is given by transporting $\mathcal{S}_{\mathrm{or}}$ by means of the isomorphism of Theorem 4.2; the second is that of [Katz], Lemma C2.1. To review, this section of [Katz] is defined as follows: For an elliptic curve defined by the equation $y^2 = 4x^3 - g_2 x - g_3$, the section $\mathcal{S}_2$ is defined by associating to a point P of U the differential

$$\frac{y + y(P)}{2\{x - x(P)\}} \cdot \frac{dx}{y}$$

(which is regular everywhere except at e and P, and has residues -1 at e and $+1$ at P). In fact, strictly speaking, Katz only defines this section for smooth $C \to S$, but one checks easily that the above definition defines a section for arbitrary C. Moreover, *both sections have a pole of order* 1 *at* e. Indeed, this follows immediately from the definition for $\mathcal{S}_1$; for $\mathcal{S}_2$, it follows from, say, [Katz], Theorem C6, (2). Thus, their difference defines a section

$$\delta \in \Gamma(C, \omega_E|_C(e))$$

We would like to show that $\delta = 0$. To do this, it suffices to consider the *universal case*: i.e., $S^{\log} = \overline{\mathcal{M}}_{1,0}^{\log} \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{6}]$. Then it follows immediately from an elementary application of Riemann-Roch on the fibers of $C \to S$ that δ in fact lies in $\Gamma(C, \omega_E|_C) = \Gamma(S, \omega_E)$. On the other hand, it is well-known (cf., e.g., [KM], p. 227) that $\Gamma(S, \omega_E) = 0$. Thus, $\delta = 0$, as desired. In other words, we have proven the following result:

Corollary 4.4. *Suppose that* 6 *is invertible on* S. *Then the isomorphism* $-T_{\mathrm{or}} \cong E_C^{\dagger}$ *maps the section* $\mathcal{S}_{\mathrm{or}} : C - \{e\} \to -T_{\mathrm{or}}$ *defined above to the section of* [Katz]*, Lemma C2.1.*

In particular, the section of [Katz], *Lemma C2.1, defines a unique section* $C - \{e\} \to E_C^\dagger$, even when 6 is not invertible.

§5. Analytic Continuation of the Canonical Splitting

The purpose of this § is to show that *the canonical splitting* κ *of Theorem 2.1 may be "analytically continued" to a splitting in the category of "entire functions" on the uniformization of a degenerating elliptic curve in the sense of [Mumf]* (i.e., the formal algebraic analogue of the *Schottky uniformization*). Although this splitting will not be periodic, it will satisfy a simple rule with respect to the period of the degenerating elliptic curve (i.e., shifting by a period affects the splitting by adding a constant). In order to analyze this splitting, we will also use the splitting $\mathcal{S}_{\mathrm{or}}$ of Corollary 4.4. The difference between κ and $\mathcal{S}_{\mathrm{or}}$ will be a sort of Weierstrass ζ-function with respect to the Schottky uniformization. Thus, we will refer to it as a *Schottky-Weierstrass* ζ*-function*. We will discuss this ζ-function (as well as its "close relatives") in more detail in §6, 7 below. In the present §, we show how to translate the material of [Katz], Appendix C, §C6, C7 into the context of Mumford's construction. Although, strictly speaking, this Schottky version of the theory of [Katz], Appendix C, §C6, C7, is not logically necessary for the proofs of the main results of the present § (or, for that matter, of the present paper), we present it here nevertheless because we feel it to be both interesting and generally culturally relevant to what we do discuss here. Also, we remark that "in principle," what we discuss here is "well-known," but I do not know an adequate reference for it.

We begin by *reviewing the set-up in [Mumf]*, especially [Mumf], §5. To do this, we must introduce some notation. Let K be a *finite extension of* $\mathbf{Q}$, and let $\mathcal{O}_K$ be its *ring of integers*. In this §, we let

$$\mathcal{O}$$

be a *Zariski localization of* $\mathcal{O}_K$, i.e., a ring such that $\mathrm{Spec}(\mathcal{O})$ is an open subscheme of $\mathrm{Spec}(\mathcal{O}_K)$. Let

$$A \stackrel{\mathrm{def}}{=} \mathcal{O}[[q]]; \quad S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A)$$

(where q is an indeterminate). Moreover, we equip S with the *log structure defined by the divisor* $D \stackrel{\mathrm{def}}{=} V(q) \subseteq S$. Let $\widehat{S}$ be the *formal scheme* obtained by regarding $\mathcal{O}[[q]]$ as a topological ring with the q-adic topology. Let

$$E \to S$$

be the *semi-abelian scheme* obtain by forming the quotient of $\mathbf{G}_{\mathrm{m}}$ by the period "q" (as in [Mumf] or [FC], Chapter III). Then $E \to S$ arises in a natural way from a *log elliptic curve* $C^{\log} \to S^{\log}$. Let us write

$$\mathcal{L}_C \stackrel{\mathrm{def}}{=} \mathcal{O}_C(e); \quad \mathcal{L}_E \stackrel{\mathrm{def}}{=} \mathcal{L}_C|_E$$

(where e is the divisor defined by the identity section).

Next, we would like to consider various kinds of *analytic functions*. First, recall that we have a natural identification

$$E_{\widehat{S}} = (\mathbf{G}_{\mathrm{m}})_{\widehat{S}}$$

Let us write $\mathcal{R}_E^{\mathrm{an}}$ for the coordinate ring of $E_{\widehat{S}}$. Thus, one can think of $\mathcal{R}_E^{\mathrm{an}}$ as:

$$\mathcal{R}_E^{\mathrm{an}} = A\{\{U, U^{-1}\}\}$$

i.e., Laurent series with coefficients converging to zero in A. Let us write

$$\mathcal{R}_E^{\mathrm{alg}} \subseteq \mathcal{R}_E^{\mathrm{an}}[\theta]; \quad (\text{respectively, } \mathcal{R}'_E \subseteq \mathcal{R}_E^{\mathrm{an}}[\theta])$$

where θ is an indeterminate, for the A-subalgebra of $\mathcal{R}_E^{\mathrm{an}}[\theta]$ generated (respectively, generated q-adically) by the elements

$$\{q^{k^2+k} \cdot U^{2k+1} \cdot \theta, q^{k^2} \cdot U^{2k} \cdot \theta, q^{k^2-k} \cdot U^{2k-1} \cdot \theta\}$$

where k ranges over all elements of $\mathbf{Z}$. That is to say, $\mathcal{R}_E^{\mathrm{alg}}$ (along with its q-adic completion $\mathcal{R}'_E$) is the ring "$R_{\phi,\Sigma}$" used by Mumford (cf. [Mumf], p. 306) to construct E as a quotient of $\mathbf{G}_{\mathrm{m}}$. In this context, it is natural to think of $\mathcal{R}_E^{\mathrm{an}}[\theta]$, $\mathcal{R}_E^{\mathrm{alg}}$ and $\mathcal{R}'_E$ as *graded rings*, in which elements of $\mathcal{R}_E^{\mathrm{an}}$ have degree zero and θ has degree one.

Next, recall from the theory of [Mumf] that there is a natural action of $\mathbf{Z}$ on

$$C^{\infty} \stackrel{\mathrm{def}}{=} \mathrm{Proj}(\mathcal{R}_E^{\mathrm{alg}}) \quad \text{and} \quad C^{\infty}_{\widehat{S}} \stackrel{\mathrm{def}}{=} \mathrm{Proj}(\mathcal{R}'_E)$$

defined by letting $1 \in \mathbf{Z}$ act by:

$$U \mapsto q \cdot U; \quad \theta \mapsto q \cdot U^2 \cdot \theta$$

In the following, we shall denote this *group of automorphisms* by

$$\mathbf{Z}_{\mathrm{et}}$$

Thus, we shall identify the elements of $\mathbf{Z}_{\text{et}}$ with $\mathbf{Z}$ and write, for instance, "$1_{\text{et}} \in \mathbf{Z}_{\text{et}}$," but we prefer to use the notation $\mathbf{Z}_{\text{et}}$ for the group of automorphisms, in order to distinguish it from, for instance, $\mathbf{Z}$ regarded as a subring of $\mathcal{O}$.

Now it follows from the theory of [Mumf] that in our notation $C_{\widehat{S}}$ is formed by taking the quotient of $C^{\infty}_{\widehat{S}}$ by this action of $\mathbf{Z}_{\text{et}}$. In fact, the *special fiber*

$$(C^{\infty})_{\text{spl}} = (C^{\infty}_{\widehat{S}})_{\text{spl}}$$

(i.e., fiber over $V(q) \subseteq S$) of C^{∞} is an *infinite chain of* $\mathbf{P}^1$*'s*, connected to each other at "0" and "∞." Thus, each irreducible component of $(C^{\infty})_{\text{spl}}$ is a copy $\mathbf{P}^1$, *labeled by an element of* $\mathbf{Z}_{\text{et}}$. The action of $\mathbf{Z}_{\text{et}}$ on the irreducible components (thought of relative to this labeling) is just the action of $\mathbf{Z}_{\text{et}}$ on $\mathbf{Z}_{\text{et}}$ by addition. (See [Mumf] for more details.) Another way to think of C^{∞} is as a sort of *Néron model* for $\mathbf{G}_{\text{m}}$ over S relative to the open immersion $S - V(q) \subseteq S$. Thus, the irreducible components of the special fiber correspond naturally to the q-adic orders of elements of $A[q^{-1}]$.

Relative to the point of view of [Mumf] (cf. especially [Mumf], p. 289), the ample line bundle "$\mathcal{O}(1)$" on $C^{\infty}_{\widehat{S}}$ obtained by regarding $\mathcal{R}'_E$ as a graded ring (as discussed above) may be identified in a natural fashion with the pull-back (relative to the quotient $C^{\infty}_{\widehat{S}} \to C_{\widehat{S}}$)

$$\mathcal{L}^{\otimes 2}_{C^{\infty}_{\widehat{S}}}$$

to $C^{\infty}_{\widehat{S}}$ of $\mathcal{L}^{\otimes 2}_{C}$. Thus, the elements of $\mathcal{R}'_E$ of degree i define sections of $\mathcal{L}^{\otimes 2i}_{C^{\infty}_{\widehat{S}}}$ over $C^{\infty}_{\widehat{S}}$. Conversely, I *claim* that all sections of $\mathcal{L}^{\otimes 2i}_{C^{\infty}_{\widehat{S}}}$ over $C^{\infty}_{\widehat{S}}$ arise in this way. Indeed, it suffices to note that:

(1) The global sections of $\mathcal{L}^{\otimes 2i}_{C^{\infty}_{\widehat{S}}}$ over the irreducible component of $(C^{\infty})_{\text{spl}}$ marked "0" are spanned over $\mathcal{O}$ by the restrictions to this irreducible component of the sections

$$U^{-i} \cdot \theta^i, U^{-i+1} \cdot \theta^i, \ldots, \theta^i, U^{i-1} \cdot \theta^i, U^i \cdot \theta^i$$

(2) The sections

$$U^{-i+1} \cdot \theta^i, U^{-i+2} \cdot \theta^i, \ldots, \theta^i, U^{i-1} \cdot \theta^i, U^i \cdot \theta^i$$

all *vanish* when restricted to any irreducible component of $(C^{\infty})_{\text{spl}}$ that lies on the same side of the component marked 0 as the point $U = 0$ of the components marked 0. (Indeed, acting on any of these sections by a positive integer $\in \mathbf{Z}_{\text{et}}$ sends these sections to zero modulo q.)

By translating these two observations over to the other irreducible components of ${C^{\infty}}_{\text{spl}}$, one sees easily that one may construct arbitrary sections of $\mathcal{L}^{\otimes 2i}_{(C^{\infty})_{\text{spl}}}$ over $(C^{\infty})_{\text{spl}}$ by means of elements in $\mathcal{R}'_E$. This completes the proof of the claim. Thus, in summary,

$$\Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}^{\otimes 2i}_{C^{\infty}_{\widehat{S}}}) = \text{degree } i \text{ portion of } \mathcal{R}'_E$$

Put another way, θ *defines a trivialization of the line bundle* $\mathcal{L}^{\otimes 2}_{C^{\infty}_{\widehat{S}}}$ *over* $E_{\widehat{S}} = (\mathbf{G}_{\text{m}})_{\widehat{S}}$ with respect to which global sections may be written as certain Laurent series in U whose coefficients ($\in A$) decay fairly rapidly.

In particular, suppose that $\sigma^{\text{alg}} \in \Gamma(C, \mathcal{L}_C)$ is the section given by the embedding $\mathcal{O}_C \hookrightarrow \mathcal{O}_C(e) = \mathcal{L}_C$. Then, by restricting σ^{alg} to $C^{\infty}_{\widehat{S}}$, we can write its square $(\sigma^{\text{alg}})^2$ analytically as some element

$$(\sigma^{\text{alg}})^2|_{C^{\infty}_{\widehat{S}}} = (\sigma^{\text{an}})^2 \cdot \theta \in \mathcal{R}'_E$$

where $(\sigma^{\text{an}})^2 \in \mathcal{R}^{\text{an}}_E$ is some (topological) A-linear combination of the elements

$$\{q^{k^2+k} \cdot U^{2k+1}, q^{k^2} \cdot U^{2k}, q^{k^2-k} \cdot U^{2k-1}\}_{\{k \in \mathbf{Z}\}}$$

The advantage of working with $(\sigma^{\text{an}})^2$ is that it is an explicit Laurent series. For convenience, in the following, *we shall often write just* σ^2 *for* $(\sigma^{\text{an}})^2$. In terms of normalizations of σ^{alg} (relative to multiplication by elements of $A^{\times}$), we would like to think of σ^{alg} as being chosen so that the square differential at the origin e of E determined by σ^2 (which has a zero of order 2 at the origin) is equal to $(d\ \log(U))^2$.

Next, let

$$\mathcal{R}^{\text{mer}}_{C^{\infty}_{\widehat{S}}}$$

denote the ring of *meromorphic functions on* $C^{\infty}_{\widehat{S}}$. By this, we mean those functions which can, Zariski locally on the formal scheme $C^{\infty}_{\widehat{S}}$, be written as a quotient of a regular function (i.e., a section of $\mathcal{O}_{C^{\infty}_{\widehat{S}}}$) by a nonzero regular function. Thus, *the zeroes and poles of a meromorphic function on* $C^{\infty}_{\widehat{S}}$ *have finite order.* Note, further, that we have a natural *injection*

$$\mathcal{R}^{\text{mer}}_{C^{\infty}_{\widehat{S}}} \hookrightarrow Q(\mathcal{R}^{\text{an}}_E)$$

(where "Q" denotes "the quotient field of"). Indeed, the fact that this morphism is an injection follows by considering the situation at the nodes of $C^{\infty}_{\widehat{S}}$, where this injectivity essentially amounts to the fact that the natural map

$$\mathcal{O}[[X,Y]] \to (\mathcal{O}[[X]](X^{-1}))[[Y]]$$

(where X and Y are indeterminates) from power series over $\mathcal{O}$ in X and Y to power series in Y with coefficients in $\mathcal{O}[[X]](X^{-1})$ is injective.

Observe that if f is a function on $C^{\infty}_{\widehat{S}}$ which can be written as a quotient of a (regular) section of $\mathcal{L}^{\otimes 2N}$ over $C^{\infty}_{\widehat{S}}$ by a nonzero (regular) section of $\mathcal{L}^{\otimes 2N}$ over $C^{\infty}_{\widehat{S}}$ (for some N), then

$$f \in \mathcal{R}^{\mathrm{mer}}_{C^{\infty}_{\widehat{S}}}$$

For instance, since both θ ($\neq 0$) and $\sigma^2 \cdot \theta$ are regular sections of $\mathcal{L}^2$ over $C^{\infty}_{\widehat{S}}$, it follows that

$$\sigma^2 = (\sigma^2 \cdot \theta) \cdot \theta^{-1} \in \mathcal{R}^{\mathrm{mer}}_{C^{\infty}_{\widehat{S}}}$$

Finally, if $\phi \in \mathcal{R}^{\mathrm{mer}}_{C^{\infty}_{\widehat{S}}}$, let us write

$$\phi' \stackrel{\mathrm{def}}{=} U \ \frac{\partial \phi}{\partial U}$$

Note that since $\phi \in \mathcal{R}^{\mathrm{mer}}_{C^{\infty}_{\widehat{S}}}$, we also have $\phi' \in \mathcal{R}^{\mathrm{mer}}_{C^{\infty}_{\widehat{S}}}$.

Next, we would like to make use of the *Weierstrass normal form* of the elliptic curve E (as in [Katz], §C2). That is to say, we fix the differential

$$d \ \log(U) = \frac{dU}{U}$$

and take the resulting rational functions x and y on E such that $dx/y = d \ \log(U)$ (as in [Katz], §C2). Thus, the elliptic curve E is defined by the equation

$$y^2 = 4x^3 - g_2 x - g_3$$

where the zero section e is the point at infinity (of the affine curve define by this equation). In order to do this, one must assume that $6^{-1} \in \mathcal{O}$. Thus, *for the rest of this §, until stated otherwise, we shall assume that* $6^{-1} \in \mathcal{O}$. (We will state explicitly (after Proposition 4.5) when this assumption is no longer in force.)

Thus, x is a rational function on C with the following properties:

(i) x has a pole of order two at the origin, and no other poles;

(ii) x is even;

(iii) the section of $\tau_E^{\otimes 2}$ defined by looking at the "leading term" of x at the origin is equal to $(U\frac{\partial}{\partial U})^2$.

Moreover, y is the rational function on C given by $U\frac{\partial x}{\partial U}$. (Note that since the *derivation* $U\frac{\partial}{\partial U}$ is *algebraic*, this statement makes sense in the algebraic category despite the fact that "U" is only defined analytically.) Then, by *analogy with the classical complex theory* (see, e.g., [Ahlf], p. 272), we write

$$\wp \qquad \text{and} \qquad \wp'$$

for the *analytic representations* (i.e., elements of $\mathcal{R}^{\mathrm{mer}}_{C^{\infty}_{\widehat{S}}}$) *of x and y*, respectively. Also, we write

$$\zeta \stackrel{\mathrm{def}}{=} (\sigma^2)'/(2\cdot\sigma^2) \in \mathcal{R}^{\mathrm{mer}}_{C^{\infty}_{\widehat{S}}}$$

for *the analogue of the classical Weierstrass ζ-function.* Note that since σ^2 is *even*, while $U\frac{\partial}{\partial U}$ is *odd*, it follows that ζ is an *odd* function. Also, note that since σ^2 has a zero of order 2 at the origin e, it follows that $\zeta \cdot d\log(U)$ has a simple pole at the origin e whose residue is 1. In fact, modulo q, it is easy to *compute ζ explicitly*: Indeed, *modulo q, σ^2 is equal to a unit multiple of $U-2+U^{-1}$* (compare zero loci!), so we obtain that (modulo q)

$$\zeta \cdot d\log(U) \equiv \frac{1}{2}\cdot d\log(U-2+U^{-1}) = \frac{dU}{U-1} - \frac{1}{2}d\log(U) = \left\{\frac{U}{U-1} - \frac{1}{2}\right\}\cdot d\log(U)$$

In particular, we see that the meromorphic function ζ is *regular* (i.e., does not have a pole) at the nodes $U = 0, \infty$ of the component of $(C^{\infty}_{\widehat{S}})_{\mathrm{spl}}$ marked 0.

Note that *the action of $\mathbf{Z}_{\mathrm{et}}$ on ζ* may be determined as follows: $1_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ acts on θ by:

$$\theta \mapsto q\cdot U^2\cdot\theta$$

Since $\sigma^2\cdot\theta$ is pulled back from $C_{\widehat{S}}$, it thus follows that $1_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ *fixes* $\sigma^2\cdot\theta$. In particular, the action of $1_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ on σ^2 is given by:

$$\sigma^2 \mapsto q^{-1} \cdot U^{-2} \cdot \sigma^2$$

This allows us to conclude that the action of $1_{\text{et}} \in \mathbf{Z}_{\text{et}}$ on $\frac{1}{2}$ times the logarithmic derivative of σ^2, i.e., on ζ, is given by:

$$\zeta \mapsto \zeta - 1$$

Observe that this shows, in particular, that ζ is *regular* except at the orbit

$$\widetilde{e} \stackrel{\text{def}}{=} \mathbf{Z}_{\text{et}}(e) \subseteq C^{\infty}_{\widehat{S}}$$

of e under the action of $\mathbf{Z}_{\text{et}}$, where it is has a *pole of order one*.

Next, we would like to check that certain classical formulas concerning the various functions just defined hold in the present context, as well. First of all, let us observe that the general nonsense concerning meromorphic functions on $C^{\infty}_{\widehat{S}}$ can be extended to the product $C^{\infty}_{\widehat{S}} \times_{\widehat{S}} C^{\infty}_{\widehat{S}}$, as well. We leave the formulation of this general nonsense to the reader. Now let us write

$$\mu, \pi_1, \pi_2 : E_{\widehat{S}} \times_{\widehat{S}} E_{\widehat{S}} \to E_{\widehat{S}}$$

for the obvious *multiplication* and *projection* maps, respectively. Write

$$\nu : E_{\widehat{S}} \times_{\widehat{S}} E_{\widehat{S}} \to E_{\widehat{S}}$$

for the map given by $(\alpha, \beta) \mapsto \alpha - \beta$. Then we have the following *analogue of a well-known classical formula*:

Proposition 5.1. *One has an equality*

$$(-\pi_1^*\wp + \pi_2^*\wp)^2 = \frac{(\mu^*\sigma^2) \cdot (\nu^*\sigma^2)}{(\pi_1^*\sigma^4) \cdot (\pi_2^*\sigma^4)}$$

of meromorphic functions on $C^{\infty}_{\widehat{S}} \times_{\widehat{S}} C^{\infty}_{\widehat{S}}$.

Proof. The proof is entirely similar to that in the classical case. Namely, one uses the *cubical structure* on $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{L}_C|_{E_{\widehat{S}}}$ to first deduce the *corresponding (natural) isomorphism of line bundles*: Indeed, if

$$\mathcal{B} \stackrel{\text{def}}{=} (\mu^*\mathcal{L}) \otimes (\pi_1^*\mathcal{L})^{-1} \otimes (\pi_2^*\mathcal{L})^{-1}$$

is the corresponding $\mathbf{G}_{\mathrm{m}}$-*biextension* on the $E \times_S E$ (cf. [MB] for more on cubical structures), then one has a natural isomorphism

$$\begin{aligned}\mathcal{O}_{E\times_S E} &\cong \mathcal{B} \otimes \left(\mathrm{id}_E \times (-\mathrm{id}_E)\right)^* \mathcal{B} \\ &= (\mu^*\mathcal{L}) \otimes (\nu^*\mathcal{L}) \otimes (\pi_1^*\mathcal{L})^{\otimes -2} \otimes (\pi_2^*\mathcal{L})^{\otimes -2}\end{aligned}$$

Taking the "square" of this isomorphism shows that both sides of the desired equality can be naturally (i.e., *algebraically* without enlisting the aid of the trivialization θ) regarded as rational functions on $E \times_S E$. Note that here, we use the fact that the trivialization θ is *compatible with the cubical structure on* $\mathcal{L}$ (which follows from the theory of [Mumf]; [FC], Chapter III). Next, observe that since $\wp$ is an even function, the left-hand side has zeroes of order 2 at $\mu^{-1}(e)$ and $\nu^{-1}(e)$. Moreover, the left-hand side has poles only at $\pi_1^{-1}(e)$ and $\pi_2^{-1}(e)$, and these poles are both of order 4. Since this enumeration exhausts all the poles and zeroes of the right-hand side, we thus conclude that the desired equality holds up to muliplication by a unit of A. In fact, for our purposes, this will be sufficient, but one can check that this unit must be $= 1$ be looking at the leading term at the origin. ○

Proposition 5.2. *One has an equality*

$$\frac{1}{2}\ \frac{\pi_1^*\wp' - \pi_2^*\wp'}{\pi_1^*\wp - \pi_2^*\wp} = (\mu^*\zeta) - (\pi_1^*\zeta) - (\pi_2^*\zeta)$$

of meromorphic functions on $C^\infty_{\widehat{S}} \times_{\widehat{S}} C^\infty_{\widehat{S}}$.

Proof. The proof is entirely similar to that in the classical case: Namely, one takes the logarithmic derivative of both sides of the equality of Proposition 5.1 with respect to the derivation $\pi_1^*(U\frac{\partial}{\partial U}) + \pi_2^*(U\frac{\partial}{\partial U})$, and then divides by 4. Note here that the push-forward of this derivation with respect to μ (respectively, ν) is equal to $2 \cdot (U\frac{\partial}{\partial U})$ (respectively, 0). ○

Next, we would like to prove the *analogue of Theorem C6 of [Katz]*. The analogue of the *first part of this theorem* (i.e., Theorem C6, (1)) consists of making explicit the differentials defined by the canonical splitting in terms of the special function ζ. Recall that *over* D, we already did this (i.e., wrote down the differential "ω_x" explicitly) in our construction of κ (cf. the discussion preceding Theorem 2.1). Thus, we would like to extend this discussion to the situation over $\widehat{S}$.

To do this, let $\alpha \in \mathbf{G}_{\mathrm{m}}(\widehat{S})$ be a point of $E_{\widehat{S}}$. Write

$$T_\alpha : E_{\widehat{S}} \to E_{\widehat{S}}$$

for the morphism given by *translating by* α. Then let us consider the differential (on $E_{\widehat{S}}$) defined by

$$\omega_\alpha \stackrel{\text{def}}{=} \left(T^*_{\alpha^{-1}}(\zeta) - \zeta\right) \cdot d\ \log(U)$$

Note that since $\zeta \cdot d\ \log(U)$ has a simple pole with residue 1 at e, the differential ω_α has a simple pole with residue 1 (respectively, -1) at α (respectively, e). Clearly, the correspondence $\alpha \mapsto \omega_\alpha$ is functorial in α. Moreover, we have the following

Lemma 5.3. *The differential* ω_α *is* algebraic *in the sense that it arises from a meromorphic differential on* C. *Moreover, this meromorphic differential on* C *has no poles except at* α *and* e.

Proof. Indeed, $1_{\text{et}} \in \mathbf{Z}_{\text{et}}$ acts on both ζ and its translate $T^*_{\alpha^{-1}}(\zeta)$ by adding -1. Thus, $1_{\text{et}} \in \mathbf{Z}_{\text{et}}$ stabilizes ω_α. This implies that ω_α descends to an algebraic meromorpic differential on C. The description of the poles of ω_α follows from the description of the poles of ζ given in the above discussion. ○

Next, let us observe that (for $\alpha, \beta \in \mathbf{G}_{\text{m}}(\widehat{S}) = E_{\widehat{S}}(\widehat{S})$) the meromorphic function

$$\frac{(T^*_{\alpha^{-1}}\sigma^2) \cdot (T^*_{\beta^{-1}}\sigma^2)}{(T^*_{\alpha^{-1}\cdot\beta^{-1}}\sigma^2) \cdot \sigma^2}$$

on $C^\infty_{\widehat{S}}$ is "algebraic," i.e., arises from a meromorphic function on E. This follows immediately (cf. the use of cubical structures in the proof of Proposition 5.1) from the fact that the corresponding line bundle

$$(T^*_{\alpha^{-1}}\mathcal{L}) \otimes (T^*_{\beta^{-1}}\mathcal{L}) \otimes (T^*_{\alpha^{-1}\cdot\beta^{-1}}\mathcal{L})^{-1} \otimes \mathcal{L}^{-1}$$

(and hence also its square) is trivial. On the other hand, if we then take the logarithmic derivative of this rational function (with respect to the derivation $U\frac{\partial}{\partial U}$) and multiply by $\frac{1}{2} \cdot d\ \log(U)$, we obtain

$$\omega_\alpha + \omega_\beta - \omega_{\alpha\cdot\beta}$$

This implies (cf. the discussion preceding Theorem 2.1) that the correspondence $\alpha \mapsto \omega_\alpha$ defines a *homomorphism* of $E_{\widehat{S}}$ into $E^\dagger_{\widehat{S}}$. By the uniqueness statement in Theorem 2.1, we thus conclude the following *Schottky analogue of [Katz], Theorem C6, (1)*:

Theorem 5.4. *The splitting $\kappa : E_{\widehat{S}} \to E^{\dagger}_{\widehat{S}}$ of Theorem 2.1 is the splitting defined by the correspondence*

$$\alpha \mapsto \omega_\alpha \stackrel{\text{def}}{=} \left(T^*_{\alpha^{-1}}(\zeta) - \zeta\right) \cdot d\,\log(U)$$

for $\alpha \in E_{\widehat{S}}(\widehat{S})$.

Next, we would like to move on to proving the analogue of the *second part of Theorem C6 of [Katz].* First, let us recall the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{T}_C \to \tau_E|_C \to 0$$

considered in the discussion following Corollary 4.3. If we tensor this exact sequence with $\omega_E \otimes_{\mathcal{O}_S} \mathcal{L}_C^{\otimes 2}$, we obtain an exact sequence

$$0 \to \omega_E \otimes \mathcal{L}_C^{\otimes 2} \to \mathcal{T}' \to \mathcal{L}_C^{\otimes 2} \to 0$$

Now let us recall the section

$$\mathcal{S}_{\text{or}} : C - e \to -T_{\text{or}} \cong E^{\dagger}_C$$

of Corollary 4.4. Since this section has a pole of order precisely 1 (≤ 2) at e, it follows from the definition of $\mathcal{T}_C$ that it defines a *lifting*

$$\xi^{\text{alg}} \in \Gamma(C, \mathcal{T}')$$

of the section $(\sigma^{\text{alg}})^2 \in \Gamma(E, \mathcal{L}_E^{\otimes 2})$. Now let us write ξ^{alg} in terms of its *components relative to the canonical splitting* (cf. Definition 2.3):

$$(\xi[0] \cdot \theta, \xi[1] \cdot \theta)$$

(Here, by trivializing ω_E by means of the section $U\frac{\partial}{\partial U}$, we may regard $\xi[0]$ as a function, rather than just a section of $\omega_E|_{E_{\widehat{S}}}$.) Thus, by definition,

$$\xi[1] = \sigma^2$$

We would like to compute $\xi[0]$. In fact, one has the following:

Proposition 5.5. *We have $\xi[0] = \frac{1}{2}(\sigma^2)'$.*

Proof. The proof of the equality $\xi[0] = \frac{1}{2}(\sigma^2)'$ is entirely formally analogous to the proof of the second part of [Katz], Theorem C6. Namely, it is a formal consequence of Proposition 5.2. Indeed, if $\alpha \in E_{\widehat{S}}(\widehat{S}) = \mathbf{G}_{\mathrm{m}}(\widehat{S})$, then $\xi[0]$ is the unique function such that adding $(\sigma^{-2} \cdot \xi[0])(\alpha)$ to the canonical splitting $\kappa(\alpha)$ of Theorem 2.1 gives rise to the splitting $\mathcal{S}_{\mathrm{or}}(\alpha)$ over α. But by Corollary 4.4 (and the explicit form of $\mathcal{S}_{\mathrm{or}}$ in terms of x and y reviewed in the discussion preceding Corollary 4.4), this splitting $\mathcal{S}_{\mathrm{or}}(\alpha)$ corresponds to the meromorphic differential

$$\frac{y + y(\alpha)}{2\{x - x(\alpha)\}} \cdot d\ \log(U)$$

while by Theorem 5.4, the meromorphic differential given by adding $(\sigma^{-2} \cdot \xi[0])(\alpha)$ to the canonical splitting $\kappa(\alpha)$ of Theorem 2.1 is precisely

$$\omega_\alpha + (\sigma^{-2} \cdot \xi[0])(\alpha) \cdot d\ \log(U) = \{\zeta(\alpha^{-1} \cdot U) - \zeta(U) + \sigma(\alpha)^{-2}(\xi[0])(\alpha)\} \cdot d\ \log(U)$$

Since y is an odd function, while x is even, it thus follows from Proposition 5.2 that these two expressions are equal precisely when $\sigma^{-2} \cdot \xi[0] = \zeta$, i.e., $\xi[0] = \frac{1}{2}(\sigma^2)'$, as desired. ○

Now let us recall that $\zeta = \frac{(\sigma^2)'}{2 \cdot \sigma^2}$ defines a *meromorphic function on* $C^\infty_{\widehat{S}}$ with a pole of order 1 at the origin e. Next, note that since the meromorphic splitting $\mathcal{S}_{\mathrm{or}}$ is *defined* over $C_{\widehat{S}}$, its pull-back to $C^\infty_{\widehat{S}}$ is a *meromorphic splitting of* $E^\dagger_C|_{C^\infty_{\widehat{S}}}$ *over* $C^\infty_{\widehat{S}}$ which is regular away from $\widetilde{e}$. On the other hand, it follows from Proposition 5.5 that the canonical splitting κ over $E_{\widehat{S}}$ is equal to the difference between $\mathcal{S}_{\mathrm{or}}|_{E_{\widehat{S}}}$ and $\zeta \cdot d\ \log(U)|_{E_{\widehat{S}}}$. Thus, it follows that *the canonical splitting* κ *extends to a meromorphic splitting of* $E^\dagger_C|_{C^\infty_{\widehat{S}}}$.

Next, let us observe that the natural action of $\mathbf{Z}_{\mathrm{et}}$ on $C^\infty_{\widehat{S}}$ and $E^\dagger_C|_{C^\infty_{\widehat{S}}}$ fixes $\mathcal{S}_{\mathrm{or}}$ (since $\mathcal{S}_{\mathrm{or}}$ is obtained by pull-back from $C_{\widehat{S}}$). The action of $\mathbf{Z}_{\mathrm{et}}$ on the meromorphic extension over $C^\infty_{\widehat{S}}$ of κ may thus be determined by looking at the action of $\mathbf{Z}_{\mathrm{et}}$ on the difference between this meromorphic extension and $\mathcal{S}_{\mathrm{or}}|_{C^\infty_{\widehat{S}}}$, i.e., the action of $\mathbf{Z}_{\mathrm{et}}$ on ζ. Moreover, as computed above, the action of $1_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ on ζ is given by:

$$\zeta \mapsto \zeta - 1$$

Thus, the action of $1_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ on the meromorphic extension of κ (which is, roughly speaking, just "$\mathcal{S}_{\mathrm{or}} - \zeta$") over $C^\infty_{\widehat{S}}$ is given by:

$$\kappa \mapsto \kappa + d\ \log(U)$$

In particular, since we know that ζ and $\mathcal{S}_{\mathrm{or}}|_{C^{\infty}_{\widehat{S}}}$ are regular everywhere except $\widetilde{e}$, while κ is regular at $e \in E_{\widehat{S}}(\widehat{S}) \subseteq C^{\infty}_{\widehat{S}}(\widehat{S})$, we conclude that the meromorphic extension of κ over $C^{\infty}_{\widehat{S}}$ is, in fact, *regular over all of* $C^{\infty}_{\widehat{S}}$. In other words, we have proven the following:

Theorem 5.6. *The canonical section κ of Theorem 2.1 extends to a (regular) section of $E^{\dagger}_{C}|_{C^{\infty}_{\widehat{S}}}$ over $C^{\infty}_{\widehat{S}}$. By abuse of notation, we also denote this (unique) extension of κ by κ. The action of $1_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ on $C^{\infty}_{\widehat{S}}$ induces the following action on κ:*

$$\kappa \mapsto \kappa + d\ \log(U)$$

The difference between κ and the pull-back to $C^{\infty}_{\widehat{S}}$ of the section $\mathcal{S}_{\mathrm{or}}$ of Corollary 4.4 is given by the function $\zeta \stackrel{\mathrm{def}}{=} \frac{(\sigma^2)'}{2\cdot\sigma^2}$, which is meromorphic on $C^{\infty}_{\widehat{S}}$, and regular everywhere on $C^{\infty}_{\widehat{S}}$ except for a pole of order 1 on $\widetilde{e}$ (the $\mathbf{Z}_{\mathrm{et}}$-orbit of the origin of $E_{\widehat{S}}$). The action of $1_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ on ζ is given by $\zeta \mapsto \zeta - 1$. Finally, the section κ of $E^{\dagger}_{C}|_{C^{\infty}_{\widehat{S}}}$ may also be thought of (relative to the isomorphism of Theorem 4.2) as ($-\frac{1}{2}$ times) the section associated to the unique connection *on $\mathcal{L}^{\otimes 2}_{C^{\infty}_{\widehat{S}}}$ for which the section θ is* horizontal*.*

Proof. It remains to prove the final assertion concerning the connection. Let us denote this connection by ∇. Then

$$\nabla(\sigma \cdot \theta) = 2 \cdot \zeta \cdot (\sigma \cdot \theta) \cdot d\ \log(U)$$

Since $\sigma \cdot \theta$ is nonzero everywhere on $C^{\infty}_{\widehat{S}}$ except $\widetilde{e}$, it thus follows that the connection ∇ is regular everywhere on $C^{\infty}_{\widehat{S}}$, except possibly at $\widetilde{e}$. On the other hand, since $1_{\mathrm{et}}(\theta) = q \cdot U^2 \cdot \theta$, it follows that $1_{\mathrm{et}}(\nabla) = \nabla - 2$; hence that ∇ (which we know to be regular at e) is also regular over all of $\widetilde{e}$, hence over all of $C^{\infty}_{\widehat{S}}$. Thus, ∇ defines a regular section κ_{∇} of $E^{\dagger}_{C}|_{C^{\infty}_{\widehat{S}}}$ which satisfies $1_{\mathrm{et}}(\kappa_{\nabla}) = \kappa_{\nabla} + 1$. In particular, $\kappa - \kappa_{\nabla} \in \Gamma(C^{\infty}_{\widehat{S}}, \omega_E|_{C^{\infty}_{\widehat{S}}}) = \omega_E$, i.e., the difference $\kappa - \kappa_{\nabla}$ is *constant*. Now we consider the automorphism α of $C^{\infty}_{\widehat{S}}$ induced by "multiplication by -1" on E. Note that since $\alpha(\sigma \cdot \theta) = \sigma \cdot \theta$, the above formula for $\nabla(\sigma \cdot \theta)$ shows that $\alpha(\nabla) = \nabla$. Thus, (since the isomorphism of Theorem 4.2 is compatible with the automorphism α) we obtain that the differential $\kappa - \kappa_{\nabla} \in \omega_E = A \cdot d\ \log(U)$ is fixed by α. On the other hand, since $\alpha(d\ \log(U)) = -d\ \log(U)$, this implies that $\kappa - \kappa_{\nabla} = 0$, as desired. ○

Corollary 5.7. *The canonical section κ induces an isomorphism:*

$$E_C^{\dagger}|_{C^{\infty}_{\widehat{S}}} \cong W_{\widehat{S}} \times_{\widehat{S}} C^{\infty}_{\widehat{S}}$$

(where $W_{\widehat{S}}$ is the geometric object corresponding to the line bundle $\omega_E|_{\widehat{S}}$ on the formal scheme $\widehat{S}$). Moreover, relative to this isomorphism, the quotient $E_C^{\dagger}|_{C^{\infty}_{\widehat{S}}} \to E_C^{\dagger}$ is given by the action of $\mathbf{Z}_{\text{et}}$ on $W \times_S C^{\infty}_{\widehat{S}}$ defined by taking the product of the usual action of $\mathbf{Z}_{\text{et}}$ on $C^{\infty}_{\widehat{S}}$ with the action of $\mathbf{Z}_{\text{et}}$ on W given by letting 1 act as addition by $d\ \log(U)$.

Corollary 5.8. *The components (cf. Definition 2.3) of any section $\in \Gamma(E_C^{\dagger}, \mathcal{L}_C^{\otimes 2i}|_{E_C^{\dagger}})$ with respect to the canonical splitting κ of Theorem 2.1 form elements of $\mathcal{R}'_E$ of degree i.*

Remark. Note that the assertions of Theorem 5.6 and Corollaries 5.7, 5.8 may be checked after inverting 6, so in fact, *these results are all valid even if one does not invert* 6. On the other hand, as computed above, modulo q, we have:

$$\zeta \equiv \frac{1}{2} \cdot d\ \log(U - 2 + U^{-1}) = \frac{dU}{U-1} - \frac{1}{2} d\ \log(U)$$

i.e., ζ itself *fails to be integral at the prime* 2. Since the sections κ (cf. Theorem 2.1) and $\mathcal{S}_{\text{or}}$ (cf. the discussion preceding Corollary 4.4) *are* integral at 2, and ζ is precisely the difference between these two sections *relative to the isomorphism of Theorem 4.2*, it thus follows that the isomorphism between $E_C^{\dagger} \to C$ and $-T_{\text{or}} \to C$ (cf. Theorem 4.2) is *not* integral at 2.

Finally, before proceeding, we apply the theory developed thus far to obtain explicit information concerning *liftings of torsion points of E to $E^{\dagger}$* (cf. Remark 1 following Definition 3.2). First, observe that since $E_{\widehat{S}}$ is essentially obtained by dividing $\mathbf{G}_{\text{m}}$ by the action of $\mathbf{Z}_{\text{et}}$ obtained by multiplying $\mathbf{G}_{\text{m}}$ powers of the period q, it follows that the inverse images of the torsion points of $E_{\widehat{S}}$ in $C^{\infty}_{\widehat{S}}$ are the points the form:

$$\alpha \cdot q^{\beta} \in \mathbf{G}_{\text{m}}(A[q^{\beta}, q^{-1}]) = C^{\infty}(A[q^{\beta}]) = C^{\infty}_{\widehat{S}}(A[q^{\beta}])$$

where "$\mathbf{G}_{\text{m}}(\sim [q^{-1}]) = C^{\infty}(\sim)$" follows from the "Néron model-like" property of C^{∞}, referred to earlier; α is a *root of unity* $\in A$; and $\beta \in \mathbf{Q}$. On the other hand, by Corollary 5.7, $E^{\dagger}_{\widehat{S}}$ is essentially obtained by dividing the abelian object by $W \times \mathbf{G}_{\text{m}}$ by the action of $\mathbf{Z}_{\text{et}}$ obtained by adding/multiplying multiples of the element $(d\ \log(U), q)$. We thus conclude the following:

Corollary 5.9. *Relative to the isomorphism*

$$E^{\dagger}_{C}|_{C^{\infty}_{\widehat{S}}} \cong W_{\widehat{S}} \times_{\widehat{S}} C^{\infty}_{\widehat{S}}$$

of Corollary 5.7, the inverse images of the torsion points of $E^{\dagger}$ are precisely the points whose coordinates (on the right-hand side of the above isomorphism) are given by

$$(\beta \cdot d\ \log(U), \alpha \cdot q^{\beta})$$

where α is a root of unity $\in A^{\times}$; and $\beta \in \mathbf{Q}$.

Thus, in particular, torsion points of order N will, in general, have coordinates with denominator N.

§6. Higher Schottky-Weierstrass Zeta Functions

In this §, we define a *new integral structure on the universal extension of a degenerating elliptic curve* at finite primes. Using this new integral structure, we construct *"higher" analogues of the Schottky-Weierstrass zeta function* considered in §5. These higher Schottky-Weierstrass zeta functions (together with their twisted analogues, to be discussed in Chapter IV) will play a fundamental role in this paper.

We work with the notation of §5. Thus, let $\mathcal{O}$ be a Zariski localization of $\mathcal{O}_K$, where K is a finite extension of $\mathbf{Q}$. Let

$$A \stackrel{\text{def}}{=} \mathcal{O}[[q]]; \quad S \stackrel{\text{def}}{=} \text{Spec}(A)$$

Then we have a one-dimensional semi-abelian scheme

$$E \to S$$

over S. Roughly speaking, one may think of E as being "$\mathbf{G}_{\text{m}}/q^{\mathbf{Z}}$." More rigorously, E may be compactified to a log elliptic curve $C^{\log} \to S^{\log}$. Moreover, $C_{\widehat{S}}$ (the result of base changing C to the q-adic completion $\widehat{S}$ of S) may be written as

$$C_{\widehat{S}} = C^{\infty}_{\widehat{S}}/\mathbf{Z}_{\text{et}}$$

Next, let us recall the contents of Theorem 5.6, and its corollaries. First of all, the *universal extension* $E^{\dagger} \to E$ of E extends naturally (cf. Corollary 4.3) to a W_E-torsor

$E_C^\dagger \to C$. (Here, $W_E \to S$ is the geometric line bundle defined by the invertible sheaf ω_E on S, i.e., it is the object denoted "W" in §2.) Then Theorem 5.6, Corollary 5.9 state that we have a natural isomorphism

$$E_C^\dagger|_{C^\infty_{\widehat{S}}} \cong (W_E)_{\widehat{S}} \times_{\widehat{S}} C^\infty_{\widehat{S}}$$

relative to which the quotient $E_C^\dagger|_{C^\infty_{\widehat{S}}} \to E_C^\dagger$ of the left-hand side corresponds to the quotient by the subgroup generated by

$$(d\ \log(U), q)$$

on the right-hand side. In the following, we shall identify $E_C^\dagger|_{C^\infty_{\widehat{S}}}$ with $(W_E)_{\widehat{S}} \times_{\widehat{S}} C^\infty_{\widehat{S}}$ via this isomorphism. Moreover, (to simplify the notation) *we shall trivialize* ω_E *by means of the section* $d\ \log(U)$. Thus, we may write

$$W_E = \mathrm{Spec}(\mathcal{O}_S[T])$$

(where T is the indeterminate corresponding to the chosen trivialization of ω_E).

Next, let us write (for $n \in \mathbf{Z}_{\geq 0}$)

$$T^{[n]} \stackrel{\mathrm{def}}{=} \frac{1}{n!} T(T-1)(T-2) \cdot \ldots \cdot (T-(n-1))$$

which we think of as a section of $\mathcal{O}_{W_E} \otimes \mathbf{Q}$. For negative n, we let $T^{[n]} \stackrel{\mathrm{def}}{=} 0$. Thus, we have

$$T^{[0]} = 1; \quad T^{[1]} = T; \quad T^{[2]} = \frac{1}{2}T(T-1)$$

Also, if $f(T)$ is a section of $\mathcal{O}_{W_E} \otimes \mathbf{Q}$, let us write $f(T+n)$ for the result of applying to $f(T)$ the automorphism of W_E induced by $n_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$, i.e., the automorphism defined by $T \mapsto T+n$. Thus, in particular, we may define:

$$\delta(f) \stackrel{\mathrm{def}}{=} f(T+1) - f(T)$$

Then we have

$$\delta(T^{[n]}) = T^{[n-1]}$$

Now we are ready to define *new integral structures*. First of all, let us write

$$\mathcal{R}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}}$$

for the push-forward via $E_C^\dagger|_{C^\infty_{\widehat{S}}} \to C^\infty_{\widehat{S}}$ of the structure sheaf $\mathcal{O}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}}$. Then we would like to define a new integral structure on $\mathcal{R}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}} \otimes \mathbf{Q}$ as follows: By the above discussion, the sheaf of algebras $\mathcal{R}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}}$ on $C^\infty_{\widehat{S}}$ may be identified with

$$\mathcal{O}_{C^\infty_{\widehat{S}}} \otimes_{\mathcal{O}_S} \mathcal{O}_S[T] = \mathcal{O}_{C^\infty_{\widehat{S}}} \otimes_{\mathcal{O}_S} \left(\bigoplus_{n \geq 0} \mathcal{O}_S \cdot T^n \right)$$

Then the new integral structure will be given by the sheaf

$$\mathcal{O}_{C^\infty_{\widehat{S}}} \otimes_{\mathcal{O}_S} \left(\bigoplus_{n \geq 0} \mathcal{O}_S \cdot T^{[n]} \right)$$

Let us denote by

$$\mathcal{R}^{\mathrm{et}}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}} \subseteq \mathcal{R}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}} \otimes \mathbf{Q}$$

the corresponding subsheaf of $\mathcal{R}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}} \otimes \mathbf{Q}$. Note that $\mathcal{R}^{\mathrm{et}}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}}$ has a natural filtration

$$\ldots \subseteq F^n(\mathcal{R}^{\mathrm{et}}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}}) \subseteq \ldots \subseteq \mathcal{R}^{\mathrm{et}}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}}$$

whose n-th member is given by the polynomials of degree $< n$. Thus, $F^n(\mathcal{R}^{\mathrm{et}}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}})$ is a vector bundle on $C^\infty_{\widehat{S}}$ of rank n such that we have a natural identification

$$(F^{n+1}/F^n)(\mathcal{R}^{\mathrm{et}}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}}) = \frac{1}{n!} \cdot \mathcal{O}_{C^\infty_{\widehat{S}}} \otimes_{\mathcal{O}_S} \tau_E^{\otimes n}$$

Moreover, let us observe that the filtered $\mathcal{O}_{C^\infty_{\widehat{S}}}$-submodule $\mathcal{R}^{\mathrm{et}}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}} \subseteq \mathcal{R}_{E_C^\dagger|_{C^\infty_{\widehat{S}}}} \otimes \mathbf{Q}$ is *preserved by the natural action of* $\mathbf{Z}_{\mathrm{et}}$ on $C^\infty_{\widehat{S}}$. Indeed, $1_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ maps $f(T) \stackrel{\mathrm{def}}{=} T^{[n]}$ to

$f(T+1) = f(T) + \delta(f) = f(T) + T^{[n-1]}$. Since $T^{[n-1]}$ also forms a section of $\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C}|_{C^{\infty}_{\widehat{S}}}}$, and $\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C}|_{C^{\infty}_{\widehat{S}}}}$ is an $\mathcal{O}_{C^{\infty}_{\widehat{S}}}$-module (hence closed under addition), we thus conclude that $\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C}|_{C^{\infty}_{\widehat{S}}}}$ (as well as its filtration) is stabilized by the action of $\mathbf{Z}_{\mathrm{et}}$, as desired. We summarize this as follows:

Proposition 6.1. *The $\mathcal{O}_{C^{\infty}_{\widehat{S}}}$-submodules*

$$\ldots \subseteq F^n(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C}|_{C^{\infty}_{\widehat{S}}}}) \subseteq \ldots \subseteq \mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C}|_{C^{\infty}_{\widehat{S}}}} \subseteq \mathcal{R}_{E^{\dagger}_{C}|_{C^{\infty}_{\widehat{S}}}} \otimes \mathbf{Q}$$

are all preserved by the natural action of $\mathbf{Z}_{\mathrm{et}}$. Thus, they descend to form natural $\mathcal{O}_C$-submodules

$$\ldots \subseteq F^n(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C}}) \subseteq \ldots \subseteq \mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C}} \subseteq \mathcal{R}_{E^{\dagger}_{C}} \otimes \mathbf{Q}$$

where the subsheaf $F^n(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C}})$ of the filtration is a rank n vector bundle on C such that $(F^{n+1}/F^n)(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C}}) = \frac{1}{n!} \cdot \mathcal{O}_C \otimes_{\mathcal{O}_S} \tau_E^{\otimes n}$.

Proof. It remains only to check that the various sheaves that were constructed on $C^{\infty}_{\widehat{S}}$ do indeed descend to C. That is to say, since the covering $C^{\infty}_{\widehat{S}} \to C_{\widehat{S}}$ is *infinite*, the reader may fear that one cannot immediately apply the usual machinery of étale descent. In fact, however, this is not a problem since $C_{\widehat{S}}$ admits a (finite) Zariski open cover $\{\mathcal{U}\}$ (in the category of formal $\widehat{S}$-schemes) such that over each $\mathcal{U}$, the covering $C^{\infty}_{\widehat{S}} \to C_{\widehat{S}}$ *splits*. Thus, there is no problem with descent. ○

Definition 6.2. The integral structures denoted by a superscript "et" will be referred to as *étale-integral, or et-integral, structures*. That is to say, "et" stands for "étale," and is the same as the "et" in $\mathbf{Z}_{\mathrm{et}}$. In particular, the "étale-integral structure" may be thought of as the integral structure arising from thinking of algebraic functions on the universal extension as set-theoretic functions on $\mathbf{Z}_{\mathrm{et}}$, i.e., on the fibers of the infinite étale covering $C^{\infty}_{\widehat{S}} \to C^{\infty}_{\widehat{S}}/\mathbf{Z}_{\mathrm{et}} = C_{\widehat{S}}$.

The original integral structure on the various "$\mathcal{R}$'s" (i.e., the integral structures without a label) will be referred to as the *de Rham-integral, or DR-integral, structures*. This is because they arise from the original natural integral structures on the universal extension $E^{\dagger}$ as a sort of de Rham cohomology – i.e., "$H^1_{\mathrm{DR}}(E, \mathcal{O}^{\times}_E)$" – associated to E.

Now we are ready to begin the construction of the *higher Schottky-Weierstrass zeta functions.* In the following, we shall refer to polynomials

$$\phi_r T^{[r]} + \phi_{r-1} T^{[r-1]} + \ldots + \phi_0$$

(where r is a natural number) whose coefficients $\phi_0, \ldots, \phi_r$ are *meromorphic functions on* $C^{\infty}_{\widehat{S}}$ as *extension polynomials.* One may also think of extension polynomials as meromorphic sections of $\mathcal{R}^{\mathrm{et}}_{E_C^{\dagger}}$ over $C^{\infty}_{\widehat{S}}$. We would like to consider those extension polynomials which are *invariant under the natural action of* $\mathbf{Z}_{\mathrm{et}}$, i.e., which descend to $C_{\widehat{S}}$. Note that since all $\mathcal{O}_{C_{\widehat{S}}}$-torsors on $C_{\widehat{S}}$ split as soon as one allows a pole of order 1 at the origin e, it follows that *there exist* $\mathbf{Z}_{\mathrm{et}}$*-invariant extension polynomials* of the form:

$$T^{[r]} + \phi_{r-1} T^{[r-1]} + \ldots + \phi_0$$

where $\phi_0, \ldots, \phi_{r-1}$ are meromorphic on $C^{\infty}_{\widehat{S}}$, regular away from $\widetilde{e}$ (the $\mathbf{Z}_{\mathrm{et}}$-orbit of the origin), and have a pole of order at most 1 at the points of $\widetilde{e}$. For instance, the *most basic example* (in the case where $2 \in \mathcal{O}^{\times}$) of such a polynomial is the polynomial

$$T + \zeta$$

(where ζ is as in §5) studied in §5. Note that whereas $1_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ acts on T by

$$T \mapsto T + 1$$

it acts on ζ by $\zeta \mapsto \zeta - 1$. Thus, $T + \zeta$ is indeed $\mathbf{Z}_{\mathrm{et}}$-invariant. In the following, we would like to study the *case of arbitrary* $r \geq 1$.

First, let us introduce some notation: If f is an extension polynomial, then let us denote the result of acting on f by $1_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ by means of the notation $\alpha(f)$. Let

$$\delta(f) \stackrel{\mathrm{def}}{=} \alpha(f) - f$$

This extends the definition of δ given in the discussion preceding Proposition 1.1 (i.e., the case of extension polynomials with constant coefficients). Also, if f and g are extension polynomials, then we have

$$\delta(f \cdot g) = \delta(f) \cdot \alpha(g) + f \cdot \delta(g)$$

Thus, to say that

$$f = \sum_{i=0}^{n} \phi_{n-i} \cdot T^{[i]}$$

is $\mathbf{Z}_{\text{et}}$-*invariant* means precisely that

$$\delta(f) = \sum_{i=0}^{n} \delta(\phi_{n-i})(T^{[i]} + T^{[i-1]}) + \phi_{n-i} \cdot T^{[i-1]} = 0$$

i.e., that for all i,

$$\delta(\phi_{n-i}) + \delta(\phi_{n-i-1}) + \phi_{n-i-1} = 0$$

(where we let $\phi_j \stackrel{\text{def}}{=} 0$ for $j < 0$, $j > n$). Thus, we obtain the following:

Lemma 6.3. *Let n be a nonnegative integer. Suppose that*

$$f = \sum_{i=0}^{n} \phi_{n-i} \cdot T^{[i]}$$

is a $\mathbf{Z}_{\text{et}}$-invariant extension polynomial such that $\phi_0 = 1$; and all the ϕ_j's are meromorphic functions on $C^{\infty}_{\widehat{S}}$ which are regular away from $\widetilde{e}$ (the $\mathbf{Z}_{\text{et}}$-orbit of the origin), and have a pole of order at most 1 at the points of $\widetilde{e}$. Then there exists a $\mathbf{Z}_{\text{et}}$-invariant extension polynomial

$$g = \sum_{i=0}^{n+1} \psi_{n+1-i} \cdot T^{[i]}$$

such that $\phi_j = \psi_j$ for $0 \leq j \leq n$; and ψ_{n+1} is a meromorphic function on $C^{\infty}_{\widehat{S}}$ which is regular away from $\widetilde{e}$ (the $\mathbf{Z}_{\text{et}}$-orbit of the origin), and has a pole of order at most 1 at the points of $\widetilde{e}$.

Proof. First note that, since for $0 \leq j \leq n$, the invariance condition discussed above (i.e., "$\delta(\phi_{n-i}) + \delta(\phi_{n-i-1}) + \phi_{n-i-1} = 0$") on the ϕ_j's is precisely the same as the invariance condition on the ψ_j's, it thus follows that

$$? \cdot T^{[0]} + \sum_{i=1}^{n+1} \phi_{n+1-i} \cdot T^{[i]}$$

defines a $\mathbf{Z}_{\text{et}}$-invariant section of the sheaf of extension polynomials *modulo* constant terms on $C^{\infty}_{\widehat{S}}$. Moreover, since $H^1(C_{\widehat{S}}, \mathcal{O}_{C_{\widehat{S}}}(e)) = 0$, it thus follows that as long as we allow a pole of order 1 at $\widetilde{e}$, this section lifts to a $\mathbf{Z}_{\text{et}}$-invariant extension polynomial

$$g = \sum_{i=0}^{n+1} \psi_{n+1-i} \cdot T^{[i]}$$

(where $\psi_j = \phi_j$ for $j = 0, \ldots, n$, and ψ_{n+1} has the desired properties). This completes the proof. ○

Theorem 6.4. *Let n be a nonnegative integer. Then there exists a $\mathbf{Z}_{\text{et}}$-invariant extension polynomial*

$$f = \sum_{i=0}^{n} \zeta_{n-i} \cdot T^{[i]}$$

such that $\zeta_0 = 1$;

$$\delta(\zeta_{n-i}) + \delta(\zeta_{n-i-1}) + \zeta_{n-i-1} = 0$$

(for all i); and all the ζ_j's are meromorphic functions on $C^{\infty}_{\widehat{S}}$ which are regular away from $\widetilde{e}$ (the $\mathbf{Z}_{\text{et}}$-orbit of the origin), and have a pole of order at most 1 at the points of $\widetilde{e}$. In particular, we have

$$\delta(\zeta_j) = -\zeta_{j-1} + \zeta_{j-2} - \ldots + (-1)^{j-1}\zeta_1 + (-1)^j \zeta_0$$

(for all j). Finally, if $\widehat{\zeta}_0, \ldots, \widehat{\zeta}_n$ satisfy the same conditions as $\zeta_0, \ldots, \zeta_n$, then for each $j = 0, \ldots, n$,

$$\zeta_j - \widehat{\zeta}_j = \text{ some } A\text{-linear combination of } \zeta_0, \ldots, \zeta_{j-1}$$

(where $A = \mathcal{O}[[q]]$).

Proof. The existence of $\zeta_0, \ldots, \zeta_n$ as stated follows by successively applying Lemma 6.3, starting with the extension polynomial 1 (in the case $r = 0$). The second formula for $\delta(\zeta_j)$ follows by induction on j from the first formula $\delta(\zeta_{n-i}) + \delta(\zeta_{n-i-1}) + \zeta_{n-i-1} = 0$.

Thus, it remains to see what happens if $\widehat{\zeta}_0, \ldots, \widehat{\zeta}_n$ satisfy the same conditions as $\zeta_0, \ldots, \zeta_n$. We prove that $\zeta_j - \widehat{\zeta}_j$ is an A-linear combination of $\zeta_0, \ldots, \zeta_{j-1}$ by induction on j. This is clear for $j = 0$ since, by assumption, $\zeta_0 = \widehat{\zeta}_0 = 1$. Now observe that, by the induction hypothesis, both

$$\delta(\zeta_j) = -\zeta_{j-1} + \zeta_{j-2} - ... + (-1)^{j-1}\zeta_1 + (-1)^j\zeta_0$$

and

$$\delta(\widehat{\zeta}_j) = -\widehat{\zeta}_{j-1} + \widehat{\zeta}_{j-2} - ... + (-1)^{j-1}\widehat{\zeta}_1 + (-1)^j\widehat{\zeta}_0$$

are of the form "$-\zeta_{j-1}$ plus an A-linear combination of $\zeta_0, \ldots, \zeta_{j-2}$." Thus, by repeated application of the formula

$$\delta(\zeta_l) = -\zeta_{l-1} + \zeta_{l-2} - ... + (-1)^{l-1}\zeta_1 + (-1)^l\zeta_0$$

for $l \leq j - 1$, it follows that there exists some

$$\gamma = \zeta_j + \eta$$

where η is an A-linear combination of $\zeta_0, \ldots, \zeta_{j-1}$, and, moreover,

$$\delta(\gamma) = \delta(\widehat{\zeta}_j)$$

In particular, it follows that $\epsilon \stackrel{\mathrm{def}}{=} \gamma - \widehat{\zeta}_j$ is a meromorphic function on $C^{\infty}_{\widehat{S}}$ which is regular away from $\widetilde{e}$, has a pole of order ≤ 1 at the points of $\widetilde{e}$, and, moreover, satisfies $\delta(\epsilon) = 0$, i.e., is $\mathbf{Z}_{\mathrm{et}}$-invariant. But this implies that ϵ arises from a meromorphic function on $C_{\widehat{S}}$ which is regular away from the origin, where it has a pole of order ≤ 1. In other words, this meromorphic function on $C_{\widehat{S}}$ forms a regular section of $\mathcal{O}_{C_{\widehat{S}}}(e)$ over $C_{\widehat{S}}$. But it follows from Riemann-Roch that such a function must be constant, i.e., $\in A$. Thus, $\epsilon = \gamma - \widehat{\zeta}_j \in A$. This completes the proof. ◯

Remark 1. Note that the ζ_j's of Theorem 6.4 are *not uniquely defined* (except for $j = 0$). Nevertheless, we shall often refer to these ζ_j's, which are useful for explicit computations, as *higher Schottky-Weierstrass ζ-functions.* Also, we observe that if $2 \in \mathcal{O}^{\times}$, then the function ζ of §5 serves as a "ζ_1" in Theorem 6.4.

Remark 2. By using the $\mathbf{G}_{\mathrm{m}}$-*splitting* discussed in §3, one may also construct *complex analytic Schottky-Weierstrass ζ-functions*, as follows. Namely, just as in §5, we may form the *complex analytic Schottky-Weierstrass ζ-function*

$$\zeta$$

by taking the difference between the $\mathbf{G}_{\mathrm{m}}$-splitting of §3 and the meromorphic splitting $\mathcal{S}_{\mathrm{or}}$ of Corollary 4.4. This function is meromorphic on $\mathbf{G}_{\mathrm{m}}$ (in the usual sense of complex analysis), regular away from $q^{\mathbf{Z}} \subseteq \mathbf{G}_{\mathrm{m}}$, and satisfies (relative to the notation in the discussion of the $\mathbf{G}_{\mathrm{m}}$-splitting in §3) the relation

$$\zeta(q \cdot U) = \zeta(U) - 1$$

(where U is the standard multiplicative coordinate on $\mathbf{G}_{\mathrm{m}}$). Indeed, to check that this relation is satisfied, it suffices (since all the functions involved depend meromorphically on q and U) to check it as $q \to 0$. But this case is precisely the case where one is working near the point at infinity of $\overline{\mathcal{M}}_{1,0}(\mathbf{C})$, i.e., the case discussed in §5. Thus, by regarding the various algebraic and formal algebraic functions of §5 as complex analytic functions, we see that this relation follows from the corresponding formal algebraic relation proven in Theorem 5.6. Finally, by performing exactly the same formal operations that we have done in the present §, but this time in the complex analytic category, we also obtain *complex analytic higher Schottky-Weierstrass ζ-functions*

$$\zeta_0, \ldots, \zeta_n$$

as in Theorem 6.4.

§7. Canonical Schottky-Weierstrass Zeta Functions

In this §, we continue the discussion of *higher Schottky-Weierstrass zeta functions* begun in §6. The purpose of the present § is to construct two collections of *canonical higher Schottky-Weierstrass zeta functions* by means of various differential operators acting on the line bundle under consideration. The first type of canonical zeta function, which we refer to as *divided power* canonical zeta functions, is the *most basic type* of canonical zeta function. These divided power canonical zeta functions do not satisfy quite the same relations (concerning the operator δ) as the functions of Theorem 6.4 (although they do satisfy certain relations similar to the relations in Theorem 6.4), and, moreover, are *not integral*, i.e., are only defined over $\mathbf{Q}$. The second type of canonical zeta function, which we refer to as *binomial*, is of the sort considered in Theorem 6.4, and, moreover, is *integral except at the prime* 2. Both the divided power and binomial canonical zeta functions of the present § (as well as various other types, to be introduced later) play an important role in the theory of this paper (cf., especially, Chapter V, §4; Chapter VII; Chapter VIII).

We maintain the notation of §6, except that in this §, we assume, until mentioned otherwise, that $\mathcal{O}$ *is a number field*. In particular, $\mathcal{O} \supseteq \mathbf{Q}$. As usual, we will write

$$\mathcal{L}_{C^{\infty}_{\widehat{S}}} = \mathcal{O}_{C^{\infty}_{\widehat{S}}}(\widetilde{e})$$

where $\widetilde{e} \stackrel{\text{def}}{=} \mathbf{Z}_{\text{et}}(e) \subseteq C^{\infty}_{\widehat{S}}$. Note that $\mathbf{Z}_{\text{et}}$ acts on the global sections of $\mathcal{L}_{C^{\infty}_{\widehat{S}}}$ over $C^{\infty}_{\widehat{S}}$. Thus, in particular, the operator δ of §6 defined by

$$\delta(f) \stackrel{\text{def}}{=} 1_{\text{et}}(f) - f$$

acts on $\Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}_{C^{\infty}_{\widehat{S}}})$. Just as in §6, *we would like to study sections of $\mathcal{L}_{C^{\infty}_{\widehat{S}}}$ over $C^{\infty}_{\widehat{S}}$ which satisfy certain properties with respect to the operator δ.*

Next, let us observe that we also have another operator on sections of $\mathcal{L}_{C^{\infty}_{\widehat{S}}}$ given by the *connection* ∇ on $\mathcal{L}_{C^{\infty}_{\widehat{S}}}$ defined by the canonical section $-\kappa$ (cf. Theorem 5.6). That is to say, by Theorem 5.6, $-2 \cdot \kappa$ determines a connection on $\mathcal{L}^{\otimes 2}_{C^{\infty}_{\widehat{S}}}$, so $-\kappa$ determines a connection on $\mathcal{L}_{C^{\infty}_{\widehat{S}}}$ which is defined whenever $2 \in \mathcal{O}^{\times}$ (as is the case here). Let us denote the operator on sections of $\mathcal{L}_{C^{\infty}_{\widehat{S}}}$ given by applying the connection ∇ in the direction $U\frac{\partial}{\partial U}$ by

$$f \mapsto \delta^*(f) \stackrel{\text{def}}{=} \nabla_{(U\frac{\partial}{\partial U})}(f)$$

Note that since (cf. Theorem 5.6), $1_{\text{et}}(\kappa) = \kappa + d\ \log(U)$, we obtain that $1_{\text{et}}(\nabla) = \nabla - d\ \log(U)$. Thus, $1_{\text{et}}(\delta^*) = \delta^* - 1$ (i.e., $1_{\text{et}}(\delta^*(f)) = \delta^*(1_{\text{et}}(f)) - 1_{\text{et}}(f)$), so we obtain:

$$\begin{aligned} [\delta^*, \delta](f) &= [\delta^*, 1_{\text{et}}](f) \\ &= \delta^*(1_{\text{et}}(f)) - 1_{\text{et}}(\delta^*(f)) \\ &= 1_{\text{et}}(f) \end{aligned}$$

i.e., $[\delta^*, \delta] = 1_{\text{et}}$. Roughly speaking,

> *It is natural to think of δ^* as "differentiation ∂ with respect to the holomorphic variable" and δ as "differentiation $\overline{\partial}$ with respect to the anti-holomorphic variable."*

Alternatively, one may think of the operator δ^* as a sort of *adjoint* to the operator δ. This point of view is motivated by the complex analytic theory, which we will discuss in more detail in Chapter VII, §4.

Next, let us observe that if we let $\zeta_0^{\text{PD}} \in \Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}_{C^{\infty}_{\widehat{S}}})$ be the section defined by "$1 \in \mathcal{O}_{C^{\infty}_{\widehat{S}}}(\widetilde{e})$," then it follows from Theorem 5.6 that

$$\delta^*(\zeta_0^{\text{PD}}) = \zeta_1$$

(where ζ_1 is as in §5). Thus, it is natural to set (for any integer $n \geq 0$)

$$\zeta_n^{\mathrm{PD}} \stackrel{\mathrm{def}}{=} \frac{1}{n!}(\delta^*)^n \zeta_0^{\mathrm{PD}}$$

Here, "PD" stands for "puissances divisés" (i.e., "divided powers" in French).

Just as was the case with the higher Schottky-Weierstrass zeta functions of Theorem 6.4, the canonical Schottky-Weierstrass zeta functions fit together to form "$\mathbf{Z}_{\mathrm{et}}$-*invariant extension polynomials*" as follows: First, observe that in the above discussion, we worked with various functions/sections of line bundles *over* $C^{\infty}_{\widehat{S}}$. In order to obtain $\mathbf{Z}_{\mathrm{et}}$-invariant extension polynomials, we must instead work with objects over $E_C^{\dagger}|_{C^{\infty}_{\widehat{S}}}$. Recall that by Corollary 5.7, $E_C^{\dagger}|_{C^{\infty}_{\widehat{S}}} \to C^{\infty}_{\widehat{S}}$ admits a natural splitting which allows us to regard the push-forward of $\mathcal{O}_{E_C^{\dagger}|_{C^{\infty}_{\widehat{S}}}}$ to $C^{\infty}_{\widehat{S}}$ as the $\mathcal{O}_{C^{\infty}_{\widehat{S}}}$-algebra

$$\mathcal{O}_{C^{\infty}_{\widehat{S}}}[T]$$

(cf. the discussion at the beginning of §6). Note that by Theorem 4.2 (which we are at liberty to apply since here, we are working in characteristic zero), the pull-back to $E_C^{\dagger}|_{C^{\infty}_{\widehat{S}}}$ of the line bundle $\mathcal{L}_C$ admits a *tautological connection* ∇^{taut}. Here, just as in the above discussion involving ∇, we use the notation "$(\delta^{\mathrm{taut}})^*$" for the tautological connection ∇^{taut} applied in the direction $U\frac{\partial}{\partial U}$. Since the natural isomorphism of Corollary 5.7 is an isomorphism of $W_{\widehat{S}}$-*torsors*, it follows easily from the definitions that, relative to the connection ∇ considered above (arising from $-\kappa$), we have:

$$(\delta^{\mathrm{taut}})^* = \delta^* + T$$

Note that unlike δ^*, the operator $(\delta^{\mathrm{taut}})^*$ *descends* (since it is "tautological") to $E_C^{\dagger}$, hence is $\mathbf{Z}_{\mathrm{et}}$-*invariant.* Now we have the following result:

Lemma 7.1. *We have (for $n \geq 0$):*

(i) $\delta^*(\zeta_n^{\mathrm{PD}}) = (n+1) \cdot \zeta_{n+1}^{\mathrm{PD}}$;

(ii) $\zeta_n^{\mathrm{PD}}[T] \stackrel{\mathrm{def}}{=} \sum_{i=0}^{n} \zeta_i^{\mathrm{PD}} \cdot \frac{T^{n-i}}{(n-i)!}$ *is* $\mathbf{Z}_{\mathrm{et}}$*-invariant.*

(iii) $\delta(\zeta_n^{\mathrm{PD}}) = \sum_{i=0}^{n-1}(-1)^{i+n}\frac{1}{(n-i)!}\zeta_i^{\mathrm{PD}} = -\zeta_{n-1}^{\mathrm{PD}} + \frac{1}{2} \cdot \zeta_{n-2}^{\mathrm{PD}} + \ldots + (-1)^n \cdot \frac{1}{n!} \cdot \zeta_0^{\mathrm{PD}}$.

In particular, $\delta^n(\zeta_n^{\mathrm{PD}}) = (-1)^n \cdot \zeta_0^{\mathrm{PD}}$.

Proof. First, we observe that: (i) follows immediately from the definition of ζ_n^{PD}; (iii) follows from the $\mathbf{Z}_{\mathrm{et}}$-invariance of (ii) by evaluating the expression of (ii) at "U" equals

$q \cdot U$, "T" equals 0 (which yields $1_{\mathrm{et}}(\zeta_n^{\mathrm{PD}})$), and noting that this is equal to the expression of (ii) at "U" equals U, "T" equals -1 (which yields $\sum_{i=0}^{n} \frac{(-1)^{i+n}}{(n-i)!} \cdot \zeta_i^{\mathrm{PD}}$). Thus, it suffices to prove (ii). But (ii) follows from the $\mathbf{Z}_{\mathrm{et}}$-*invariance* of $(\delta^{\mathrm{taut}})^*$: Indeed, since $[\ \delta^*,\ T\ ] = 0$, if we apply $\frac{1}{n!} \cdot ((\delta^{\mathrm{taut}})^*)^n$ to ζ_0^{PD} (which is clearly $\mathbf{Z}_{\mathrm{et}}$-invariant), we obtain:

$$\begin{aligned}\frac{1}{n!} \cdot ((\delta^{\mathrm{taut}})^*)^n(\zeta_0^{\mathrm{PD}}) &= \frac{1}{n!} \cdot \sum_{i=0}^{n} \binom{n}{i} (\delta^*)^i(\zeta_0^{\mathrm{PD}}) \cdot T^{n-i} \\ &= \sum_{i=0}^{n} \zeta_i^{\mathrm{PD}} \cdot \frac{T^{n-i}}{(n-i)!}\end{aligned}$$

as desired. ○

Lemma 7.2. *Any $\phi \in \Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}_{C^{\infty}_{\widehat{S}}})$ such that $\delta^{n+1}(\phi) = 0$ for some nonnegative integer n may be written as a* unique *A-linear combination of $\zeta_0^{\mathrm{PD}}, \ldots, \zeta_n^{\mathrm{PD}}$. In particular, $\zeta_0^{\mathrm{PD}}, \ldots, \zeta_n^{\mathrm{PD}}$ are* linearly independent over *A.*

Proof. This follows by induction on n. It is clear when $n = 0$ (since $\Gamma(C, \mathcal{L}_C)$ ($\subseteq \Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}_{C^{\infty}_{\widehat{S}}})$) is a free A-module of rank 1 generated by ζ_0^{PD}). For arbitrary $n \geq 1$, if $\delta^{n+1}(\phi) = 0$, then by the induction hypothesis, $\delta(\phi)$ may be written as an A-linear combination $\sum_{i=0}^{n-1} c_i \cdot \zeta_i^{\mathrm{PD}}$. Thus, by Lemma 7.1, it follows that for some appropriate $c_i' \in A$, the function $\phi - \sum_{i=0}^{n-1} c_i' \cdot \zeta_{i+1}^{\mathrm{PD}}$ is contained in the kernel of δ, hence can be written as an A-multiple of ζ_0^{PD}, as desired. Linear independence follows similarly by applying δ, using the induction hypothesis and Lemma 7.1. ○

Definition 7.3. We shall refer the functions $\zeta_n^{\mathrm{PD}} \in \Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}_{C^{\infty}_{\widehat{S}}})$ as *divided power (canonical Schottky-Weierstrass) zeta functions.*

Thus, putting everything together, we see that we have proven the following result:

Theorem 7.4. (Divided Power Canonical Schottky-Weierstrass Functions) *Let $\zeta_0^{\mathrm{PD}} \in \Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}_{C^{\infty}_{\widehat{S}}})$ be the function "1" $\in \mathcal{O}_{C^{\infty}_{\widehat{S}}}(\widetilde{e})$. Write (for $n \in \mathbf{Z}_{\geq 0}$)*

$$\zeta_n^{\mathrm{PD}} \stackrel{\mathrm{def}}{=} \frac{1}{n!} (\delta^*)^n \zeta_0^{\mathrm{PD}}$$

(and let $\zeta_n^{\mathrm{PD}} \stackrel{\mathrm{def}}{=} 0$ if $n < 0$). (Thus, in particular, the function ζ_1^{PD} here is the same as the function "ζ_1" of Theorem 5.6.) Then $\delta^*(\zeta_n^{\mathrm{PD}}) = (n+1) \cdot \zeta_{n+1}^{\mathrm{PD}}$, $\delta^n(\zeta_n^{\mathrm{PD}}) = (-1)^n \cdot \zeta_0^{\mathrm{PD}}$ (if $n \geq 0$);

$\delta(\zeta_n^{\mathrm{PD}}) = \sum_{i=0}^{n-1}(-1)^{i+n}\frac{1}{(n-i)!}\zeta_i^{\mathrm{PD}} = -\zeta_{n-1}^{\mathrm{PD}} + \frac{1}{2}\cdot\zeta_{n-2}^{\mathrm{PD}} + \ldots + (-1)^n\cdot\frac{1}{n!}\cdot\zeta_0^{\mathrm{PD}}$

(for all $n \in \mathbf{Z}$). Moreover, the A-submodule of $\Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}_{C^{\infty}_{\widehat{S}}})$ generated by $\zeta_0^{\mathrm{PD}}, \ldots, \zeta_n^{\mathrm{PD}}$ is equal to the A-submodule of sections of $\Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}_{C^{\infty}_{\widehat{S}}})$ which are annihilated by δ^{n+1}. In particular, this submodule is equal to the A-submodule of $\Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}_{C^{\infty}_{\widehat{S}}})$ generated by the functions denoted "$\zeta_0, \ldots, \zeta_n$" in Theorem 6.4. Finally, the polynomial

$$\zeta_n^{\mathrm{PD}}[T] \stackrel{\mathrm{def}}{=} \sum_{i=0}^{n} \zeta_i^{\mathrm{PD}} \cdot \frac{T^{n-i}}{(n-i)!} = \zeta_0^{\mathrm{PD}} \cdot \frac{T^n}{n!} + \zeta_1^{\mathrm{PD}} \cdot \frac{T^{(n-1)}}{(n-1)!} + \ldots + \zeta_{n-1}^{\mathrm{PD}} \cdot T + \zeta_n^{\mathrm{PD}}$$

($\in \Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}_{C^{\infty}_{\widehat{S}}}[T])$) is $\mathbf{Z}_{\mathrm{et}}$-invariant (relative to the natural action of $\mathbf{Z}_{\mathrm{et}}$ on $C^{\infty}_{\widehat{S}}$, $\mathcal{L}_{C^{\infty}_{\widehat{S}}}$, and the action of $\mathbf{Z}_{\mathrm{et}}$ on T given by $1_{\mathrm{et}}(T) = T + 1$).

Remark 1. Just as was the case with Theorem 6.4, Theorem 7.4 also has a *complex analytic version.* We leave the routine details to the reader.

Remark 2. It is clear from the formula for $\delta(\zeta_n^{\mathrm{PD}})$ that the denominators that occur are "essential" (i.e., they cannot be eliminated as in the case of Theorem 6.4 simply by "redefining the integral structure" as in Definition 6.2). In fact, we shall see (cf. Chapter VI) that the functions of Theorem 6.4 are more suited to the study of the universal extension at *finite primes* (i.e., in mixed characteristic), where as the functions of Theorem 7.4 are more suited to the study of the universal extension at *infinite primes* (cf. Chapter VII; Chapter VIII).

Remark 3. One "*generating function-theoretic*" way to summarize the content of Theorem 7.4 is the following: Let $\mathbf{s}$ be an indeterminate. Note that

$$[\ \delta,\ (\delta^{\mathrm{taut}})^*\ ] \ = \ [\ \delta,\ \delta^* + T\ ] \ = \ 0$$

Thus, since $\delta(\zeta_0^{\mathrm{PD}}) = 0$, δ also annihilates

$$\begin{aligned} e^{(\delta^*+T)\cdot\mathbf{s}} \cdot \zeta_0^{\mathrm{PD}} &= \sum_{n\geq 0} \left(\sum_{i=0}^{n} \zeta_i^{\mathrm{PD}} \cdot \frac{T^{n-i}}{(n-i)!}\right) \cdot \mathbf{s}^n \\ &= \zeta_0^{\mathrm{PD}}[T] + \zeta_1^{\mathrm{PD}}[T]\cdot\mathbf{s} + \zeta_2^{\mathrm{PD}}[T]\cdot\mathbf{s}^2 + \ldots + \zeta_n^{\mathrm{PD}}[T]\cdot\mathbf{s}^n + \ldots \end{aligned}$$

The coefficients of this series are precisely the functions discussed in Theorem 7.4.

Next, we would like to construct the *binomial canonical Schottky-Weierstrass functions.* First, recall the polynomials

$$T^{[n]} \stackrel{\text{def}}{=} \frac{1}{n!}T(T-1)(T-2)\cdot\ldots\cdot(T-(n-1))$$

discussed in §6. If D is any operator, we shall denote by

$$\binom{D}{n}$$

the operator obtained by substituting D into the polynomial $T^{[n]}$.

Lemma 7.5. *If D_1 and D_2 are any two commuting operators, we have (for n a positive integer)*

$$\binom{D_1+D_2}{n} = \sum_{m=0}^{n} \binom{D_1}{m}\cdot\binom{D_2}{n-m}$$

Proof. This identity is essentially an equality between two polynomials (i.e., the expressions on the left- and right-hand sides) in two variables (i.e., D_1, D_2). Thus, it holds if and only if holds when D_1 and D_2 are arbitrary positive integers. But for such D_1 and D_2, this identity may be checked by considering the coefficients of x^n in the equality $(1+x)^{D_1+D_2} = (1+x)^{D_1}\cdot(1+x)^{D_2}$ (where x is an indeterminate). ○

Now we define $\zeta_0^{\text{BI}} \stackrel{\text{def}}{=} \zeta_0^{\text{PD}}$; (for n a positive integer) $\zeta_n^{\text{BI}} \stackrel{\text{def}}{=} \binom{\delta^*}{n}(\zeta_0^{\text{PD}})$;

$$\zeta_n^{\text{BI}}[T] \stackrel{\text{def}}{=} \binom{\delta^*+T}{n}(\zeta_0^{\text{PD}}) = \sum_{m=0}^{n} \zeta_m^{\text{BI}}\cdot T^{[n-m]}$$

Note that the $\mathbf{Z}_{\text{et}}$-invariance of $(\delta^{\text{taut}})^* = \delta^*+T$ implies that $\zeta_n^{\text{BI}}[T]$ is also $\mathbf{Z}_{\text{et}}$-invariant. In particular, the ζ_n^{BI} are *higher Schottky-Weierstrass zeta functions in the sense of the Theorem 6.4.* From a "generating function-theoretic" point of view, one can write

$$\begin{aligned}(1+\mathbf{s})^{(\delta^*+T)}\cdot\zeta_0^{\text{BI}} &= \sum_{n\geq 0} \zeta_n^{\text{BI}}[T]\cdot\mathbf{s}^n \\ &= \zeta_0^{\text{BI}}[T] + \zeta_1^{\text{BI}}[T]\cdot\mathbf{s} + \zeta_2^{\text{BI}}[T]\cdot\mathbf{s}^2 + \ldots + \zeta_n^{\text{BI}}[T]\cdot\mathbf{s}^n + \ldots\end{aligned}$$

Finally, we observe that the ζ_n^{BI} are *integral* over $\mathbf{Z}[\frac{1}{2}]$. Indeed, to see this, we reason as follows: Recall the section $\sigma^{\mathrm{alg}} \in \Gamma(C, \mathcal{L}_C)$ of §5. If we apply $\binom{\delta^*}{n}$ to $(\sigma^{\mathrm{alg}})^2|_{C^\infty_{\widehat{S}}} = (\sigma^{\mathrm{an}})^2 \cdot \theta$, which is a series with **Z**-*integral coefficients* in the monomials

$$\{q^{k^2+k} \cdot U^{2k+1} \cdot \theta, q^{k^2} \cdot U^{2k} \cdot \theta, q^{k^2-k} \cdot U^{2k-1} \cdot \theta\}$$

then since $\binom{\delta^*}{n}$ acts on U^k by multiplication by $\binom{k}{n}$, we obtain that $\binom{\delta^*}{n}(\sigma^{\mathrm{alg}})^2|_{C^\infty_{\widehat{S}}}$ is **Z**-integral. On the other hand, ζ_n^{BI} is essentially $\binom{\delta^*}{n}(\sigma^{\mathrm{alg}})|_{C^\infty_{\widehat{S}}}$, so the $\mathbf{Z}[\frac{1}{2}]$-integrality of ζ_n^{BI} follows formally by solving for $2 \cdot \mathbf{s}^n \binom{\delta^*}{n}(\sigma^{\mathrm{alg}})|_{C^\infty_{\widehat{S}}}$ (and applying induction on n) in the formal identity

$$(1+\mathbf{s})^{\delta^*}\{(\sigma^{\mathrm{alg}})^2\} = \{(1+\mathbf{s})^{\delta^*}(\sigma^{\mathrm{alg}})\}^2$$

(where we note that this formal identity follows from the general fact that if X is any "operator" on "(commuting) functions" A, B which satisfies the Leibniz rule, then the identity $e^X(A \cdot B) = e^X(A) \cdot e^X(B)$ is a formal consequence of the Leibniz rule).

Definition 7.6. We shall refer the functions $\zeta_n^{\mathrm{BI}} \in \Gamma(C^\infty_{\widehat{S}}, \mathcal{L}_{C^\infty_{\widehat{S}}})$ as *binomial (canonical Schottky-Weierstrass) zeta functions.*

Theorem 7.7. (Binomial Canonical Schottky-Weierstrass Functions) *Let* $\zeta_0^{\mathrm{BI}} \in \Gamma(C^\infty_{\widehat{S}}, \mathcal{L}_{C^\infty_{\widehat{S}}})$ *be the function "1"* $\in \mathcal{O}_{C^\infty_{\widehat{S}}}(\widetilde{e})$. *Write (for* $n \in \mathbf{Z}_{\geq 0}$*)* $\zeta_n^{\mathrm{BI}} \stackrel{\mathrm{def}}{=} \binom{\delta^*}{n}(\zeta_0^{\mathrm{BI}})$*;*

$$\zeta_n^{\mathrm{BI}}[T] \stackrel{\mathrm{def}}{=} \binom{\delta^* + T}{n}(\zeta_0^{\mathrm{BI}}) = \sum_{m=0}^{n} \zeta_m^{\mathrm{BI}} \cdot T^{[n-m]}$$

Then the ζ_n^{BI}, $\zeta_n^{\mathrm{BI}}[T]$*'s satisfy the properties of Theorem 6.4 (where we take "*ζ_n*" of Theorem 6.4 to be* ζ_n^{BI}*; "*f*" of Theorem 6.4 to be* $\zeta_n^{\mathrm{BI}}[T]$*). In particular, the* $\zeta_n^{\mathrm{BI}}[T]$*'s are* $\mathbf{Z}_{\mathrm{et}}$*-invariant. Finally, the* $\zeta_n^{\mathrm{BI}}[T]$*'s are integral over* $\mathbf{Z}[\frac{1}{2}]$.

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Chapter IV: Theta Groups and Theta Functions

§0. Introduction

In this Chapter, we discuss *theta groups* in various contexts. These groups will play a key role in this paper. In §1, we review the "classical theory" of theta groups and *algebraic theta functions* of [Mumf1,2,3], specialized to the case of elliptic curves. In §2, we discuss the relationship between this sort of theta action, and the theta action that one considers in the case of the *Schottky uniformization* of a degenerating elliptic curve. In §3, we use the theory of §2 to define *twisted Schottky-Weierstrass zeta functions*, which will play an important role in the calculations of Chapter V. In §4, we review Zhang's theory ([Zh]) of metrized line bundles. Finally, in §5, we discuss the *relationship between Zhang's theory and the theory of Mumford* reviewed in §1. We then apply this discussion to compute various *degrees of push-forwards of metrized line bundles*.

§1. Mumford's Algebraic Theta Functions

In this §, we review the definitions and basic properties of *theta groups* (cf., e.g., [MB], Chapitres V, VI) and *algebraic theta functions* (cf. [Mumf1,2,3]). See also [Mumf4] for basic facts concerning abelian varieties. Let

$$f : E \to S$$

be an *elliptic curve* over a scheme S, with identity section $e : S \to E$. Let $\mathcal{L}$ be a *relatively ample, symmetric* line bundle on E, of relative degree d (over S). Then it is not difficult to show that $\mathcal{L}$ is necessarily of the form:

$$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E((d-1) \cdot [e] + [\tau]) \otimes_{\mathcal{O}_S} \mathcal{M}$$

where $\tau \in E(S)$ is a section such that $2 \cdot \tau = 0$, and $\mathcal{M}$ is a line bundle on S.

For any point $\alpha \in E(T)$ over an S-scheme T, the line bundle

$$(\mathcal{T}_\alpha^* \mathcal{L}) \otimes \mathcal{L}^{-1}$$

(where $E_T \stackrel{\text{def}}{=} E \times_S T$; $\mathcal{T}_\alpha : E_T \to E_T$ is translation by α) on E_T is of relative degree 0 over T, hence defines (by $\alpha \mapsto (\mathcal{T}_\alpha^* \mathcal{L}) \otimes \mathcal{L}^{-1}$) a homomorphism

$$\phi_{\mathcal{L}} : E \to E$$

whose kernel

$$K_{\mathcal{L}} = \mathrm{Ker}([d] : E \to E)$$

is the kernel of multiplication by d on E. In particular, $K_{\mathcal{L}}$ is a *finite, flat group scheme over* S.

Next, we define the scheme

$$\mathcal{G}_{\mathcal{L}}$$

to be the *scheme that represents pairs* (α, ι), where $\alpha \in E(T)$ belongs to $K_{\mathcal{L}}(T)$, and $\iota : \mathcal{T}_\alpha^* \mathcal{L} \cong \mathcal{L}$ is an isomorphism of line bundles on E_T. Then $\mathcal{G}_{\mathcal{L}}$ has a natural structure of *group scheme over* S which fits into an exact sequence (in, say, the finite flat topology of S):

$$1 \to (\mathbf{G}_{\mathrm{m}})_S \to \mathcal{G}_{\mathcal{L}} \to K_{\mathcal{L}} \to 1$$

where the projection $\mathcal{G}_{\mathcal{L}} \to K_{\mathcal{L}}$ is given by $(\alpha, \iota) \mapsto \alpha$, and the inclusion $(\mathbf{G}_{\mathrm{m}})_S \hookrightarrow \mathcal{G}_{\mathcal{L}}$ is given by the natural action of $(\mathbf{G}_{\mathrm{m}})_S$ on $\mathcal{L}$. Although $(\mathbf{G}_{\mathrm{m}})_S$ lies in the center of $\mathcal{G}_{\mathcal{L}}$, in general, $\mathcal{G}_{\mathcal{L}}$ will not be commutative. Indeed, the commutator map $(\alpha, \beta) \mapsto \alpha \cdot \beta \cdot \alpha^{-1} \cdot \beta^{-1}$ (for $\alpha, \beta \in \mathcal{G}_{\mathcal{L}}(T)$) defines a *bilinear nondegenerate pairing*

$$[-, -] : K_{\mathcal{L}} \times K_{\mathcal{L}} \to (\mathbf{G}_{\mathrm{m}})_S$$

The group scheme $\mathcal{G}_{\mathcal{L}}$ is called the *theta group associated to the line bundle* $\mathcal{L}$.

Next, we would like to consider *quasi-coherent* $\mathcal{O}_S$*-modules equipped with an action of* $\mathcal{G}_{\mathcal{L}}$ such that $(\mathbf{G}_{\mathrm{m}})_S \subseteq \mathcal{G}_{\mathcal{L}}$ acts via the "identity character" $(\mathbf{G}_{\mathrm{m}})_S \to (\mathbf{G}_{\mathrm{m}})_S$. We shall refer to such modules as "$\mathcal{O}_S[\mathcal{G}_{\mathcal{L}}]$-modules," for short. Note that it follows from the definition of $\mathcal{G}_{\mathcal{L}}$ that

$$\mathcal{V} \stackrel{\mathrm{def}}{=} f_* \mathcal{L}$$

has a natural structure of $\mathcal{O}_S[\mathcal{G}_{\mathcal{L}}]$*-module*. Moreover, one has the following (cf. [MB], Chapitre V, Corollaire 2.4.3):

Theorem 1.1. *The operation "$\otimes_{\mathcal{O}_S} \mathcal{V}$" induces an equivalence of categories between the category of quasi-coherent* $\mathcal{O}_S$*-modules and the category of* $\mathcal{O}_S[\mathcal{G}_{\mathcal{L}}]$*-modules. Moreover,* *up to tensor product with a line bundle on S, $\mathcal{V}$ is the unique $\mathcal{O}_S[\mathcal{G}_{\mathcal{L}}]$-module with this property.*

Example 1.2. Now suppose that we have an *isomorphism*

$$K_{\mathcal{L}} \cong \boldsymbol{\mu}_d \times (\mathbf{Z}/d\mathbf{Z})$$

such that the pairing discussed above is given by $(\alpha, \beta) \mapsto \alpha^\beta$, for $\alpha \in \boldsymbol{\mu}_d$, $\beta \in \mathbf{Z}/d\mathbf{Z}$, $\lambda \in \mathbf{G}_{\mathrm{m}}(T)$. Then one may construct $\mathcal{G}_{\mathcal{L}}$, as well as the $\mathcal{G}_{\mathcal{L}}$-module $\mathcal{V}$ *explicitly*, as follows. First of all, the T-valued points (for an S-scheme T) $\mathcal{G}_{\mathcal{L}}(T)$ of the group scheme $\mathcal{G}_{\mathcal{L}}$ may be thought of as the set of triples (α, β, λ), where $\alpha \in \boldsymbol{\mu}_d$, $\beta \in \mathbf{Z}/d\mathbf{Z}$, $\lambda \in \mathbf{G}_{\mathrm{m}}(T)$, and the multiplication law is given by:

$$(\alpha_1, \beta_1, \lambda_1) \cdot (\alpha_2, \beta_2, \lambda_2) = (\alpha_1 \cdot \alpha_2, \beta_1 + \beta_2, \lambda_1 \cdot \lambda_2 \cdot \alpha_2^{-\beta_1})$$

Now define the $\mathcal{G}_{\mathcal{L}}$-module $\mathcal{W}$ as the free $\mathcal{O}_S$-module

$$W \stackrel{\text{def}}{=} \mathcal{O}_S \cdot e_0 \oplus \mathcal{O}_S \cdot e_1 \oplus \ldots \oplus \mathcal{O}_S \cdot e_{d-1}$$

(where we think of the indices of the e_i as elements of $\mathbf{Z}/d\mathbf{Z}$) on which $(0, 0, \lambda) \in \mathcal{G}_{\mathcal{L}}(T)$ acts via the $\mathcal{O}_S$-module structure; $(\alpha, 0, 1)$ acts on e_i via $e_i \mapsto \alpha^i \cdot e_i$; and $(0, \beta, 1)$ maps e_i to $e_{i+\beta}$. Then the resulting $\mathcal{O}_S[\mathcal{G}_{\mathcal{L}}]$-module $\mathcal{W}$ is isomorphic to

$$\mathcal{M} \otimes_{\mathcal{O}_S} f_* \mathcal{L}$$

where $\mathcal{M}$ is a line bundle on S.

Next, we consider *Lagrangian subgroups* of $\mathcal{G}_{\mathcal{L}}$. Let

$$K_H \subseteq K_{\mathcal{L}}$$

be a subgroup scheme of $K_{\mathcal{L}}$ which is étale locally (on S) isomorphic to *either* to $\boldsymbol{\mu}_d$ *or* to $\mathbf{Z}/d\mathbf{Z}$, and, moreover, has the property that the restriction of the pairing $K_{\mathcal{L}} \times K_{\mathcal{L}} \to (\mathbf{G}_{\mathrm{m}})_S$ discussed above to K_H is trivial. Then the inverse image $\mathcal{G}_H \subseteq \mathcal{G}_{\mathcal{L}}$ of K_H defines an *abelian* group scheme over S which fits into an exact sequence

$$1 \to (\mathbf{G}_{\mathrm{m}})_S \to \mathcal{G}_H \to K_H \to 1$$

If K_H is *étale locally isomorphic to* $\boldsymbol{\mu}_d$, then $(\mathbf{G}_{\mathrm{m}})_S$ and K_H are both group schemes *of multiplicative type*. Hence, it follows that the splittings of this exact sequence are equivalent to splittings of the corresponding exact sequence of *character groups* (i.e., the exact sequence given by applying $\mathrm{Hom}(-, (\mathbf{G}_{\mathrm{m}})_S)$ to the above exact sequence). In particular, we see that the above exact sequence splits étale locally on S. If K_H is *étale locally isomorphic to* $\mathbf{Z}/d\mathbf{Z}$, then the issue of whether or not this sequence splits is more delicate, but in the cases that we are interested in, this sequence will, in fact, split.

Suppose that

$$H \subseteq \mathcal{G}_H$$

is a splitting of the above exact sequence. Then we have a natural isomorphism $H \cong K_H$.

Definition 1.3. We shall refer to a subgroup $K_H \subseteq K_{\mathcal{L}}$ as above as a *Lagrangian subgroup of* $K_{\mathcal{L}}$. We shall refer to splittings $H \subseteq \mathcal{G}_H$ as *Lagrangian subgroups of* $\mathcal{G}_{\mathcal{L}}$.

Remark. Note that the definition of "Lagrangian" here is somewhat different from that of [MB]. Nevertheless, when applied to H (as opposed to K_H), any Lagrangian subgroup relative to Definition 1.3 is also Lagrangian for [MB], Chapitre V, Définition 2.5.1.

The following result is proven in [MB], Chapitre V, Théorème 3.2, (i); Chapitre VI, Propositions 1.2, 1.4.5:

Theorem 1.4. *Let* $H \subseteq \mathcal{G}_{\mathcal{L}}$ *be* Lagrangian. *Then the correspondence*

$$\mathcal{M} \mapsto \mathcal{M}^H$$

defines a equivalence of categories between the category of $\mathcal{O}_S[\mathcal{G}_{\mathcal{L}}]$*-modules and the category of quasi-coherent* $\mathcal{O}_S$*-modules. Moreover,* $H \subseteq \mathcal{G}_{\mathcal{L}}$ *defines a line bundle* $\mathcal{L}_H$ *on the elliptic curve* $E_H \stackrel{\mathrm{def}}{=} E/K_H$ *together with an isomorphism* $\mathcal{L}_H|_E \cong \mathcal{L}$*. Finally, this isomorphism induces an isomorphism*

$$(f_*\mathcal{L})^H \cong (f_H)_*(\mathcal{L}_H)$$

(where $f_H : E_H \to S$ *is the structure morphism).*

Remark. In fact, to give a line bundle $\mathcal{L}_H$ on $E_H = E/K_H$ together with an isomorphism $\mathcal{L}_H|_E \cong \mathcal{L}$ is equivalent to giving the datum of the lifting $H \subseteq \mathcal{G}_{\mathcal{L}}$ of $K_H \subseteq K_{\mathcal{L}}$. Thus, often instead of specifying (respectively, showing the existence of) the lifting $H \subseteq \mathcal{G}_{\mathcal{L}}$, we will specify (respectively, show the existence of) the line bundle $\mathcal{L}_H$ on $E_H = E/K_H$.

Theorem 1.4 is convenient for reducing proofs concerning $f_*\mathcal{L}$ to proofs concerning $(f_H)_*(\mathcal{L}_H)$.

We are now ready to review Mumford's *construction of algebraic theta functions* (cf. [Mumf1,2,3]). In the remainder of this §, we assume, for simplicity, that

$$\tau = e$$

If $N \in \mathbf{Z}$, let us write $[N] : E \to E$ for the map given by multiplication by N. Note that the map $[-1] : E \to E$ is an automorphism of E of order 2 such that $[-1]^*\mathcal{L} = \mathcal{L}$, i.e., by "=," we mean the isomorphism between $[-1]^*\mathcal{L}$ and $\mathcal{L}$ arising from the fact that (up to tensor product with a line bundle on S) $\mathcal{L}$ is the line bundle associated to the divisor $d \cdot [e]$, which is fixed by $[-1]$. Thus, from the definition of $\mathcal{G}_\mathcal{L}$, it follows that $[-1]$ gives rise to an *automorphism*

$$\delta_\mathcal{L} : \mathcal{G}_\mathcal{L} \to \mathcal{G}_\mathcal{L}$$

of order 2 whose restriction to $(\mathbf{G}_\mathrm{m})_S$ is the identity and which induces the inverse morphism on the quotient $K_\mathcal{L}$ of $\mathcal{G}_\mathcal{L}$. Let us write

$$\mathcal{S}_\mathcal{L} \subseteq \mathcal{G}_\mathcal{L}$$

for the subscheme of T-valued points (where T is an S-scheme) $\gamma \in \mathcal{G}_\mathcal{L}$ satisfying $\delta_\mathcal{L}(\gamma) = \gamma^{-1}$. Such points γ are often referred to as *symmetric* (cf. [Mumf1], Definition, p. 309). One checks easily that the projection

$$\mathcal{S}_\mathcal{L} \to K_\mathcal{L}$$

is surjective, finite, and flat, and that it is, in fact, a *torsor* over $\boldsymbol{\mu}_2$ ($\subseteq \mathbf{G}_\mathrm{m}$). In the following discussion, we would like to show that *this torsor admits a natural section* over $2 \cdot K_\mathcal{L} \subseteq K_\mathcal{L}$.

The construction of this section, as well as the sort of section obtained, differ slightly, depending on whether d is *even* or *odd*. We begin by making the following construction, which is valid regardless of the parity of d: For $\alpha \in K_\mathcal{L}(T)$ (where T is an S-scheme), if $\widetilde{\alpha} \in \mathcal{G}_\mathcal{L}(T)$ is any element that lifts α, then the assignment

$$\alpha \mapsto \widetilde{\alpha} \cdot \delta_\mathcal{L}(\widetilde{\alpha}^{-1})$$

(where we note that the right-hand side is independent of the choice of $\widetilde{\alpha}$ since $\delta_\mathcal{L}$ induces the identity on $(\mathbf{G}_\mathrm{m})_S \subseteq \mathcal{G}_\mathcal{L}$) defines a *morphism*

$$\sigma_1 : K_\mathcal{L} \to \mathcal{G}_\mathcal{L}$$

which factors through $\mathcal{S}_\mathcal{L} \subseteq \mathcal{G}_\mathcal{L}$, and whose composite with the projection $\mathcal{G}_\mathcal{L} \to K_\mathcal{L}$ is the morphism $[2] : K_\mathcal{L} \to K_\mathcal{L}$ (multiplication by 2 on $K_\mathcal{L}$). (Note, that here we write the group law of $K_{\mathcal{L}}$ *additively*.) Indeed, these facts follow from the fact that $\delta_{\mathcal{L}}$ is an automorphism of the group scheme $\mathcal{G}_{\mathcal{L}}$ of order 2 which induces the automorphism $[-1]$ on the quotient $K_{\mathcal{L}}$ of $\mathcal{G}_{\mathcal{L}}$. This morphism σ_1 is *not* (in general) *a group homomorphism*, but does satisfy the following property:

Lemma 1.5. *Let $\alpha, \beta \in K_{\mathcal{L}}(T)$ (where T is an S-scheme). Then*

$$\sigma_1(\alpha+\beta) = [\alpha,\beta]^{-2} \cdot \sigma_1(\alpha) \cdot \sigma_1(\beta)$$

(where we write the group law of $K_{\mathcal{L}}$ (respectively, $\mathcal{G}_{\mathcal{L}}$) additively (respectively, multiplicately)). In particular, it follows that: (i.) if $[\alpha,\beta]^2 = 1 \in \mathbf{G}_{\mathrm{m}}$, then $\sigma_1(\alpha+\beta) = \sigma_1(\alpha) \cdot \sigma_1(\beta)$; (ii.) the restriction of σ_1 to any subgroup scheme of $K_{\mathcal{L}}$ on which the pairing $[-,-]$ (discussed above) is trivial — we shall call such subgroup schemes "$[-,-]$-trivial" *— is a* group homomorphism.

Proof. Indeed, if $\widetilde{\beta} \in \mathcal{G}_{\mathcal{L}}(T)$ lifts β, then $\widetilde{\alpha} \cdot \widetilde{\beta}$ lifts $\alpha + \beta$, so we have:

$$\begin{aligned}
\sigma_1(\alpha+\beta) &= \widetilde{\alpha} \cdot \widetilde{\beta} \cdot \delta_{\mathcal{L}}(\widetilde{\beta}^{-1} \cdot \widetilde{\alpha}^{-1}) \\
&= \widetilde{\alpha} \cdot (\widetilde{\beta} \cdot \delta_{\mathcal{L}}(\widetilde{\beta}^{-1})) \cdot \delta_{\mathcal{L}}(\widetilde{\alpha}^{-1}) \\
&= [\alpha,\beta]^{-2} \cdot \widetilde{\alpha} \cdot \delta_{\mathcal{L}}(\widetilde{\alpha}^{-1}) \cdot (\widetilde{\beta} \cdot \delta_{\mathcal{L}}(\widetilde{\beta}^{-1})) \\
&= [\alpha,\beta]^{-2} \cdot \sigma_1(\alpha) \cdot \sigma_1(\beta)
\end{aligned}$$

as desired. The remaining assertions follow immediately. ○

Thus, at any rate, if d is *odd*, then multiplication by [2] defines an *automorphism* on $K_{\mathcal{L}}$, so we obtain by σ_1 defines a *section*

$$\sigma : K_{\mathcal{L}} \ (= 2 \cdot K_{\mathcal{L}}) \ \to \mathcal{G}_{\mathcal{L}}$$

(such that $\sigma \cdot [2] = \sigma_1$), as desired.

Next, we consider the case when d is *even*. First, we introduce some notation: Let us write

$$d_0 \stackrel{\mathrm{def}}{=} \frac{1}{2} d$$

For $N \in \mathbf{Z}$, let us denote the kernel of $[N] : E \to E$ by

$${}_N E \stackrel{\mathrm{def}}{=} \ker([N] : E \to E)$$

Then I *claim* that the morphism $\sigma_1 : K_{\mathcal{L}} = {}_dE \to \mathcal{G}_{\mathcal{L}}$ *factors through* $K_{\mathcal{L}}/{}_2E \cong {}_{d_0}E$. Indeed, it suffices to prove that for any $\alpha, \beta \in K_{\mathcal{L}}(T)$ (for T an S-scheme) such that $2\beta = 0$, we have $\sigma_1(\alpha+\beta) = \sigma_1(\alpha)$. But by Lemma 1.5, we have $\sigma_1(\alpha+\beta) = \sigma_1(\alpha)\cdot\sigma_1(\beta)$ (since $[\alpha,\beta]^2 = [\alpha, 2\beta] = [\alpha, 0] = 1$). Thus, it suffices to show that $\sigma_1(\beta) = 1$. But this follows from the fact that since the line bundle $\mathcal{L}$ is *totally symmetric* (cf. [Mumf1], §2, Propositions 3, 6; Corollary 2), after finite flat localization, we may choose the lift $\widetilde{\beta} \in \mathcal{G}_{\mathcal{L}}(T)$ of $\beta \in K_{\mathcal{L}}(T)$ to lie inside $\mathcal{S}_{\mathcal{L}}(T) \subseteq \mathcal{G}_{\mathcal{L}}(T)$ and (at the same time) satisfy $\widetilde{\beta}^2 = 1$. Thus, $\sigma_1(\beta) = \widetilde{\beta}\cdot\delta_{\mathcal{L}}(\widetilde{\beta})^{-1} = \widetilde{\beta}^2 = 1$, as desired. This completes the proof of the claim.

Note that we have:

$$2\cdot K_{\mathcal{L}} \stackrel{\text{def}}{=} {}_{d_0}E \subseteq {}_dE = K_{\mathcal{L}}$$

In particular, we obtain a morphism

$$\sigma : 2\cdot K_{\mathcal{L}} \to \mathcal{G}_{\mathcal{L}}$$

which factors through $\mathcal{S}_{\mathcal{L}} \subseteq \mathcal{G}_{\mathcal{L}}$, and whose composite with the projection $\mathcal{G}_{\mathcal{L}} \to K_{\mathcal{L}}$ is the natural injection $2\cdot K_{\mathcal{L}} \subseteq K_{\mathcal{L}}$. That is to say, in the even case, we obtain a section of $\mathcal{G}_{\mathcal{L}} \to K_{\mathcal{L}}$ *over the subgroup scheme* $2\cdot K_{\mathcal{L}} \subseteq K_{\mathcal{L}}$. This completes our discussion of the even case.

Observe that in both the even and odd cases, the section σ over $2\cdot K_{\mathcal{L}}$ that we constructed satisfies the following property: If $\alpha = 2\alpha_1, \beta = 2\beta_1 \in 2\cdot K_{\mathcal{L}}(T)$ (where T is an S-scheme), then

$$\sigma(\alpha+\beta)\cdot\sigma(\alpha)^{-1}\cdot\sigma(\beta)^{-1} = [\alpha_1, \beta_1]^2$$

(where we note that since $[K_{\mathcal{L}}, {}_2E]^2 = [K_{\mathcal{L}}, 0] = 1$, it follows (a priori!) that $[\alpha_1, \beta_1]^2$ depends only on α, β). Indeed, this relation follows immediately from Lemma 1.5. Let us observe that although the proof of Lemma 1.5 depends on the explicit definition of σ in terms of $\delta_{\mathcal{L}}$, the *square* of the above relation, i.e.,

$$\{\sigma(\alpha+\beta)\cdot\sigma(\alpha)^{-1}\cdot\sigma(\beta)^{-1}\}^2 = [\alpha, \beta]$$

may be proven *solely from the assumption that* σ *maps into* $\mathcal{S}_{\mathcal{L}} \subseteq \mathcal{G}_{\mathcal{L}}$: Indeed, let us denote $\sigma(\alpha+\beta)\cdot\sigma(\alpha)^{-1}\cdot\sigma(\beta)^{-1} \in \mathbf{G}_{\mathrm{m}}(T)$ by λ. Then since $\sigma(\alpha+\beta) \in \mathcal{S}_{\mathcal{L}}(T)$, we obtain $\delta_{\mathcal{L}}(\sigma(\beta))\cdot\delta_{\mathcal{L}}(\sigma(\alpha))\cdot\lambda = \lambda^{-1}\cdot\sigma(\alpha)^{-1}\cdot\sigma(\beta)^{-1}$. If we combine this with $\delta_{\mathcal{L}}(\sigma(\alpha)) = \sigma(\alpha)^{-1}$, $\delta_{\mathcal{L}}(\sigma(\beta)) = \sigma(\beta)^{-1}$, then the result follows immediately.

Next, note that if $\alpha \in E(S)$, then transport of structure defines a natural isomorphism

$$\mathcal{G}_\alpha : \mathcal{G}_{\mathcal{T}_\alpha^*\mathcal{L}} \cong \mathcal{G}_{\mathcal{L}}$$

which is compatible with the injections of $\mathbf{G}_{\mathrm{m}}$, and projections to $K_{\mathcal{L}} = K_{\mathcal{T}_\alpha^* \mathcal{L}}$ on both sides. Thus, by conjugating by $\mathcal{G}_\alpha$, we see that $\sigma : 2 \cdot K_{\mathcal{L}} \to \mathcal{G}_{\mathcal{L}}$ defines a section

$$\sigma_\alpha : 2 \cdot K_{\mathcal{T}_\alpha^* \mathcal{L}} \to \mathcal{G}_{\mathcal{T}_\alpha^* \mathcal{L}}$$

In summary, we have proven the following result (which is implicit in the theory of [Mumf1], §2):

Theorem 1.6. *Let $E \to S$ be an elliptic curve over a scheme S. Let d be a positive integer. Let $\mathcal{M}$ be a line bundle on S. Write*

$$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(d \cdot [e]) \otimes_{\mathcal{O}_S} \mathcal{M}$$

Then:

(1) *There is a* canonical section

$$\sigma : 2 \cdot K_{\mathcal{L}} \to \mathcal{G}_{\mathcal{L}}$$

of the projection $\mathcal{G}_{\mathcal{L}} \to K_{\mathcal{L}}$ over $2 \cdot K_{\mathcal{L}} \subseteq K_{\mathcal{L}}$ that maps into $\mathcal{S}_{\mathcal{L}} \subseteq \mathcal{G}_{\mathcal{L}}$, is functorial in S, and whose restriction to $[-,-]$-trivial subgroup schemes of $2 \cdot K_{\mathcal{L}}$ is a group homomorphism. Moreover, these properties uniquely determine σ over $4 \cdot K_{\mathcal{L}}$.

(2) *The section σ is related to the pairing $[-,-] : K_{\mathcal{L}} \times K_{\mathcal{L}} \to (\mathbf{G}_{\mathrm{m}})_S$ by the following formula: If $\alpha = 2\alpha_1, \beta = 2\beta_1 \in 2 \cdot K_{\mathcal{L}}(T)$ (where T is an S-scheme), then*

$$\sigma(\alpha + \beta) \cdot \sigma(\alpha)^{-1} \cdot \sigma(\beta)^{-1} = [\alpha_1, \beta_1]^2$$

In particular, we have: $\{\sigma(\alpha + \beta) \cdot \sigma(\alpha)^{-1} \cdot \sigma(\beta)^{-1}\}^2 = [\alpha, \beta]$.

(3) *Finally, if $\alpha \in E(S)$, then σ defines a section*

$$\sigma_\alpha : 2 \cdot K_{\mathcal{T}_\alpha^* \mathcal{L}} \to \mathcal{G}_{\mathcal{T}_\alpha^* \mathcal{L}}$$

by transport of structure via the translation morphism $\mathcal{T}_\alpha : E \to E$.

Now suppose that d is *odd* (respectively, *even*). Then $K_{\mathcal{L}}^0 \stackrel{\text{def}}{=} K_{\mathcal{L}}/2 \cdot K_{\mathcal{L}}$ is $= S$ (respectively, $\cong {}_2E$). If one thinks of σ as a $2 \cdot K_{\mathcal{L}}$-valued point of $\mathcal{G}_{\mathcal{L}}$, then it follows from the definition of $\mathcal{G}_{\mathcal{L}}$ that if $\alpha \in E(S)$ is any S-valued point, then σ induces an *isomorphism*

$$\mathcal{L}|_{\mathcal{T}_\alpha^* K_{\mathcal{L}}} \cong (\mathcal{L}|_{\alpha^0}) \otimes_{\mathcal{O}_{K^0_{\mathcal{L}}}} \mathcal{O}_{K_{\mathcal{L}}}$$

for some line bundle $\mathcal{L}|_{\alpha^0}$ on the quotient $K^0_{\mathcal{L}}$ of $K_{\mathcal{L}}$. Thus, put another way, the canonical section σ determines natural *descent data* for $\mathcal{L}|_{\mathcal{T}_\alpha^* K_{\mathcal{L}}}$ from $K_{\mathcal{L}}$ to $K^0_{\mathcal{L}}$. (Thus, if d is odd, then we may think of this isomorphism as a *trivialization* of the restriction of the line bundle $\mathcal{L}$ to the subscheme $\mathcal{T}_\alpha^* K_{\mathcal{L}}$.) In particular, *restriction to* $\mathcal{T}_\alpha^* K_{\mathcal{L}}$ defines (by composing with the above isomorphism) a morphism of $\mathcal{O}_{K^0_{\mathcal{L}}}$-modules with natural $\mathcal{G}_{\mathcal{L}}$-action:

$$f_* \mathcal{L} \to (\mathcal{L}|_{\alpha^0}) \otimes_{\mathcal{O}_{K^0_{\mathcal{L}}}} \mathcal{O}_{K_{\mathcal{L}}}$$

In other words, this morphism allows us to think of global sections of $\mathcal{L}$ over E as being (essentially) *functions* on the subscheme $\mathcal{T}_\alpha^* K_{\mathcal{L}}$. These functions are the *algebraic theta functions* of [Mumf1,2,3].

§2. Theta Actions and the Schottky Uniformization

In this §, we discuss the relationship between *torsion actions on line bundles* (i.e., as in §1) and the *uniformization theory* reviewed in Chapter III, §5. Thus, in this §, we will use the same notation as in Chapter III, §5. We begin by reviewing this notation. First of all, $\mathcal{O}$ is a Zariski localization of the ring of integers of a finite extension of $\mathbf{Q}$;

$$A \stackrel{\text{def}}{=} \mathcal{O}[[q]]; \quad S \stackrel{\text{def}}{=} \text{Spec}(A); \quad \widehat{S} \stackrel{\text{def}}{=} \text{Spf}(A)$$

(where q is an indeterminate, and we regard A as equipped with the q-adic topology). Over S, we had a *natural elliptic curve* $E \to S$, together with a *compactification* $C \to S$ whose pull-back $C_{\widehat{S}} \to \widehat{S}$ to $\widehat{S}$ may be obtained as a *quotient of an object* $C^\infty_{\widehat{S}}$ *with respect to the natural action of the group* $\mathbf{Z}_{\text{et}}$ on $C^\infty_{\widehat{S}}$. Moreover, we have a natural identification $E_{\widehat{S}} = (\mathbf{G}_{\text{m}})_{\widehat{S}}$. The *special fiber* $(C^\infty_{\widehat{S}})_{\text{spl}}$ of $C^\infty_{\widehat{S}}$ (i.e., the zero locus of the function q) is a *chain of* $\mathbf{P}^1$*'s* over $\text{Spec}(\mathcal{O})$ indexed by $\mathbf{Z}$ and permuted by the action of $\mathbf{Z}_{\text{et}}$ in a fashion which is compatible with the indexing by $\mathbf{Z}$ and the natural action of $\mathbf{Z}_{\text{et}} = \mathbf{Z}$ on $\mathbf{Z}$ (by addition). Recall that we denoted the origin of E by e. This origin, regarded as a divisor in E or C, gives rise to line bundles $\mathcal{L}_E \stackrel{\text{def}}{=} \mathcal{O}_E(e)$, $\mathcal{L}_C \stackrel{\text{def}}{=} \mathcal{O}_C(e)$. The inverse image of e in $C^\infty_{\widehat{S}}$ (via the quotient map $C^\infty_{\widehat{S}} \to C_{\widehat{S}}$) is denoted by $\widetilde{e} \subseteq C^\infty_{\widehat{S}}$. In Chapter III, §5, we also discussed how

$$\Gamma(C^\infty_{\widehat{S}}, \mathcal{L}^{\otimes 2i}_{C^\infty_{\widehat{S}}})$$

(where $\mathcal{L}_{C^{\infty}_{\widehat{S}}} \stackrel{\text{def}}{=} \mathcal{L}_C|_{C^{\infty}_{\widehat{S}}}$) may be described explicitly as *the degree i portion of a certain graded ring* $\mathcal{R}'_E$. This explicit description will play a key role in this §, and, indeed, in the proof of the main results of this paper.

In the following, we fix a *positive even number* $n = 2m$. Then let us recall (cf. [Mumf5], p. 289) that there is a *natural action* of $(\mathbf{G}_{\mathrm{m}})_{\widehat{S}} = E_{\widehat{S}}$ on $C^{\infty}_{\widehat{S}}$ and $\mathcal{L}^{\otimes 2}_{C^{\infty}_{\widehat{S}}}$. In particular, we obtain a natural action of $(\boldsymbol{\mu}_n)_{\widehat{S}}$ (i.e., the kernel of multiplication by n on the abelian group object $(\mathbf{G}_{\mathrm{m}})_{\widehat{S}}$) on $C^{\infty}_{\widehat{S}}$ and $\mathcal{L}_{C^{\infty}_{\widehat{S}}}$. On the other hand, one also has a natural action of $\mathbf{Z}_{\mathrm{et}}$ in $C^{\infty}_{\widehat{S}}$ and $\mathcal{L}^{\otimes 2}_{C^{\infty}_{\widehat{S}}}$. Moreover, it is easy to see from the definitions of these actions of $\mathbf{Z}_{\mathrm{et}}$ and $(\boldsymbol{\mu}_n)_{\widehat{S}}$ (i.e., $(\boldsymbol{\mu}_n)_{\widehat{S}}$ acts *trivially* on θ – cf. [Mumf5], p. 289; the action of $\mathbf{Z}_{\mathrm{et}}$, namely, $1_{\mathrm{et}}(\theta) = q \cdot U^2 \cdot \theta$, is discussed in Chapter III, §5) that they *commute* on $C^{\infty}_{\widehat{S}}$ and $\mathcal{L}^{\otimes n}_{C^{\infty}_{\widehat{S}}}$. Thus, it follows that *we may form the quotients*

$$\widetilde{C}^{\infty}_{\widehat{S}} \stackrel{\text{def}}{=} C^{\infty}_{\widehat{S}}/\boldsymbol{\mu}_n; \quad \widetilde{C}_{\widehat{S}} \stackrel{\text{def}}{=} C_{\widehat{S}}/\boldsymbol{\mu}_n; \quad \widetilde{E}_{\widehat{S}} \stackrel{\text{def}}{=} E_{\widehat{S}}/\boldsymbol{\mu}_n; \quad \widetilde{\mathcal{M}}_{\widehat{S}} \stackrel{\text{def}}{=} \mathcal{L}^{\otimes n}_{C_{\widehat{S}}}/\boldsymbol{\mu}_n$$

of $C^{\infty}_{\widehat{S}}$, $C_{\widehat{S}}$, $E_{\widehat{S}}$, *and* $\mathcal{L}^{\otimes n}_{C_{\widehat{S}}}$, *respectively, by* $(\boldsymbol{\mu}_n)_{\widehat{S}}$. Indeed, in the case of $C^{\infty}_{\widehat{S}}$, if we think of $C^{\infty}_{\widehat{S}}$ as the q-adic completion of a sort of *Néron model* of $\mathbf{G}_{\mathrm{m}}$ over $A[q^{-1}]$ (cf. the discussion in Chapter III, §5), then the quotient $C^{\infty}_{\widehat{S}} \to \widetilde{C}^{\infty}_{\widehat{S}}$ is just the quotient induced on completions of Néron models by the morphism $\mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ given by multiplication by n. Finally, note that $\widetilde{\mathcal{M}}_{\widehat{S}}$ defines a line bundle of degree 1 on $\widetilde{C}_{\widehat{S}}$ whose pull-back to $C_{\widehat{S}}$ is $\mathcal{L}^{\otimes n}_{C_{\widehat{S}}}$.

Next, let us write

$$C^{[n]}_{\widehat{S}} \stackrel{\text{def}}{=} C^{\infty}_{\widehat{S}}/(n \cdot \mathbf{Z}_{\mathrm{et}})$$

Thus, we obtain covering morphisms

$$C^{\infty}_{\widehat{S}} \to C^{[n]}_{\widehat{S}} \to C_{\widehat{S}} \to \widetilde{C}_{\widehat{S}}$$

where the first two morphisms are étale with Galois groups $n \cdot \mathbf{Z}_{\mathrm{et}}$ and $\mathbf{Z}_{\mathrm{et}}/n \stackrel{\text{def}}{=} \mathbf{Z}_{\mathrm{et}}/n \cdot \mathbf{Z}_{\mathrm{et}}$, respectively, and the last morphism is the quotient by the action of $(\boldsymbol{\mu}_n)_{\widehat{S}}$. Moreover, the last two morphisms may be *algebraized* (along with the line bundle $\widetilde{\mathcal{M}}_{\widehat{S}}$ on $\widetilde{C}_{\widehat{S}}$) into morphisms

$$C^{[n]} \to C \to \widetilde{C}$$

(and a line bundle $\widetilde{\mathcal{M}}$ on $\widetilde{C}$) which compactify *isogenies of smooth group schemes*

$$E^{[n]} \stackrel{\mathbf{Z}_{\mathrm{et}}/n}{\longrightarrow} E \stackrel{\boldsymbol{\mu}_n}{\longrightarrow} \widetilde{E}$$

over S. Here, the groups over the arrows denote the kernels of the respective arrows, or, alternatively, the finite group schemes with respect to which the arrows are torsors. The group schemes E and $\widetilde{E}$ have connected special fibers, while the group of connected components of the special fiber of $E^{[n]}$ may be identified with $\mathbf{Z}_{\mathrm{et}}/n$. Moreover,

$$\Pi_n \stackrel{\mathrm{def}}{=} (\mathbf{Z}_{\mathrm{et}}/n) \times \boldsymbol{\mu}_n$$

acts on $C^{[n]}$ and $E^{[n]}$, and $E^{[n]} \to \widetilde{E}$ is naturally a Π_n-torsor.

Next, observe that the *group of line bundles on* $\widetilde{E}$ *whose pull-back to* $E^{[n]}$ *is trivial* may be identified with the group of characters

$$\mathrm{Hom}(\Pi_n, (\mathbf{G}_{\mathrm{m}})_S) = \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$$

(where the "Hom" is with respect to group schemes over S, or equivalently in this case, over $\mathrm{Spec}(A[q^{-1}])$). Indeed, this correspondence may be given explicitly as follows. Given a *character*

$$\chi \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$$

one obtains an action of Π_n on the trivial $\mathcal{O}_{E^{[n]}}$-module $\mathcal{O}_{E^{[n]}}$ that covers the natural action of Π_n on $E^{[n]}$ by multiplying the trivial action of Π_n on $\mathcal{O}_{E^{[n]}}$ by χ. Let us denote $\mathcal{O}_{E^{[n]}}$ equipped with this action by $\mathcal{O}^{\chi}_{E^{[n]}}$. Then taking Π_n-invariants of $\mathcal{O}^{\chi}_{E^{[n]}}$ gives us (since $E^{[n]} \to \widetilde{E}$ is a Π_n-torsor) a line bundle $\mathcal{O}^{\chi}_{\widetilde{E}}$ on $\widetilde{E}$ whose restriction to $E^{[n]}$ is equal to $\mathcal{O}^{\chi}_{E^{[n]}}$ and whose n-th power is trivial. Put another way, $\mathcal{O}^{\chi}_{\widetilde{E}}$ may be thought of as the *sheaf of functions* $f(\epsilon)$ on $E^{[n]}$ which satisfy the *transformation rule*:

$$f(\alpha \cdot \epsilon) = \chi(\alpha)^{-1} \cdot f(\epsilon)$$

(for all points α (valued in some scheme) of Π_n). This correspondence

$$\chi \mapsto \mathcal{O}^{\chi}_{\widetilde{E}}$$

induces the natural bijection between characters and line bundles referred to above. Also, note that although it is not clear whether or not $\mathcal{O}^{\chi}_{\widetilde{E}}$ extends to a line bundle on $\widetilde{C}$, it is clear that the line bundle with Π_n-action $\mathcal{O}^{\chi}_{E^{[n]}}$ on $E^{[n]}$ extends to a line bundle with Π_n-action $\mathcal{O}^{\chi}_{C^{[n]}}$ on $C^{[n]}$.

Next, let us observe that the line bundle

$$\widetilde{\mathcal{L}}_{\widetilde{C}} \stackrel{\mathrm{def}}{=} \mathcal{O}_{\widetilde{C}}(\widetilde{e})$$

(where $\widetilde{e}$ is the identity of $\widetilde{E}$) satisfies

$$\widetilde{\mathcal{L}}_{\widetilde{C}}|_{C^{[n]}} = \mathcal{O}_{C^{[n]}}([\Pi_n]) \cong \mathcal{O}_{C^{[n]}}(n \cdot [\mathbf{Z}_{\mathrm{et}}/n]) = \mathcal{L}_C^{\otimes n}|_{C^{[n]}} = \widetilde{\mathcal{M}}|_{C^{[n]}}$$

(where $[\Pi_n]$ and $[\mathbf{Z}_{\mathrm{et}}/n]$ denote the divisors in $C^{[n]}$ defined by the respective subgroup schemes of $E^{[n]}$). Here, the isomorphism "$\cong$" in the middle follows from the fact that, since the image of $\boldsymbol{\mu}_n$ forms a subgroup scheme in $C^{[n]}$ annihilated by n, and $\mathcal{O}_{C^{[n]}}([\mathbf{Z}_{\mathrm{et}}/n])$ is a line bundle of degree n on $C^{[n]}$, we obtain that translation by $\boldsymbol{\mu}_n$ fixes the isomorphism class of $\mathcal{O}_{C^{[n]}}([\mathbf{Z}_{\mathrm{et}}/n])$, hence that $\mathcal{O}_{C^{[n]}}([\Pi_n]) \cong \mathcal{O}_{C^{[n]}}(n \cdot [\mathbf{Z}_{\mathrm{et}}/n])$, as desired. Thus, it follows from the discussion above concerning characters that there exists a *unique* $\chi \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$ such that

$$\widetilde{\mathcal{L}}_{\widetilde{E}} = \mathcal{O}^{\chi}_{\widetilde{E}} \otimes_{\mathcal{O}_{\widetilde{E}}} (\widetilde{\mathcal{M}}|_{\widetilde{E}})$$

We are now ready to state the main result of this §:

Theorem 2.1. *The character $\chi \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$ such that*

$$\widetilde{\mathcal{L}}_{\widetilde{E}} = \mathcal{O}^{\chi}_{\widetilde{E}} \otimes_{\mathcal{O}_{\widetilde{E}}} (\widetilde{\mathcal{M}}|_{\widetilde{E}})$$

(is the unique character which) satisfies: $\chi(1_{\mathrm{et}}) = -1$; $\chi|_{\boldsymbol{\mu}_n} : \boldsymbol{\mu}_n \to \boldsymbol{\mu}_n$ is the character given by raising to the m-th power.

Proof. To simplify notation, write $\psi \stackrel{\mathrm{def}}{=} \theta^m$. Recall (cf. [Mumf5], especially p. 289) that the action of $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ on ψ which gives rise to the line bundle $\widetilde{\mathcal{M}}$ is the *trivial* action. In the following, we shall wish to take objects which have some action of $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ defined in [Mumf5], and *twist* this action by the character χ described in the statement of Theorem 2.1. We shall denote the resulting objects with twisted action by means of a superscript or subscript χ (whichever is more convenient), and write, for instance,

$$[-1](\psi) = \psi; \quad [-1](\psi_\chi) = -\psi_\chi$$

(where $[-1]$ is a generator of $\boldsymbol{\mu}_n/\boldsymbol{\mu}_m = \boldsymbol{\mu}_2$). Since $\widetilde{\mathcal{L}}$ has a unique (up to constant multiples) nonzero global section whose zero locus is precisely $\widetilde{e}$, it suffices to check that the set of $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$-invariant sections of $(\mathcal{L}^{\otimes m}_{C^{\infty}_{\widehat{S}}})_\chi$ on $(C^{\infty}_{\widehat{S}})_{\mathrm{spl}}$ has a generator whose zero locus on the

$\mathbf{G}_{\mathrm{m}}$ inside $(C^{\infty}_{\widehat{S}})_{\mathrm{spl}}$ which contains the identity is precisely the zero locus of $U^n - 1$ (i.e., the inverse image of $\widetilde{e} \subseteq \widetilde{E}$).

To show this, recall from Chapter III, §5, that the image of $\Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}^{\otimes m}_{C^{\infty}_{\widehat{S}}})$ under restriction to this $\mathbf{G}_{\mathrm{m}}$ is the $\mathcal{O}$-module generated by

$$U^m \cdot \psi_\chi,\ U^{m-1} \cdot \psi_\chi,\ \ldots,\ U \cdot \psi_\chi,\ \psi_\chi,\ U^{-1} \cdot \psi_\chi,\ \ldots,\ U^{-m+1} \cdot \psi_\chi,\ U^{-m} \cdot \psi_\chi$$

Since the only $\boldsymbol{\mu}_m$-invariant (where $\boldsymbol{\mu}_m \subseteq \boldsymbol{\mu}_n$) U^i are those for which i is divisible by m, and, moreover, $[-1](\psi_\chi) = -\psi_\chi \neq \psi_\chi$ (where $[-1]$ is a generator of $\boldsymbol{\mu}_n/\boldsymbol{\mu}_m = \boldsymbol{\mu}_2$), it follows that the $\boldsymbol{\mu}_n$-invariant subspace of this image is given by the $\mathcal{O}$-module generated by the two elements:

$$U^m \cdot \psi_\chi, U^{-m} \cdot \psi_\chi$$

Moreover, we have

$$1_{\mathrm{et}}(U^{-m} \cdot \psi_\chi) = \chi(1_{\mathrm{et}}) \cdot (q \cdot U)^{-m} \cdot (q \cdot U^2)^m \cdot \psi_\chi = \chi(1_{\mathrm{et}}) \cdot U^m \cdot \psi_\chi = -U^m \cdot \psi_\chi$$

Thus, the $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$-invariant subspace is generated by

$$(U^m - U^{-m}) \cdot \psi_\chi$$

whose zero locus is the same as that of $U^n - 1$, as desired. ◯

Before concluding this §, we make some final remarks which will be helpful for the computations that we will perform later. First, observe that it is a consequence of the discussion at the beginning of Chapter III, §5, that $\Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}^{\otimes n}_{C^{\infty}_{\widehat{S}}})$ is topologically A-generated by the sections

$$U^m \cdot \theta^m,\ U^{m-1} \cdot \theta^m,\ \ldots,\ U \cdot \theta^m,\ \theta^m,\ U^{-1} \cdot \theta^m,\ \ldots,\ U^{-m+1} \cdot \theta^m,\ U^{-m} \cdot \theta^m$$

and their $\mathbf{Z}_{\mathrm{et}}$-translates. Let us compute these $\mathbf{Z}_{\mathrm{et}}$-translates. Since $1_{\mathrm{et}}(U) = q \cdot U$, $1_{\mathrm{et}}(\theta) = q \cdot U^2 \cdot \theta$, we obtain, for $k, i \in \mathbf{Z}$, $|i| \leq m$:

$$k_{\mathrm{et}}(U^i) = q^{ik} \cdot U^i; \qquad k_{\mathrm{et}}(\theta^m) = q^{m \cdot k^2} \cdot U^{2mk} \cdot \theta^m$$

Thus, we obtain the following:

Proposition 2.2. *The A-module $\Gamma(C^\infty_{\widehat{S}}, \mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}})$ is topologically A-generated by the sections*

$$k_{\mathrm{et}}(U^i \cdot \theta^m) = q^{m \cdot k^2 + ik} \cdot U^{2mk+i} \cdot \theta^m$$

where $k, i \in \mathbf{Z}$, $|i| \leq m$.

(For instance, the case $m = 1$ of Proposition 2.2 is essentially stated in the discussion at the beginning of Chapter III, §5.)

We will ultimately use Proposition 2.2 to compute the image in $E_{\widehat{S}} = (\mathbf{G}_{\mathrm{m}})_{\widehat{S}}$ of sections of $\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}} \otimes_{\mathcal{O}_{C^\infty_{\widehat{S}}}} \mathcal{O}^{\chi}_{C^\infty_{\widehat{S}}}$ for various characters $\chi \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$.

§3. Twisted Schottky-Weierstrass Zeta Functions

In this §, we generalize the discussion of *Schottky-Weierstrass zeta functions* of Chapter III, §6, 7, to the *"twisted" context* of §2. We maintain the notation of §2.

First, let us recall the *isogeny*

$$E \to \widetilde{E}$$

discussed in §2. This kernel of this isogeny is $\boldsymbol{\mu}_n$. Moreover, this isogeny may be compactified to a morphism $C \to \widetilde{C}$. Recall (Chapter III, §1) that the *universal extension* $E^\dagger$ of E is defined by considering certain line bundles with connection on C. Given such a line bundle with connection on C, by taking the *norm* of this line bundle (relative to the finite morphism $C \to \widetilde{C}$), we obtain a line bundle on $\widetilde{C}$; moreover, taking the *trace* of the connection on the line bundle on C, we get a connection on the norm on this line bundle on $\widetilde{C}$. Thus, by performing these operations, we obtain a *push-forward homomorphism*

$$E^\dagger \to \widetilde{E}^\dagger$$

which covers (since taking the norm of the line bundle defined by a divisor on E is the same as taking the line bundle associated to the image of this divisor in $\widetilde{E}$) the given isogeny $E \to \widetilde{E}$. Moreover, since taking the trace of differentials on $E_{\widehat{S}} = (\mathbf{G}_{\mathrm{m}})_{\widehat{S}}$ via what amounts to the multiplication by n map $(\mathbf{G}_{\mathrm{m}})_{\widehat{S}} = E_{\widehat{S}} \to (\mathbf{G}_{\mathrm{m}})_{\widehat{S}} = \widetilde{E}_{\widehat{S}}$ on $(\mathbf{G}_{\mathrm{m}})_{\widehat{S}}$ induces an isomorphism of the invariant differentials on $E_{\widehat{S}}$ onto the invariant differentials on $\widetilde{E}_{\widehat{S}}$ (i.e., the trace of dU/U is $n \cdot dU/U = d(U^n)/(U^n)$), it follows that the push-forward map defined above maps $W_E \subseteq E^\dagger$ (i.e., W_E is what we called "W" when E was the only elliptic curve under discussion) *isomorphically* onto $W_{\widetilde{E}} \subseteq \widetilde{E}^\dagger$. That is to say, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_E & \longrightarrow & E^\dagger & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_{\widetilde{E}} & \longrightarrow & \widetilde{E}^\dagger & \longrightarrow & \widetilde{E} & \longrightarrow & 0
\end{array}$$

where the vertical arrow on the left is an isomorphism, and

$$\mathrm{Ker}(E^\dagger \to \widetilde{E}^\dagger) \cong \mathrm{Ker}(E \to \widetilde{E}) \cong \boldsymbol{\mu}_n$$

Next, we would like to consider the *canonical splittings* $\kappa : E_{\widehat{S}} \to E^\dagger_{\widehat{S}}$, $\widetilde{\kappa} : \widetilde{E}_{\widehat{S}} \to \widetilde{E}^\dagger_{\widehat{S}}$ of Chapter III, §2, Theorem 2.1. Now I *claim* that

$$\kappa(\mathrm{Ker}(E \to \widetilde{E})) \subseteq \mathrm{Ker}(E^\dagger \to \widetilde{E}^\dagger)$$

Indeed, since κ is a *homomorphism*, it follows that the image

$$\mathrm{Im}(\kappa(\mathrm{Ker}(E \to \widetilde{E}))) \subseteq \widetilde{E}^\dagger_{\widehat{S}}$$

is annihilated by n and vanishes when projected to $\widetilde{E}_{\widehat{S}}$, hence is contained in $(W_{\widetilde{E}})_{\widehat{S}}$. On the other hand, the image in $(W_{\widetilde{E}})_{\widehat{S}} \cong \mathbf{A}^1_{\widehat{S}}$ of any A-flat group scheme annihilated by n is necessarily trivial. Indeed, this may be checked (by A-flatness) in characteristic zero, where it is obvious. This completes the proof of the claim. Note that the claim then implies that $\kappa : E_{\widehat{S}} \to E^\dagger_{\widehat{S}}$ descends to a homomorphism $\widetilde{E}_{\widehat{S}} \to \widetilde{E}^\dagger_{\widehat{S}}$ which splits the projection $\widetilde{E}^\dagger_{\widehat{S}} \to \widetilde{E}_{\widehat{S}}$. Since $\widetilde{\kappa}$ is the *unique* such homomorphism (cf. Chapter III, §2), it thus follows that we have a commutative diagram:

$$\begin{array}{ccc}
E_{\widehat{S}} & \xrightarrow{\kappa} & E^\dagger_{\widehat{S}} \\
\downarrow & & \downarrow \\
\widetilde{E}_{\widehat{S}} & \xrightarrow{\widetilde{\kappa}} & \widetilde{E}^\dagger_{\widehat{S}}
\end{array}$$

Next, we consider *line bundles*. For convenience, *we assume that $\mathcal{O}$ has enough primes inverted so that*

$$\mathrm{Pic}(\mathcal{O}) = \{1\}$$

(Here, we use the well-known fact from elementary algebraic number theory that "Pic" of the ring of integers of a number field is a *finite group*, i.e., "the finiteness of the class group.") Note that since $\mathrm{Pic}(\mathcal{O})$ is trivial, it follows immediately that $\mathrm{Pic}(A)$ is also trivial. Now recall the line bundle

$$\widetilde{\mathcal{M}}$$

on $\widetilde{C}$. If $\chi \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$, then we shall write

$$\widetilde{\mathcal{M}}^{\chi}_{\widetilde{E}} \stackrel{\mathrm{def}}{=} \mathcal{O}^{\chi}_{\widetilde{E}} \otimes_{\mathcal{O}_{\widetilde{E}}} (\widetilde{\mathcal{M}}|_{\widetilde{E}})$$

Note that since C is *regular*, and $C - E$ is of codimension 2 in C, it follows that the line bundle $\widetilde{\mathcal{M}}^{\chi}_{\widetilde{E}}|_E$ extends (uniquely) to a line bundle $\widetilde{\mathcal{M}}^{\chi}_C$ on C. Moreover, since $E \rightarrow \widetilde{E}$ is a $\boldsymbol{\mu}_n$-torsor, we obtain an action of $\boldsymbol{\mu}_n$ on $\widetilde{\mathcal{M}}^{\chi}_C$. Since $\boldsymbol{\mu}_n$ is of multiplicative type, hence *reductive* (i.e., its representations split up into direct sums of representations by characters), we thus obtain that the $\boldsymbol{\mu}_n$*-invariant subspace*

$$\Gamma(C, \widetilde{\mathcal{M}}^{\chi}_C)^{\boldsymbol{\mu}_n} \subseteq \Gamma(C, \widetilde{\mathcal{M}}^{\chi}_C)$$

is a direct summand of the A-module $\Gamma(C, \widetilde{\mathcal{M}}^{\chi}_C)$ (which, by Riemann-Roch, is a projective A-module of rank n). Since, after one inverts q, this invariant subspace may be identified with $\Gamma(\widetilde{E}, \widetilde{\mathcal{M}}^{\chi}_{\widetilde{E}}) \otimes_A A[q^{-1}]$ (which, by Riemann-Roch, is a projective $A[q^{-1}]$-module of rank 1), we thus obtain that this invariant subspace $\Gamma(C, \widetilde{\mathcal{M}}^{\chi}_C)^{\boldsymbol{\mu}_n}$ is a projective A-module of rank 1. Since $\mathrm{Pic}(A)$ is trivial, we thus obtain that $\Gamma(C, \widetilde{\mathcal{M}}^{\chi}_C)^{\boldsymbol{\mu}_n}$ is a *free A-module of rank* 1. Thus, we may choose a generator s_χ of this module, i.e.:

$$A \cdot s_\chi = \Gamma(C, \widetilde{\mathcal{M}}^{\chi}_C)^{\boldsymbol{\mu}_n}$$

Thus, if one pulls s_χ back to $C^{\infty}_{\widehat{S}}$, we see that

$$s_\chi|_{C^{\infty}_{\widehat{S}}}$$

may be thought of as a topological A-linear combination

$$\sigma_\chi$$

of the elements discussed in Proposition 2.2 which satisfies, for all points α of $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ (valued in some scheme),

$$\alpha(\sigma_\chi) = \chi^{-1}(\alpha) \cdot \sigma_\chi$$

where the action of α on the left-hand side is via the usual action of $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ on expressions as in Proposition 2.2, and *we regard the character* χ *as a character on* $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ *via the natural surjection* $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n \to (\mathbf{Z}_{\mathrm{et}}/n) \times \boldsymbol{\mu}_n = \Pi_n$.

Next, let us recall the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{T}_C \to \tau_E|_C \to 0$$

of the discussion following Chapter III, Corollary 4.3. The extension class of this exact sequence is precisely the torsor defined by the universal extension $E_C^\dagger \to C$. By the discussion above concerning the relationship between the universal extensions $E^\dagger$ and $\widetilde{E}^\dagger$, it follows that this exact sequence descends naturally to an exact sequence

$$0 \to \mathcal{O}_{\widetilde{C}} \to \widetilde{\mathcal{T}}_{\widetilde{C}} \to \tau_E|_{\widetilde{C}} \to 0$$

Thus, we see that we obtain a natural action of $\boldsymbol{\mu}_n$ on $\Gamma(C, \mathcal{T}_C \otimes_{\mathcal{O}_C} \widetilde{\mathcal{M}}_C^\chi)$. Then just as in the discussion of sections of $\widetilde{\mathcal{M}}^\chi$, we obtain (by applying Riemann-Roch and the reductiveness of $\boldsymbol{\mu}_n$) that the $\boldsymbol{\mu}_n$-invariant subspace

$$\Gamma(C, \mathcal{T}_C \otimes \widetilde{\mathcal{M}}_C^\chi)^{\boldsymbol{\mu}_n} \subseteq \Gamma(C, \mathcal{T}_C \otimes \widetilde{\mathcal{M}}_C^\chi)$$

fits into an exact sequence of A-modules:

$$0 \to \Gamma(C, \widetilde{\mathcal{M}}_C^\chi)^{\boldsymbol{\mu}_n} \to \Gamma(C, \mathcal{T}_C \otimes \widetilde{\mathcal{M}}_C^\chi)^{\boldsymbol{\mu}_n} \to \Gamma(C, \widetilde{\mathcal{M}}_C^\chi)^{\boldsymbol{\mu}_n} \otimes_A (A \cdot U\frac{\partial}{\partial U}) \to 0$$

In particular, we may choose a section

$$\mathcal{S}_\chi \in \Gamma(C, \mathcal{T}_C \otimes \widetilde{\mathcal{M}}_C^\chi)^{\boldsymbol{\mu}_n}$$

whose image in $\Gamma(C, \widetilde{\mathcal{M}}_C^\chi)^{\boldsymbol{\mu}_n} \otimes_A (A \cdot U\frac{\partial}{\partial U})$ is equal to $s_\chi \cdot U\frac{\partial}{\partial U}$. If we pull-back $\mathcal{S}_\chi$ to $C^\infty_{\widehat{S}}$, then we may think of $\mathcal{S}_\chi^\infty$ as a section of $\mathcal{T}_C \otimes \mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}}$ which satisfies

$$\alpha(\mathcal{S}_\chi^\infty) = \chi(\alpha)^{-1} \cdot \mathcal{S}_\chi^\infty$$

for all points α of $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ (valued in some scheme).

On the other hand, κ defines a section

$$\kappa_{\mathcal{T}} \in \Gamma(C^\infty_{\widehat{S}}, \mathcal{T}_C|_{C^\infty_{\widehat{S}}})$$

whose image in $\Gamma(C^{\infty}_{\widehat{S}}, \tau_E|_{C^{\infty}_{\widehat{S}}})$ is $U\frac{\partial}{\partial U}$. Since κ descends to $\widetilde{\kappa}$ (as discussed above), the natural action of $\boldsymbol{\mu}_n$ on $\kappa_{\mathcal{T}}$ is *trivial.* Moreover, by Chapter III, Theorem 5.6, it follows that

$$1_{\mathrm{et}}(\kappa_{\mathcal{T}}) = \kappa_{\mathcal{T}} + 1$$

In particular, the section

$$\sigma_{\chi} \cdot \kappa_{\mathcal{T}} \in \Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}^{\otimes n}_{C^{\infty}_{\widehat{S}}} \otimes_{\mathcal{O}_C} \mathcal{T}_C)$$

satisfies

$$1_{\mathrm{et}}(\sigma_{\chi} \cdot \kappa_{\mathcal{T}}) = \chi(1_{\mathrm{et}})^{-1} \cdot \sigma_{\chi} \cdot (\kappa_{\mathcal{T}} + 1); \quad \alpha(\sigma_{\chi} \cdot \kappa_{\mathcal{T}}) = \chi(\alpha)^{-1} \cdot \sigma_{\chi} \cdot \kappa_{\mathcal{T}}$$

(where α is a point of $\boldsymbol{\mu}_n$ valued in some scheme).

Thus, if we form the difference

$$\zeta^{\chi} \stackrel{\mathrm{def}}{=} \mathcal{S}^{\infty}_{\chi} - \sigma_{\chi} \cdot \kappa_{\mathcal{T}} \in \Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}^{\otimes n}_{C^{\infty}_{\widehat{S}}})$$

then we have:

$$1_{\mathrm{et}}(\zeta^{\chi}) = \chi(1_{\mathrm{et}})^{-1} \cdot (\zeta^{\chi} - \sigma_{\chi}); \quad \alpha(\zeta^{\chi}) = \chi(\alpha)^{-1} \cdot \zeta^{\chi}$$

(where α is a point of $\boldsymbol{\mu}_n$ valued in some scheme). Let us define

$$\delta^{\chi}(f) \stackrel{\mathrm{def}}{=} \{\chi(1_{\mathrm{et}}) \cdot 1_{\mathrm{et}}(f)\} \ - \ f$$

(Thus, formally, $\delta^{\chi}(f \cdot g) = \delta^{\chi}(f) \cdot 1_{\mathrm{et}}(g) + f \cdot \delta(g)$, where $\delta(g) = 1_{\mathrm{et}}(g) - g$ is as in Chapter III, §6.) Then we have

$$\delta^{\chi}(\zeta^{\chi}) = -\sigma_{\chi}$$

Note that if χ is the trivial character, $n = 2$, and $2 \in \mathcal{O}^{\times}$, then ζ^{χ} is simply (in the notation of Chapter III, §5)

$$\text{“}(\zeta + C) \cdot \sigma^2\text{”}$$

for some $C \in A$.

Now we would like to *generalize* ζ^χ *in precisely the way we generalized* ζ *in Chapter III, §6.* To do this, we would like to consider *extension polynomials* – i.e., sections of the sheaf $\mathcal{R}^{\mathrm{et}}_{E_C^\dagger}$ of Chapter III, Proposition 6.1 – but this time (unlike in Chapter III, §6) with *coefficients that are sections of* $\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}}$ *over* $C^\infty_{\widehat{S}}$. In other words, we would like to consider sections of $\mathcal{R}^{\mathrm{et}}_{E_C^\dagger} \otimes_{\mathcal{O}_C} \mathcal{L}^{\otimes n}_C$ over $C^\infty_{\widehat{S}}$. Also, whereas before we considered $\mathbf{Z}_{\mathrm{et}}$-invariant extension polynomials, this time we would like to consider extension polynomials on which $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ acts via the character χ – i.e., *polynomials which are* $(\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n)$*-invariant when considered as sections of* $\mathcal{R}^{\mathrm{et}}_{E_C^\dagger} \otimes_{\mathcal{O}_C} (\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}})^\chi$. Other than these formal changes, however, the proofs proceed just as for Chapter III, §6, Lemma 6.3, Theorem 6.4. The end result is the following:

Theorem 3.1. *Let r be a nonnegative integer, and $\chi \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$. Then there exists a $(\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n)$-invariant extension polynomial*

$$f = \sum_{i=0}^{r} \zeta^\chi_{r-i} \cdot T^{[i]}$$

(where the coefficients lie in $\Gamma(C^\infty_{\widehat{S}}, (\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}})^\chi)$), such that $\zeta^\chi_0 = \sigma_\chi$;

$$\delta^\chi(\zeta^\chi_{r-i}) + \delta^\chi(\zeta^\chi_{r-i-1}) + \zeta^\chi_{r-i-1} = 0$$

(for all i); and all the $\zeta^\chi_j \in \Gamma(C^\infty_{\widehat{S}}, (\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}})^\chi)$ are $\boldsymbol{\mu}_n$-invariant. In particular, we have

$$\delta^\chi(\zeta^\chi_j) = -\zeta^\chi_{j-1} + \zeta^\chi_{j-2} - \ldots + (-1)^{j-1}\zeta^\chi_1 + (-1)^j\zeta^\chi_0$$

(for all j). Finally, if $\widehat{\zeta}^\chi_0, \ldots, \widehat{\zeta}^\chi_r$ satisfy the same conditions as $\zeta^\chi_0, \ldots, \zeta^\chi_r$, then for each $j = 0, \ldots, r$,

$$\zeta^\chi_j - \widehat{\zeta}^\chi_j = \text{ some } A\text{–linear combination of } \zeta^\chi_0, \ldots, \zeta^\chi_{j-1}$$

(where $A = \mathcal{O}[[q]]$).

Remark. Just as in the non-twisted case (cf. Chapter III, §6, Theorem 6.4), we may take $\zeta^\chi_1 = \zeta^\chi$. Also, (just as in Chapter III, §6, Remark 2) everything we did here can also be done in the *complex analytic context.*

Similarly, one may generalize the *canonical Schottky-Weierstrass zeta functions* of Chapter III, §7, to the present twisted context, as follows. First, we assume for the remainder of the § that $\mathcal{O}$ *is a finite extension of* $\mathbf{Q}$. Then we define the operator δ^* on $\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}}$ by

$$f \mapsto \delta^*(f) \stackrel{\mathrm{def}}{=} \frac{1}{n} \cdot \nabla_{(U\frac{\partial}{\partial U})}(f)$$

where ∇ is the connection induced on $\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}} = \mathcal{L}^{\otimes 2m}_{C^\infty_{\widehat{S}}}$ by the connection on $\mathcal{L}^{\otimes 2}_{C^\infty_{\widehat{S}}}$ of Chapter III, Theorem 5.6; and U is the standard multiplicative coordinate on $E_{\widehat{S}}$ (as in §2). One checks easily that the natural action of $\boldsymbol{\mu}_n$ on δ^* is *trivial* and that:

$$[\delta^*, \delta^\chi] = \chi(1_{\mathrm{et}}) \cdot 1_{\mathrm{et}}$$

Also, just as in Chapter III, §7, we may also define a *tautological connection* (on the universal extension), as well as a corresponding differential operator $(\delta^{\mathrm{taut}})^*$. Thus, by the same formal arguments as those used to derive Chapter III, Theorem 7.4, in Chapter III, we obtain the following result:

Theorem 3.2. (Divided Power Twisted Canonical Schottky-Weierstrass Functions) *Let* $\chi \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$, $\zeta_0^{\mathrm{PD},\chi} \stackrel{\mathrm{def}}{=} \sigma_\chi \in \Gamma(C^\infty_{\widehat{S}}, (\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}})^\chi)$. *Write (for* $r \in \mathbf{Z}_{\geq 0}$*)*

$$\zeta_r^{\mathrm{PD},\chi} \stackrel{\mathrm{def}}{=} \frac{1}{r!}(\delta^*)^r \zeta_0^{\mathrm{PD},\chi}$$

(and let $\zeta_r^{\mathrm{PD},\chi} \stackrel{\mathrm{def}}{=} 0$ *if* $r < 0$*). Then* $\delta^*(\zeta_r^{\mathrm{PD},\chi}) = (r+1) \cdot \zeta_{r+1}^{\mathrm{PD},\chi}$, $(\delta^\chi)^r(\zeta_r^{\mathrm{PD},\chi}) = (-1)^r \cdot \zeta_0^{\mathrm{PD},\chi}$ *(if* $r \geq 0$*);*

$$\delta^\chi(\zeta_r^{\mathrm{PD},\chi}) = \textstyle\sum_{i=0}^{r-1}(-1)^{i+r}\frac{1}{(r-i)!}\zeta_i^{\mathrm{PD},\chi} = -\zeta_{r-1}^{\mathrm{PD},\chi} + \frac{1}{2} \cdot \zeta_{r-2}^{\mathrm{PD},\chi} + \ldots + (-1)^r \cdot \frac{1}{r!} \cdot \zeta_0^{\mathrm{PD},\chi}$$

(for all $r \in \mathbf{Z}$*). Moreover, all of the* $\zeta_i^{\mathrm{PD},\chi}$ *(for* $i \in \mathbf{Z}$*) are* $\boldsymbol{\mu}_n$*-invariant, and, in fact, the* A*-submodule of* $\Gamma(C^\infty_{\widehat{S}}, (\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}})^\chi)$ *generated by* $\zeta_0^{\mathrm{PD},\chi}, \ldots, \zeta_r^{\mathrm{PD},\chi}$ *is equal to the* A*-submodule of* $\boldsymbol{\mu}_n$*-invariant sections of* $\Gamma(C^\infty_{\widehat{S}}, (\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}})^\chi)$ *which are annihilated by* $(\delta^\chi)^{r+1}$*. In particular, this submodule is equal to the* A*-submodule of* $\Gamma(C^\infty_{\widehat{S}}, (\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}})^\chi)$ *generated by the functions denoted "*$\zeta_0^{\mathrm{PD},\chi}, \ldots, \zeta_r^{\mathrm{PD},\chi}$*" in Theorem 3.1. Finally, the polynomial*

$$\zeta_r^{\mathrm{PD},\chi}[T] \stackrel{\mathrm{def}}{=} \sum_{i=0}^{r} \zeta_i^{\mathrm{PD},\chi} \cdot \frac{T^{r-i}}{(r-i)!} = \zeta_0^{\mathrm{PD},\chi} \cdot \frac{T^r}{r!} + \zeta_1^{\mathrm{PD},\chi} \cdot \frac{T^{(r-1)}}{(r-1)!} + \ldots + \zeta_{r-1}^{\mathrm{PD},\chi} \cdot T + \zeta_r^{\mathrm{PD},\chi}$$

($\in \Gamma(C^{\infty}_{\widehat{S}}, (\mathcal{L}^{\otimes n}_{C^{\infty}_{\widehat{S}}})^{\chi}[T])$) *is* $(\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n)$*-invariant relative to the natural action of* $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ *on* $C^{\infty}_{\widehat{S}}$, $(\mathcal{L}^{\otimes n}_{C^{\infty}_{\widehat{S}}})^{\chi}$, *and the action of* $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ *on* T *given by* $1_{\mathrm{et}}(T) = T + 1$, $\alpha(T) = T$ ($\forall \alpha \in \boldsymbol{\mu}_n$).

Remark. Just as was the case with Theorem 3.1, Theorem 3.2 also has a *complex analytic version.* We leave the routine details to the reader. Also, just as was the case for Chapter III, Theorem 7.4, it is clear from the formula for $\delta^{\chi}(\zeta_r^{\mathrm{PD},\chi})$ that the denominators that occur are "essential" (i.e., they cannot be eliminated as in the case of Theorem 3.1 simply by "redefining the integral structure").

Finally, just as in Chapter III, §7, we may also define the twisted version of *binomial canonical Schottky-Weierstrass zeta functions* as follows. First, let us observe that there is a unique integer i_{χ} satisfying $-m \leq i_{\chi} < m$ such that the monomials of Propostion 2.2, *when considered as sections of* $(\mathcal{L}^{\otimes n}_{C^{\infty}_{\widehat{S}}})^{\chi}$), are $\boldsymbol{\mu}_n$*-invariant* if and only if the integer i ($\in \{-m, -m+1, \dots, 0, \dots, m-1\}$ – cf. Proposition 2.2) is equal to i_{χ}. Then we have the following result:

Theorem 3.3. (Binomial Twisted Canonical Schottky-Weierstrass Functions) *Let* $\chi \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$, $\zeta_0^{\mathrm{BI},\chi} \stackrel{\mathrm{def}}{=} \sigma_{\chi} \in \Gamma(C^{\infty}_{\widehat{S}}, (\mathcal{L}^{\otimes n}_{C^{\infty}_{\widehat{S}}})^{\chi})$. *Write (for* $r \in \mathbf{Z}_{\geq 0}$*)* $\zeta_r^{\mathrm{BI},\chi} \stackrel{\mathrm{def}}{=} \binom{\delta^* - \frac{i_{\chi}}{n}}{r}(\zeta_0^{\mathrm{BI},\chi})$;

$$\zeta_r^{\mathrm{BI},\chi}[T] \stackrel{\mathrm{def}}{=} \binom{\delta^* + T - \frac{i_{\chi}}{n}}{r}(\zeta_0^{\mathrm{BI},\chi}) = \sum_{j=0}^{r} \zeta_j^{\mathrm{BI},\chi} \cdot T^{[r-j]}$$

Then the $\zeta_n^{\mathrm{BI},\chi}$, $\zeta_n^{\mathrm{BI},\chi}[T]$*'s satisfy the properties of Theorem 3.1 (where we take "*ζ_n^{χ}*" of Theorem 6.4 to be* $\zeta_n^{\mathrm{BI},\chi}$*; "*$f$*" of Theorem 6.4 to be* $\zeta_n^{\mathrm{BI},\chi}[T]$*). In particular, the* $\zeta_n^{\mathrm{BI},\chi}[T]$*'s are* $\mathbf{Z}_{\mathrm{et}}$*-invariant. Finally, the* $\zeta_n^{\mathrm{BI},\chi}[T]$*'s are integral over* $\mathbf{Z}$.

Proof. It remains only to verify integrality over $\mathbf{Z}$. But this follows from observing that, by the definition of i_{χ}, $\zeta_0^{\mathrm{BI},\chi}$ may be thought of as a series in the monomials $k_{\mathrm{et}}(U^{i_{\chi}} \cdot \theta^m)$ (for $k \in \mathbf{Z}$) of Proposition 2.2. Moreover, the operator $\delta^* - \frac{i_{\chi}}{n}$ acts on the monomial

$$k_{\mathrm{et}}(U^{i_{\chi}} \cdot \theta^m) = q^{m \cdot k^2 + i_{\chi} \cdot k} \cdot U^{2mk + i_{\chi}} \cdot \theta^m$$

by multiplication by $\frac{1}{n}(2mk + i_{\chi}) - \frac{i_{\chi}}{n} = k$. Thus, the operator $\binom{\delta^* - \frac{i_{\chi}}{n}}{r}$ acts on this monomial by multiplication by $\binom{k}{n} \in \mathbf{Z}$, as desired. ○

§4. Zhang's Theory of Metrized Line Bundles

In this §, we review the theory of [Zh] in the case of elliptic curves. Roughly speaking, *this theory allows one to compute intersection numbers of vertical divisors in the special fiber of an elliptic curve with multiplicative reduction using the techniques of classical harmonic analysis on a circle.* In fact, here, we will use a slight generalization of Zhang's 1-dimensional theory to the 2-dimensional case, so in the following, we will give precise definitions and statements of basic facts. The proofs, however, will be omitted since they are entirely the same as those of [Zh].

Let R be a *valuation ring* whose valuation group is an ordered submodule of $\mathbf{R}$. We denote its valuation

$$| - |_R : R \to \mathbf{R}$$

and assume that we are given an element $\pi \in R$ such that $|\pi|_R = e^{-1}$, and all positive rational powers of the ideal $\pi \cdot R$ exist (as ideals of R). In other words, we want to think of the copy of $\mathbf{R}$ that contains the valuation group of R as being "$-\mathbf{R} \cdot \log(\pi)$" (where $\log(\pi)$ is to be regarded as a formal symbol). We denote the *quotient field* of R by F.

Let V be a finite dimensional F-vector space. Then we will refer to as a *metric on* V any map

$$| - |_V : V \to \mathbf{R}$$

such that: (i) $|\lambda \cdot v|_V = |\lambda|_R \cdot |v|_V$, for all $\lambda \in R$, $v \in V$; (ii) $|v + w|_V \leq \max(|v|_V, |w|_V)$, for all $v, w \in V$; (iii) the set

$$M_V \stackrel{\text{def}}{=} \{v \in V \mid |v|_V \leq 1\}$$

is *bounded* (i.e., given any basis $v_1, \ldots, v_n$ of V, there exists a $\lambda \in R$ such that $M_V \subseteq \sum_{i=1}^n R \cdot \lambda \cdot v_i$). Thus, M_V forms an *R-submodule* of V which is *bounded*, *saturated* — in the sense that

$$M = \bigcap_{r \in \mathbf{Q}_{>0}} (\pi^{-r} M)$$

— and *generates V over F*. Conversely, given any bounded, saturated R-submodule M of V which generates V over F, one can naturally define a metric $| - |_V$ on V such that $M = M_V$ as follows: For any nonzero rational section $v \in V$, we let

$$|v|_V \stackrel{\text{def}}{=} e^{-r_0}$$

where $r_0 \in \mathbf{R}$ is the supremum of the set of rational numbers r such that the section $\pi^{-r} \cdot v \in M$. In other words, *metrics on V are equivalent to bounded, saturated R-submodules of V that generate V over F*. This makes it clear, for instance, how to define tensor products of metrics. Indeed, the tensor product of $|-|_V$, $|-|_W$ on $V \otimes_F W$ is the metric corresponding to the *saturation* (i.e., the smallest saturated module containing the given module) of the bounded R-submodule $\mathrm{Image}(M_V \otimes_R M_W) \subseteq V \otimes_F W$.

If $\dim_F(V) = 1$, then a metric $|-|_V$ on V is completely determined by its value on any fixed nonzero element $v \in V$. Thus, it follows that the metrics on V naturally form an (additive) *torsor over* $\mathbf{R}$. Moreover, the metrics on the F-vector space F may be naturally identified (by looking at the value of the metric on $1 \in F$) with $\mathbf{R}$ itself.

Next, let us return to the set-up of §2. Recall that we have a one-dimensional semi-abelian variety

$$E \to S$$

equipped with a *compactification* $C \to S$. Here, $S = \mathrm{Spec}(A)$, $A = \mathcal{O}[[q]]$. Now let (for $N \geq 1$ an integer)

$$A_N \stackrel{\mathrm{def}}{=} A[q^{1/N}]; \quad A_\infty \stackrel{\mathrm{def}}{=} \bigcup_{N \geq 1} A_N; \quad S_N \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A_N); \quad S_\infty \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A_\infty)$$

Thus, S_N is a finite, flat S-scheme, and S_∞ is the projective limit of the S_N's. Also, let us write

$$U_S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A[q^{-1}]) \subseteq S$$

and U_{S_N}, U_{S_∞}, etc. for the inverse images in S_N, S_∞, etc. of U_S. Note that $E|_{U_S} \times_{U_S} U_{S_N}$ has a unique *regular semi-stable model*

$$C_N \to S_N$$

over S_N. We denote the complement of the nodes of C_N by

$$E_N \to S_N$$

Thus, the special fiber (i.e., the zero locus of $q^{1/N}$) of E_N is a union of N copies of $(\mathbf{G}_{\mathrm{m}})_{\mathcal{O}}$. Write

$$C_\infty \stackrel{\mathrm{def}}{=} \varprojlim C_N \to S_\infty = \varprojlim S_N$$

for the inverse limit of the $C_N \to S_N$, where N ranges over the *multiplicative* semigroup of positive integers. Thus, note that we have natural open immersions

$$E_N \times_{S_N} S_\infty \hookrightarrow C_\infty$$

If we take the union of these open subschemes of C_∞ over N, we obtain an open subscheme

$$E_\infty \subseteq C_\infty$$

which has the structure of a smooth group scheme over S_∞.

Let us refer to the prime of A_∞ generated by the q^r for all positive rational numbers r as the *central prime* of A_∞. The localization of A_∞ at the central prime is a valuation ring with value group $\mathbf{Q}$. For geometric objects over S_∞, we shall refer to the fiber of such an object over the central prime of S_∞ as the *special, or central, fiber* of the object.

Observe that the connected components of the special fiber of E_N may be naturally identified with

$$\frac{1}{N}\mathbf{Z}/\mathbf{Z}$$

Here, the $\mathbf{Z}$ should be thought of as "$\mathbf{Z} \cdot \log(q)$" (where $\log(q)$ is a formal symbol). Indeed, just as in §2, $C^\infty_{\widehat{S}}$ formed a Galois cover of $C^{[n]}_{\widehat{S}}$ with Galois group $n \cdot \mathbf{Z}_{\text{et}}$ such that the (irreducible) components of the special fiber of $C^{[n]}$ could be naturally identified with $\mathbf{Z}_{\text{et}}/N \cdot \mathbf{Z}_{\text{et}}$, there exists a natural infinite Galois cover of $(C_N)_{\widehat{S}_N}$ with Galois group $\mathbf{Z}$ (which is a blow-up of the pull-back to $\widehat{S}_N$ of the $\mathbf{Z}$-Galois cover $C^\infty_{\widehat{S}} \to C_{\widehat{S}}$) such that the components of the special fiber of C_N may be identified with $\frac{1}{N}\mathbf{Z}/\mathbf{Z}$, as desired. By taking the limit over N, we thus obtain that the components of the special fiber of C_∞ may be identified with

$$\mathbf{Q}/\mathbf{Z} \subseteq \mathbf{R}/\mathbf{Z} = \mathbf{S^1}$$

In the following, we would like to do *functional analysis* on $\mathbf{S^1}$. To this end, we define

$$\mathbf{Func}(\mathbf{S^1})$$

to be the set of *piecewise smooth continuous functions on* $\mathbf{S^1}$, i.e., continuous functions that are infinitely differentiable, except at a finite number of points of $\mathbf{S^1}$.

A crucial ingredient of the theory of [Zh] is the notion of "*divisors on* $\mathbf{S^1}$" (cf. [Zh], §2.1). Let us write

$$\mathrm{Div}(\mathbf{S^1}) \stackrel{\mathrm{def}}{=} \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$$

for the set of finite formal sums of elements of $\mathbf{Q}/\mathbf{Z}$ with $\mathbf{Z}$-coefficients. We will regard such formal sums as *divisors on* $\mathbf{S^1}$. Next, we let

$$\overline{\mathrm{Div}}(\mathbf{S^1}) \stackrel{\mathrm{def}}{=} \mathrm{Div}(\mathbf{S^1}) \oplus \mathbf{Func}(\mathbf{S^1})$$

be the group of *compactified divisors on* $\mathbf{S^1}$. Given a compactified divisor $D + g$ (i.e., $D \in \mathrm{Div}(\mathbf{S^1})$, $g \in \mathbf{Func}(\mathbf{S^1})$), we define the *curvature* of $D + g$ to be

$$h_{D+g} \stackrel{\mathrm{def}}{=} \delta_D - \Delta(g)$$

Here, δ_D is the *delta distribution* associated to the divisor D, i.e., if $f \in \mathbf{Func}(\mathbf{S^1})$, and $D = \sum_i c_i \cdot \mathfrak{p}_i$ (for $c_i \in \mathbf{Z}$; $\mathfrak{p}_i \in \mathbf{Q}/\mathbf{Z}$), then

$$< \delta_D, f > \stackrel{\mathrm{def}}{=} \sum_i c_i \cdot f(\mathfrak{p}_i)$$

while $\Delta(g)$ is the distribution obtained by applying the *Laplacian operator* Δ to g, i.e., if $f \in \mathbf{Func}(\mathbf{S^1})$, then

$$< \Delta(g), f > \stackrel{\mathrm{def}}{=} - \int_{\mathbf{S^1}} f \cdot \frac{\partial^2 g}{\partial \theta^2}$$

(where $\partial/\partial\theta$ is the standard unit tangent vector on $\mathbf{S^1}$, i.e., the tangent vector whose reciprocal $d\theta$ satisfies $\int_{\mathbf{S^1}} d\theta = 1$). One also defines an *intersection pairing* on compactified divisors on $\mathbf{S^1}$ by:

$$(D_1 + g_1, D_2 + g_2) \stackrel{\mathrm{def}}{=} < \delta_{D_1}, g_2 > + < \delta_{D_2}, g_1 > - \int_{\mathbf{S^1}} g_1 \Delta(g_2)$$

Next, we would like to consider *line bundles on* $E_\infty|_{U_S}$. First, if $\mathfrak{p}$ denotes a prime of the special fiber of E_∞, then we shall write

$$\mathcal{O}_{\mathfrak{p}}$$

for the *completion* of $\mathcal{O}_{E_\infty}$ at $\mathfrak{p}$, and $K_{\mathfrak{p}}$ for its quotient field. Thus, $\mathcal{O}_{\mathfrak{p}}$ is a valuation ring with valuation group equal to $\mathbf{Q}$ (times "log(q)"). By the above discussion, such $\mathfrak{p}$ are in natural bijective correspondence with the set $\mathbf{Q}/\mathbf{Z}$.

Definition 4.1. We define a *pre-metrized line bundle* $(\mathcal{L}, | - |_{\mathcal{L}})$ *on* E_∞ to be a pair $(\mathcal{L}, | - |_{\mathcal{L}})$ consisting of a line bundle $\mathcal{L}$ on $E_\infty|_{U_S}$, together with a set of *metrics* (as defined above)

$$| - |_{\mathcal{L}} = \{| - |_{\mathcal{L}_\mathfrak{p}}\}_\mathfrak{p}$$

(where $\mathfrak{p}$ ranges over all primes in the special fiber of E_∞) for the one-dimensional $K_\mathfrak{p}$-vector space $\mathcal{L}_\mathfrak{p} \stackrel{\text{def}}{=} \mathcal{L} \otimes_{\mathcal{O}_{E_\infty}} K_\mathfrak{p}$.

We will call a pre-metrized line bundle $(\mathcal{L}, | - |_{\mathcal{L}})$ a *metrized line bundle* if for some nonzero rational section ϕ of $\mathcal{L}$ over E_∞, the real-valued function defined on $\mathbf{Q}/\mathbf{Z} \subseteq \mathbf{S^1}$ by

$$\mathfrak{p} \mapsto -\log(|\phi|_{\mathcal{L}_\mathfrak{p}})$$

is the restriction to $\mathbf{Q}/\mathbf{Z} \subseteq \mathbf{S^1}$ of a function in $\mathbf{Func}(\mathbf{S^1})$. We denote the group of metrized line bundles on E_∞ by $\overline{\mathrm{Pic}}(E_\infty)$.

Remark 1. The definition of a metrized line bundle given here is slightly different from that of [Zh]. Nevertheless, the metrized line bundles that are ultimately dealt with in [Zh] (cf. [Zh], §2.5) are precisely the sort defined in Definition 4.1. In [Zh], however, such bundles are not given an explicit name.

Remark 2. If $\overline{\mathcal{L}}$ is a line bundle (in the usual sense) on E_∞ obtained by pull-back from a line bundle on C_N for some N, then the metrics $| - |_{\mathcal{L}_\mathfrak{p}}$ given by letting the absolute value of a generator of $\overline{\mathcal{L}}$ at $\mathfrak{p}$ be equal to 1 define on $\mathcal{L}$ the structure of a metrized line bundle. Thus, *the notion of a metrized line bundle on* E_∞ *generalizes the notion of a line bundle on* E_∞ *obtained by pull-back from a line bundle on* C_N.

One also has a notion of *compactified divisors on* E_∞, corresponding to the notion of a metrized line bundle. This group of compactified divisors is defined by

$$\overline{\mathrm{Div}}(E_\infty) \stackrel{\text{def}}{=} \mathrm{Div}(E_\infty|_{U_S}) \oplus \mathbf{Func}(\mathbf{S^1})$$

(where "$\mathrm{Div}(E_\infty|_{U_S})$" is the usual group of Weil divisors on the U_S-smooth scheme $E_\infty|_{U_S}$). Any divisor compactified divisor $D+g$ on E_∞ defines a *metrized line bundle* $\mathcal{O}_{E_\infty}(D+g) = (\mathcal{L}, | - |_{\mathcal{L}})$ as follows: First of all, if D is a Weil divisor on $E_\infty|_{U_S}$, then taking the closures of its irreducible components defines a Weil divisor $\overline{D} \subseteq E_\infty$ on E_∞. Moreover, since $E_\infty|_{U_S}$ is quasi-compact, and E_∞ is an inverse limit of regular schemes, it follows that $\overline{D}$ is, in fact, a *Cartier divisor* (cf. the discussion of the morphism ρ below for more details). Thus, we obtain a line bundle $\overline{\mathcal{L}}$ on E_∞. Let $\mathcal{L}$ be the restriction of $\overline{\mathcal{L}}$ to $E_\infty|_{U_S}$. Then it remains to define, for each $\mathfrak{p} \in \mathbf{Q}/\mathbf{Z}$, a metric $|-|_{\mathcal{L}_\mathfrak{p}}$ on $\mathcal{L}_\mathfrak{p}$. This metric is obtained by letting generators of $\overline{\mathcal{L}}$ at $\mathfrak{p}$ have absolute value $\exp(-g(\mathfrak{p}))$.

This correspondence $D + g \mapsto \mathcal{O}_{E_\infty}(D + g)$ thus defines a *surjection*

$$\overline{\mathrm{Div}}(E_\infty) \to \overline{\mathrm{Pic}}(E_\infty)$$

Under this correspondence, if $v \in \frac{1}{N}\mathbf{Z}/\mathbf{Z}$ is a component in the special fiber of C_N – which thus defines a *line bundle on* C_N – then the element of $\overline{\mathrm{Pic}}(E_\infty)$ corresponding to this line bundle (cf. Remark 2 above) is the image under the above surjection of the compactified divisor $g \in \mathbf{Func}(\mathbf{S^1})$, where g is the piecewise linear function on $\mathbf{S^1}$ which is linear on the complement of $\frac{1}{N}\mathbf{Z}/\mathbf{Z}$, equal to $\frac{1}{N}$ at v, and equal to 0 at all points of $(\frac{1}{N}\mathbf{Z}/\mathbf{Z})\backslash\{v\}$.

Next, we recall that one has a natural morphism

$$\rho : \mathrm{Div}(E_\infty|_{U_S}) \to \mathrm{Div}(\mathbf{S^1})$$

defined as follows: To define ρ, it suffices to define ρ on each prime divisor in $E_\infty|_{U_S}$. First, we set ρ to be 0 on all prime divisors which are *not* U_S*-flat*. If $D_{U_S} \subseteq E_\infty|_{U_S}$ is U_S*-flat*, then since $E_\infty|_{U_S}$ is an inverse limit of regular schemes, D_{U_S} will be a Cartier divisor, hence, in particular, a U_S-scheme of *finite presentation*. Thus, D_{U_S} will arise by base change from some divisor $(D_{U_S})_N \subseteq E_N|_{U_S}$. Write

$$D_N \subseteq C_N$$

for the *closure* of $(D_{U_S})_N$ in C_N. Let $D'_N \to D_N$ be the normalization of D_N. Thus, $D'_N \to S_N$ is finite and *flat* (since S_N is regular of dimension 2 and D'_N is normal, hence has depth 2). Note that since the prime $V(q^{1/N}) \subseteq S_N$ is regular of height 1 and has characteristic zero residue field, it follows (by Abhyankar's Lemma) that the ramification of any ramified extension of the localization of $\mathcal{O}_{S_N}$ at this prime may be annihilated by adjoining roots of q to $\mathcal{O}_{S_N}$. In particular, by taking N sufficiently large, we may assume that $D'_N \to S_N$ is *étale* at all characteristic 0 primes of height 1. But then it follows from the fact that C_N is *regular*, that *no characteristic* 0 *height* 1 *prime of* D'_N *can map to a node of* C_N. (Indeed, if R is a complete discrete valuation ring, and R' is an unramified extension of R (so R' is also a complete discrete valuation ring), then any R'-valued point of $R[[x, y]]/(xy - \pi)$ (where x, y are indeterminates, π is a uniformizer of R) would imply the existence of an R-homomorphism $R[[x, y]]/(xy - \pi) \to R'$ for which the images of x and y lie in the maximal ideal $\mathfrak{m}_{R'}$. But this implies that the image of π in R' lies in $\mathfrak{m}^2_{R'}$, which contradicts the assumption that R' is *unramified* over R.) Thus, there exists a closed subscheme $F_N \subseteq D_N$ of positive characteristic (i.e., such that $F_N \otimes \mathbf{Q} = \emptyset$) such that $D_N \backslash F_N \subseteq E_N$. In particular, the closure D_∞ of D_{U_S} in C_∞ will also satisfy:

$$D_\infty \backslash F_\infty \subseteq E_\infty$$

for some closed subscheme of positive characteristic $F_\infty \subseteq D_\infty$. For each characteristic 0 height 1 prime $\mathfrak{p}$ of D'_N, let us write $\mathrm{Comp}(\mathfrak{p}) \in \mathbf{Q}/\mathbf{Z}$ for the irreducible component of the special fiber E_∞ that contains $\mathfrak{p}$. Then we define

$$\rho(D_{U_S}) \stackrel{\text{def}}{=} \sum_{\mathfrak{p}} [k(\mathfrak{p}) : Q(\mathcal{O})] \cdot \mathrm{Comp}(\mathfrak{p})$$

where $\mathfrak{p}$ ranges over the characteristic 0 height one primes of D'_N; $k(\mathfrak{p})$ is the residue field of $\mathfrak{p}$; $Q(\mathcal{O})$ is the quotient field of $\mathcal{O} = A_N/(q^{1/N})$; and "$[\sim:\sim]$" denotes the degree of a field extension. One checks easily that (for N sufficiently large) this definition is independent of N. This completes the definition of ρ.

Now that we have the morphism ρ, we may define an *intersection pairing on* $\overline{\mathrm{Div}}(E_\infty)$

$$\overline{\mathrm{Div}}(E_\infty) \times' \overline{\mathrm{Div}}(E_\infty) \to \overline{\mathrm{Div}}(S_\infty) \stackrel{\text{def}}{=} \mathrm{Div}(S_\infty) \oplus \mathbf{R}$$

(where "$\times'$" denotes pairs of compactified divisors $D_1 + g_1$, $D_2 + g_2$ such that the generic points of the supports of D_1 and D_2 do not intersect) as follows: If D_1, D_2 are (Cartier) divisors on $E_\infty|_{U_S}$ such that the generic points of their supports do not intersect, then taking the closures of their supports defines (Cartier) divisors $\overline{D}_1, \overline{D}_2$ on $E_\infty|_{U_S}$ (cf. the discussion above). Thus, we define

$$i(D_1, D_2) \in \mathrm{Div}(S_\infty)$$

as the usual intersection divisor $\overline{D}_1 \cdot \overline{D}_2$ of $\overline{D}_1$ and $\overline{D}_2$. Then the *intersection pairing for compactified divisors* is defined by:

$$(D_1 + g_1, D_2 + g_2) \stackrel{\text{def}}{=} i(D_1, D_2) + (\rho(D_1) + g_1, \rho(D_2) + g_2)$$

(where the second "$(-,-)$" is an intersection number of compactified divisors on $\mathbf{S^1}$, hence $\in \mathbf{R}$).

Ultimately, we shall wish to do intersection theory not over "local objects" such as S, but over, for instance, $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ or finite flat coverings of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$. In that sort of global context, one can "glue together" the usual intersection theory over $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ with the intersection theory of [Zh] to obtain a *global intersection theory* for line bundles on the tautological log elliptic curve over coverings of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ which are equipped with a metric "at infinity." We leave the "general nonsense" details to the reader.

Next, we would like to consider the *curvatures* of compactified divisors and metrized line bundles. If $D + g \in \overline{\mathrm{Div}}(E_\infty)$, then its curvature is defined to be:

$$h_{D+g} \stackrel{\text{def}}{=} h_{\rho(D)+g}$$

(where the right-hand side is the curvature of the compactified divisor $\rho(D)+g \in \overline{\mathrm{Div}}(\mathbf{S^1})$). If $\overline{\mathcal{L}} = \mathcal{O}_{E_\infty}(D+g)$ is a metrized line bundle, then we define the curvature of $\overline{\mathcal{L}}$ to be:

$$h_{\overline{\mathcal{L}}} \stackrel{\text{def}}{=} h_{D+g}$$

This definition is independent of the choice of $D+g$ (cf. [Zh], §2.5). Note that in both cases, the curvature is a *distribution on* $\mathbf{S^1}$. Its integral over $\mathbf{S^1}$ is given by (cf. [Zh], §2.5):

$$\int_{\mathbf{S^1}} h_{\overline{\mathcal{L}}} = \deg(\overline{\mathcal{L}}|_{U_S})$$

the degree of $\overline{\mathcal{L}}$ on the generic fiber of E_∞. Intuitively speaking,

> *The value of the curvature $h_{\overline{\mathcal{L}}}$ at a point $a \in \mathbf{Q}/\mathbf{Z} \subseteq \mathbf{S^1}$ should be thought of as the degree of the "restriction" of $\overline{\mathcal{L}}$ to the irreducible component of the special fiber of C_∞ corresponding to a.*

Definition 4.2. A metrized line bundle or compactified divisor on E_∞ is said to be *admissible* if its curvature is the distribution given by a constant function on $\mathbf{S^1}$.

If a metrized line bundle $\overline{\mathcal{L}}$ is *admissible*, then its curvature $h_{\overline{\mathcal{L}}}$ is, in fact, the constant given by $\deg(\overline{\mathcal{L}}|_{U_S})$.

Proposition 4.3. *If $\overline{\mathcal{L}}_1 = (\mathcal{L}_1, |-|_{\mathcal{L}_1})$ and $\overline{\mathcal{L}}_2 = (\mathcal{L}_2, |-|_{\mathcal{L}_2})$ are metrized line bundles on E_∞ such that $\mathcal{L}_1 \cong \mathcal{L}_2$ (as line bundles on $E_\infty|_{U_S}$) and $h_{\overline{\mathcal{L}}_1} = h_{\overline{\mathcal{L}}_2}$, then $\overline{\mathcal{L}}_1 \cong \overline{\mathcal{L}}_2 \otimes \mathcal{O}_{E_\infty}(C)$ (as metrized line bundles), where $C \in \mathbf{Func}(\mathbf{S^1})$ is a constant function. In particular, if $\overline{\mathcal{L}}_1 = (\mathcal{L}_1, |-|_{\mathcal{L}_1})$ and $\overline{\mathcal{L}}_2 = (\mathcal{L}_2, |-|_{\mathcal{L}_2})$ are both* admissible *metrized line bundles on E_∞ such that $\mathcal{L}_1 \cong \mathcal{L}_2$ (as line bundles on $E_\infty|_{U_S}$), then $\overline{\mathcal{L}}_1 \cong \overline{\mathcal{L}}_2 \otimes \mathcal{O}_{E_\infty}(C)$ (as metrized line bundles), where $C \in \mathbf{Func}(\mathbf{S^1})$ is a constant function.*

Proof. By considering $\overline{\mathcal{L}}_1 \otimes \overline{\mathcal{L}}_2^{-1}$, it suffices to consider the case where $\overline{\mathcal{L}}_1$ is the trivial metrized line bundle. Then $\overline{\mathcal{L}}_2$ is defined by some compactified divisor $g \in \mathbf{Func}(\mathbf{S^1})$. Moreover, since $h_{\overline{\mathcal{L}}_1} = 0$, we have $h_g = -\Delta(g) = 0$, i.e., g is a continuous function on $\mathbf{S^1}$ whose second (distributional) derivative is zero. But it is easy to see that such a function g is necessarily constant. ○

The notion of admissible metrized line bundles will be of fundamental importance in the following discussion. Thus, it is of interest to *construct admissible metrized line bundles*. For instance, given a horizontal (i.e., S_∞-flat) divisor

$$D \subseteq E_\infty$$

one would like to know explicitly which function $g_D \in \mathbf{Func}(\mathbf{S^1})$ is the unique function such that

(i) $\int_{\mathbf{S^1}} g_D = 0$

(ii) $D + g_D$ is admissible $\iff h_{D+g_D} = \delta_{\rho(D)} - \Delta(g_D) = \delta_{\rho_D} + \frac{\partial^2 g_D}{\partial \theta^2}$ is constant

Such a function g_D will be called the *Green's function* for D. Note that g_D depends only on the divisor $\rho(D)$ on $\mathbf{S^1}$. Thus, we will also write $g_{\rho(D)}$ for g_D.

Let $\mathcal{D} \in \mathrm{Div}(\mathbf{S^1})$ be a divisor on $\mathbf{S^1}$. We would like to consider its Green function $g_{\mathcal{D}}$. The simplest case is the case where

$$\mathcal{D} = [0]$$

i.e., the origin of $[0] \in \mathbf{Q}/\mathbf{Z} \subseteq \mathbf{R}/\mathbf{Z} = \mathbf{S^1}$. In this case, we will denote the associated Green's function by ϕ_1.

Proposition 4.4. *The Green's function ϕ_1 associated to the origin $[0]$ is given, for $|\theta| \leq \frac{1}{2}$, by:*

$$\phi_1(\theta) = \frac{1}{2}\theta^2 - \frac{1}{2}|\theta| + \frac{1}{12}$$

where θ is the standard coordinate on $\mathbf{R}$ (regarded as a covering of $\mathbf{R}/\mathbf{Z} = \mathbf{S^1}$). In particular, $\phi_1(\theta) \in \mathbf{Q}$, for all $\theta \in \mathbf{Q}$. Alternatively, in terms of Fourier expansions on $\mathbf{S^1}$, it is given by:

$$\phi_1(\theta) = \frac{1}{4\pi^2} \cdot \sum_{0 \neq n \in \mathbf{Z}} \frac{1}{n^2} e^{2\pi i n \theta}$$

In particular, $\phi_1(0) = \frac{1}{12}$ (where we recall that we think of the values of ϕ_1 as being in "$\log(q)$" units) is the maximum value attained by ϕ_1, and $\phi_1(\frac{1}{2}) = -\frac{1}{24}$ is the minimum value attained by ϕ_1.

Proof. The polynomial representation of the Green's function ϕ_1 is given in [Zh], §a.8, p. 193. The Fourier expansion may be derived as the unique (topological) linear combination of $e^{2\pi i n}$'s (for $n \in \mathbf{Z}$) whose first derivative is square integrable on $\mathbf{S^1}$ (hence is continuous), whose value at 0 is $\frac{1}{12}$, and whose second derivative is the distribution defined by

$$1 - \delta_{[0]} = - \sum_{0 \neq n \in \mathbf{Z}} e^{2\pi i n}$$

on $\mathbf{S^1}$. The assertions concerning the maximum and minimum values of ϕ_1 follow from elementary calculus. ◯

Corollary 4.5. *Let $a, b \in \mathbf{Q}/\mathbf{Z}$. Then the Green's function $\phi_{a,b}$ associated to the divisor $a - b$ on $\mathbf{S^1}$ is the unique piecewise linear function on $\mathbf{S^1}$ which is linear (with respect to θ) away from a and b, has second (distributional) derivative equal to $\delta_b - \delta_a$, and is such that $\int_{\mathbf{S^1}} \phi_{a,b} = 0$. Moreover, $\phi_{a,b}$ has a global minimum at b and a global maximum at a.*

Proof. This follows from the polynomial representation of Proposition 4.4 by direct computation. ◯

Corollary 4.6. *Let N be a positive integer. Write ϕ_N for the Green's function of the divisor $\sum_{i=0}^{N-1} [\frac{i}{N}]$. Then the Fourier expansion of ϕ_N is given by:*

$$\phi_N(\theta) = \frac{1}{4\pi^2 \cdot N} \cdot \sum_{0 \neq n \in \mathbf{Z}} \frac{1}{n^2} e^{2\pi i n \theta \cdot N} = \frac{1}{N} \phi_1(N \cdot \theta)$$

In particular, $\phi_N(0) = \frac{1}{12 \cdot N}$ (where we recall that we think of the values of ϕ_1 as being in "$\log(q)$" units) is the maximum value attained by ϕ_N, and $\phi_N(\frac{1}{2N}) = -\frac{1}{24N}$ is the minimum value attained by ϕ_1.

Proof. The assertion concerning the Fourier expansion follows by adding up the translates of the Fourier expansion of Proposition 4.4. The assertions concerning the maximum and minimum values of ϕ_N follow by thinking of $\phi_N(\theta)$ as $\frac{1}{N} \cdot \phi_1(N \cdot \theta)$. ◯

Note that the use of Green's functions allows one to *construct metrized line bundles with arbitrary prescribed restrictions to the special fiber of E_∞*. Indeed, suppose we start out with two horizontal divisors $D_1, D_2 \subseteq E_\infty$, both of degree d (on $E_\infty|_{U_S}$). Then consider the metrized line bundle

$$\overline{\mathcal{L}} \stackrel{\text{def}}{=} \mathcal{O}_{E_\infty}(D_1 + g_{D_1} - g_{D_2})$$

The restriction of $\overline{\mathcal{L}}$ to a component of the special fiber of E_∞, which amounts to the curvature $h_{\overline{\mathcal{L}}}$ of $\overline{\mathcal{L}}$ evaluated at the point of $\mathbf{S^1}$ corresponding to that component, is given (cf. Definition 4.2) by:

$$
\begin{aligned}
h_{\overline{\mathcal{L}}} &= \delta_{D_1} - \Delta(g_{D_1}) + \Delta(g_{D_2}) = h_{D_1+g_{D_1}} - h_{D_2+g_{D_2}} + \delta_{D_2} \\
&= d - d + \delta_{D_2} = \delta_{D_2}
\end{aligned}
$$

i.e., despite the fact that on the generic fiber $E_\infty|_{U_S}$ of E_∞, $\overline{\mathcal{L}}$ is just the line bundle associated to the divisor D_1, *on the special fiber of* E_∞, $\overline{\mathcal{L}}$ *looks as if it is the line bundle associated to the divisor* D_2*!* This sort of metrized line bundle will play a key role in this paper.

§5. Theta Groups and Metrized Line Bundles

We maintain the notation of §4. Let d be a positive integer; $g \in \mathbf{Func}(\mathbf{S^1})$. In this §, we would like to study the *metrized line bundle* on E_∞ given by:

$$
\overline{\mathcal{L}} \stackrel{\mathrm{def}}{=} \mathcal{O}_{E_\infty}(d[e] + g)
$$

(where $e \in E_\infty(S_\infty)$ is the origin of the group object $E_\infty \to S_\infty$, and $[e]$ is the horizontal divisor defined by its image). In particular, we will discuss the case where the *theta groups* of §1 act on $\overline{\mathcal{L}}$, and compute (when the base is global) the *degree of the push-forward* of $\overline{\mathcal{L}}$ to such a global base. We observe that although the issue of the behavior of theta groups for degenerating elliptic curves is also discussed in [MB], it is the opinion of the author that *the use of Zhang's theory of metrized line bundles substantially clarifies the behavior of theta groups for degenerating elliptic curves.*

Note first of all that when we restrict to U_S, we are in the situation discussed in §1. In particular, there is a *theta group* $(\mathcal{G}_{\mathcal{L}})_{U_S}$ associated to $\mathcal{O}_{E|_{U_S}}(d \cdot [e])$. The group scheme $(K_{\mathcal{L}})_{U_S}$ of d-torsion points of $E|_{U_S}$ fits into an exact sequence

$$
0 \to (\mu_d)_{U_S} \to (K_{\mathcal{L}})_{U_S} \to (\mathbf{Z}/d\mathbf{Z})_{U_S} \to 0
$$

Although this exact sequence does not extend over S, if one base-changes to S_∞, then it extends naturally to an exact sequence of group schemes over S_∞:

$$
0 \to \mu_d|_{S_\infty} \to K_{\mathcal{L}}|_{S_\infty} \to (\mathbf{Z}/d\mathbf{Z})|_{S_\infty} \to 0
$$

where $K_{\mathcal{L}}|_{S_\infty} \subseteq E_\infty$ is a closed subgroup scheme. In fact, this exact sequence splits, so we have a noncanonical isomorphism

$$
K_{\mathcal{L}}|_{S_\infty} \cong \{\mu_d \times (\mathbf{Z}/d\mathbf{Z})\}|_{S_\infty}
$$

In the following, we will denote $K_{\mathcal{L}}|_{S_\infty}$ by

$$K_{\overline{\mathcal{L}}}$$

Thus, the action of $K_{\overline{\mathcal{L}}}$ on E_∞ preserves the line bundle $\overline{\mathcal{L}}|_{U_S}$. We would like to investigate the extent to which it preserves $\overline{\mathcal{L}}$:

Proposition 5.1. *The action of $K_{\overline{\mathcal{L}}}$ on E_∞ preserves $\overline{\mathcal{L}}$ if and only if translation by $\frac{1}{d} \in \frac{1}{d}\mathbf{Z}/\mathbf{Z} \subseteq \mathbf{S^1}$ on $\mathbf{S^1}$ preserves the curvature $h_{\overline{\mathcal{L}}}$. We shall call such distributions on $\mathbf{S^1}$ d-invariant. For instance, if $g = d \cdot \phi_1$ or $g = d \cdot \phi_1 - \phi_d$ (cf. Corollary 4.6), then the resulting curvature will be d-invariant.*

Proof. The automorphisms of $\mathbf{S^1}$ induced by the action of $K_{\overline{\mathcal{L}}}$ on E_∞ are precisely those given by adding (integer) multiples of $\frac{1}{d}$. Thus, the necessity of the condition $h_{\overline{\mathcal{L}}}(\theta) = h_{\overline{\mathcal{L}}}(\theta + \frac{1}{d})$ is clear. To see that it is sufficient, we reason as follows. First of all, since S_N is regular of dimension 2, it suffices to prove the result in characteristic 0, i.e., in the case where $\mathcal{O}$ is a finite extension of $\mathbf{Q}$. But then $K_{\overline{\mathcal{L}}}$ is étale over S_∞, so it suffices to prove that the "physical" automorphisms of E_∞ induced by S_∞-rational points of $K_{\overline{\mathcal{L}}}$ preserve $\overline{\mathcal{L}}$. Let α be such an automorphism of E_∞ that induces the automorphism $\theta \mapsto \theta + \frac{1}{d}$ on $\mathbf{S^1}$. Then $\overline{\mathcal{L}}_\alpha \stackrel{\mathrm{def}}{=} \alpha^* \overline{\mathcal{L}}$ is a metrized line bundle on E_∞ which is isomorphic to $\overline{\mathcal{L}}$ over U_S. Moreover, by hypothesis, $h_{\overline{\mathcal{L}}} = h_{\overline{\mathcal{L}}_\alpha}$. Thus, by Proposition 4.3, it follows that $\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}_\alpha$ differ by a constant $C \in \mathbf{R}$. If this constant $C \in \mathbf{Q}$, then multiplying by q^C shows that $\overline{\mathcal{L}} \cong \overline{\mathcal{L}}_\alpha$. If not, then since α has order d, it follows that C defines a class in $H^1(\mathbf{Z}/d\mathbf{Z}, \mathbf{R}/\mathbf{Q}) = \mathrm{Hom}(\mathbf{Z}/d\mathbf{Z}, \mathbf{R}/\mathbf{Q}) = 0$. In order words, it follows from the fact that the automorphism α has finite order (while $\mathbf{R}/\mathbf{Q}$ is torsion-free) that $C \in \mathbf{Q}$, so $\overline{\mathcal{L}} \cong \overline{\mathcal{L}}_\alpha$, as desired.

Thus, it remains to see that if $g = d \cdot \phi_1$ or $d\phi_1 - \phi_d$, then $h_{\overline{\mathcal{L}}}$ is invariant under $\theta \mapsto \theta + \frac{1}{d}$. But if $g = d \cdot \phi_1$, then $h_{\overline{\mathcal{L}}}$ is a constant, so this is clear. Moreover, ϕ_d itself is invariant under $\theta \mapsto \theta + \frac{1}{d}$, so adding $\Delta(\phi_d)$ does not effect the invariance under $\theta \mapsto \theta + \frac{1}{d}$. This completes the proof. ○

Thus, if $h_{\overline{\mathcal{L}}}$ is *d-invariant*, then by considering pairs (α, ι), where α is a point of $K_{\overline{\mathcal{L}}}$ and $\iota : \mathcal{T}_\alpha^* \overline{\mathcal{L}} \cong \overline{\mathcal{L}}$ (cf. §1), we obtain a *theta group scheme*

$$\mathcal{G}_{\overline{\mathcal{L}}}$$

over S_∞ which extends $(\mathcal{G}_{\mathcal{L}})_{U_{S_\infty}}$. In particular, if $f : E_\infty \to S_\infty$ is the structure morphism, then we may define the *push-forward*

$$f_* \overline{\mathcal{L}}$$

as the (quasi-coherent) $\mathcal{O}_S$-submodule of the rank d vector bundle $(f|_{U_S})_*(\overline{\mathcal{L}}|_{U_S})$ on U_{S_∞} consisting of sections whose absolute value is ≤ 1 with respect to the $|-|_{\mathcal{L}_\mathfrak{p}}$ for all $\mathfrak{p} \in \mathbf{Q}/\mathbf{Z}$. Then $f_*\overline{\mathcal{L}}$ is equipped with a *natural action of* $\mathcal{G}_{\overline{\mathcal{L}}}$.

Since it is not difficult to construct a $\mathcal{G}_{\overline{\mathcal{L}}}$-module $\mathcal{V}$ which is a rank d vector bundle on S_∞ and on which $\mathbf{G}_\mathrm{m} \subseteq \mathcal{G}_{\overline{\mathcal{L}}}$ acts via the standard character (cf. Example 1.2), it follows (cf. Theorem 1.1 of §1; [MB], Chapitre V, Corollaire 2.4.3) that

$$f_*\overline{\mathcal{L}} \cong \mathcal{M} \otimes_{\mathcal{O}_{S_\infty}} \mathcal{V}$$

Here, $\mathcal{M}$ and its restriction $\mathcal{M}_U$ to U_{S_∞} satisfy the following: $\mathcal{M}_U$ is a line bundle on U_{S_∞}; and $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_{S_\infty}$-submodule of $j_*\mathcal{M}_U$ (where $j : U_{S_\infty} \hookrightarrow S_\infty$ is the natural inclusion). Note that since U_{S_∞} is quasi-compact and $\mathcal{M}_U$ (being a line bundle) is of finite presentation, it follows that $\mathcal{M}_U$ arises as the pull-back of a line bundle on some U_{S_N}. Thus, $\mathcal{M}_U$ extends to a line bundle $\mathcal{M}'$ on S_∞. In fact, we may even assume that $\mathcal{M}' \subseteq \mathcal{M}$ (in such a way that $\mathcal{M}'|_{U_S} = \mathcal{M}|_{U_S}$). Next, let us observe that $\mathcal{M}$ is *bounded* in the sense that there exists some positive rational number C such that

$$\mathcal{M}' \subseteq \mathcal{M} \subseteq q^{-C} \cdot \mathcal{M}'$$

(Indeed, this follows from the fact that $\overline{\mathcal{L}}$ itself is "bounded.") Moreover, $\mathcal{M}$ has the property that a rational section of $\mathcal{M}$ is integral over S_∞ if and only if it is integral over U_{S_∞} as well as at the *central prime* $V(\{q^r\}_{r \in \mathbf{Q}_{>0}}) \subseteq S_\infty$. Since the localization of A_∞ at the central prime is a valuation ring with value group $\mathbf{Q}$, it thus follows that $\mathcal{M}$ is of the form

$$\mathcal{M} = \bigcap_{r_0 \leq r \in \mathbf{Q}} q^{-r} \cdot \mathcal{M}'$$

where $r_0 \in \mathbf{R}_{\geq 0}$. In other words, the datum of $\mathcal{M}$ is equivalent to the datum of $\mathcal{M}_U$, together with a *metric* on $\mathcal{M}_U$ at the central prime of S_∞.

Definition 5.2. We shall refer to as a *metrized vector bundle* on S_∞ any pair $(\mathcal{F}, |-|_\mathcal{F})$ consisting of a vector bundle $\mathcal{F}$ on U_{S_∞}, together with a metric on $\mathcal{F}$ at the central prime of S_∞. The *rank of a metrized vector bundle* $(\mathcal{F}, |-|_\mathcal{F})$ is defined to be the rank of the vector bundle $\mathcal{F}$.

If $(\mathcal{F}, |-|_\mathcal{F})$ is a *metrized vector bundle*, then by considering the sheaf of (nonzero) sections of $\mathcal{F}$ whose $|-|_\mathcal{F} \leq 1$ at the central prime, we naturally obtain a *quasi-coherent* $\mathcal{O}_{S_\infty}$*-module*. Moreover, it follows from the discussion of metrics at the beginning of §4 – i.e., "metrics are equivalent to bounded, saturated submodules that generate over the quotient field" – that the metric $|-|_\mathcal{F}$ may be recovering from this quasi-coherent module.

In other words, *metrized vector bundles may be thought of as a special kind of quasi-coherent* $\mathcal{O}_{S_\infty}$*-module.*

On the other hand, any vector bundle $\mathcal{F}$ on S_∞ (in the usual sense) naturally defines a *metrized vector bundle on* S_∞ $(\mathcal{F}_U, |-|_{\mathcal{F}_U})$ as follows: we let $\mathcal{F}_U$ be the restriction of $\mathcal{F}$ to U_{S_∞}; $|-|_{\mathcal{F}_U}$ be the metric corresponding to the module of integral sections at the central prime of S_∞. Thus, the (nonzero) sections of $\mathcal{F}$ may be recovered as those sections of $\mathcal{F}_U$ whose $|-|_{\mathcal{F}_U} \leq 1$ at the central prime. In particular, we see that *the notion of a metrized vector bundle on* S_∞ *is a generalization of the usual notion of a vector bundle on* S_∞.

Corollary 5.3. *Let* $\overline{\mathcal{L}} \stackrel{\text{def}}{=} \mathcal{O}_{E_\infty}(d[e]+g)$ *be a metrized line bundle on* E_∞ *whose curvature* $h_{\overline{\mathcal{L}}}$ *is d-invariant. Then the push-forward sheaf* $f_*\overline{\mathcal{L}}$ *on* S_∞ *has a natural structure of* metrized vector bundle of rank d on S_∞ equipped with an action of $\mathcal{G}_{\overline{\mathcal{L}}}$. *Moreover, there exists an isomorphism*

$$f_*\overline{\mathcal{L}} \cong \mathcal{M} \otimes_{\mathcal{O}_{S_\infty}} \mathcal{V}$$

of $\mathcal{G}_{\overline{\mathcal{L}}}$*-modules where* $\mathcal{M}$ *is a metrized line bundle on* S_∞ *with trivial* $\mathcal{G}_{\overline{\mathcal{L}}}$*-action, and* $\mathcal{V}$ *is a vector bundle (in the usual sense) on* S_∞ *with* $\mathcal{G}_{\overline{\mathcal{L}}}$*-action on which* $\mathbf{G}_{\mathrm{m}} \subseteq \mathcal{G}_{\overline{\mathcal{L}}}$ *acts via the standard character. Finally, if* $\overline{\mathcal{L}}$ *is, in fact, a line bundle (i.e., is the metrized line bundle arising from some line bundle on some* C_N*), then* $f_*\overline{\mathcal{L}}$ *(respectively,* $\mathcal{M}$*) is a vector bundle (respectively, line bundle) in the usual sense.*

Now suppose (just for the remainder of this paragraph) that $\overline{\mathcal{L}}$ is *symmetric*, i.e., preserved (up to isomorphism) by the automorphism of E_∞ given by multiplication by -1. It is easy to see that this is equivalent to the assertion that the curvature $h_{\overline{\mathcal{L}}}$ satisfies $h_{\overline{\mathcal{L}}}(-\theta) = h_{\overline{\mathcal{L}}}(\theta)$ (cf. the proof of Proposition 5.1). Then (just as in the discussion of §1) the automorphism $[-1]$ of E_∞ given by multiplication by -1 induces an automorphism of $\mathcal{G}_{\overline{\mathcal{L}}}$. Thus, just as in §1 (cf. especially the discussion following Theorem 1.6), if $\alpha \in E_\infty(S_\infty)$ is any S_∞-valued point, then we get an *isomorphism* (of metrized vector bundles on S_∞)

$$\overline{\mathcal{L}}|_{\mathcal{T}^*_\alpha K_{\overline{\mathcal{L}}}} \cong (\overline{\mathcal{L}}|_{\alpha^0}) \otimes_{\mathcal{O}_{K^0_{\overline{\mathcal{L}}}}} \mathcal{O}_{K_{\overline{\mathcal{L}}}}$$

(where $K^0_{\overline{\mathcal{L}}} \stackrel{\text{def}}{=} K_{\overline{\mathcal{L}}}/2 \cdot K_{\overline{\mathcal{L}}}$, and $\overline{\mathcal{L}}|_{\alpha^0}$ is the line bundle on $K^0_{\overline{\mathcal{L}}}$ obtained by descending $\overline{\mathcal{L}}|_{\mathcal{T}^*_\alpha K_{\overline{\mathcal{L}}}}$). Thus, for instance, if d is odd, then we obtain a *trivialization* of the restriction of $\overline{\mathcal{L}}$ to the subscheme $\mathcal{T}^*_\alpha K_{\overline{\mathcal{L}}}$. In particular, *restriction to* $\mathcal{T}^*_\alpha K_{\overline{\mathcal{L}}}$ defines (by composing with the above isomorphism) a morphism of metrized vector bundles on S_∞ with $\mathcal{G}_{\overline{\mathcal{L}}}$-action:

$$f_*\overline{\mathcal{L}} \to (\overline{\mathcal{L}}|_{\alpha^0}) \otimes_{\mathcal{O}_{K^0_{\overline{\mathcal{L}}}}} \mathcal{O}_{K_{\overline{\mathcal{L}}}}$$

In other words, this morphism allows us to think of global sections of $\overline{\mathcal{L}}$ over E as being (essentially) *functions* on the subscheme $\mathcal{T}_\alpha^* K_{\overline{\mathcal{L}}}$. *These functions may thus be thought of as a sort of "metrized" or "Zhang-theoretic" version of the algebraic theta functions of [Mumf1,2,3].*

The following *standard line bundles* will be important in this paper, so we give them explicit names. First, let us write

$$\phi_d^{\mathrm{ev}}(\theta) \stackrel{\mathrm{def}}{=} \phi_d(\theta + \frac{1}{2d})$$

(notation as in Corollary 4.6). In the following, the objects with a superscript "ev" (for "even") will be important in the case when d is *even*. Nevertheless, all the definitions (of both the objects with a superscript "ev" and the objects without a superscript "ev") may be made regardless of whether d is even or odd. Let

$$g_{\mathrm{st}} \stackrel{\mathrm{def}}{=} d \cdot \phi_1 - \phi_d; \quad g_{\mathrm{st}}^{\mathrm{ev}} \stackrel{\mathrm{def}}{=} d \cdot \phi_1 - \phi_d^{\mathrm{ev}}$$

(notation as in Corollary 4.6). Write

$$\overline{\mathcal{L}}_{\mathrm{st}} \stackrel{\mathrm{def}}{=} \mathcal{O}_{E_\infty}(d[e] + g_{\mathrm{st}}); \quad \overline{\mathcal{L}}_{\mathrm{st}}^{\mathrm{ev}} \stackrel{\mathrm{def}}{=} \mathcal{O}_{E_\infty}(d[e] + g_{\mathrm{st}}^{\mathrm{ev}})$$

In other words (cf. the discussion at the end of §4), $\overline{\mathcal{L}}_{\mathrm{st}}$ (respectively, $\overline{\mathcal{L}}_{\mathrm{st}}^{\mathrm{ev}}$) is the metrized line bundle whose restriction to the special fiber of C_∞ looks like the divisor

$$\sum_{i=0}^{d-1} [\frac{i}{d}] \quad \text{(respectively, } \sum_{i=0}^{d-1} [(\frac{1}{2d}) + (\frac{i}{d})] \text{)}$$

In particular, it is easy to see that if one takes a point $\alpha \in K_{\overline{\mathcal{L}}}(S_\infty)$ whose image under the projection $K_{\overline{\mathcal{L}}} \to (\mathbf{Z}/d\mathbf{Z})$ is $1 \in \mathbf{Z}/d\mathbf{Z}$, and a point $\beta \in E_\infty(S_\infty)$ such that $2 \cdot \beta = \alpha$, then we have the following:

Lemma 5.4. *If d is* odd, *then* $\overline{\mathcal{L}}_{\mathrm{st}} \cong \mathcal{O}_{E_\infty}(\sum_{i=0}^{d-1} [i \cdot \alpha])$. *If d is* even, *then* $\overline{\mathcal{L}}_{\mathrm{st}}^{\mathrm{ev}} \cong \mathcal{O}_{E_\infty}(\sum_{i=0}^{d-1} [\beta + i \cdot \alpha])$.

Proof. Indeed, one calculates easily that both sides are isomorphic generically and have the same curvatures. (Note here that in the case of d even, if one does not shift by β, then both sides will not even be isomorphic generically. Indeed, they will differ generically by a line bundle of order precisely 2.) Thus, we conclude by Proposition 4.3 that the two sides differ by some $\mathcal{O}_{E_\infty}(C)$, for $C \in \mathbf{R}$ a constant. But now observe (cf. Proposition 4.4) that g_{st} (respectively, $g_{\mathrm{st}}^{\mathrm{ev}}$) takes *rational* values on $\mathbf{Q}/\mathbf{Z}$. On the other hand, for any irreducible component $\mathfrak{p}$ of the special fiber of E_∞, the order $\mathrm{ord}_{\mathfrak{p}}$ at $\mathfrak{p}$ of a rational function on E_∞

whose divisor on $E_\infty|_{U_S}$ is $d[e] - \sum_{i=0}^{d-1} [i \cdot \alpha]$ (respectively, $d[e] - \sum_{i=0}^{d-1} [\beta + i \cdot \alpha]$) lies in $\mathbf{Q}$. Thus, it follows that C may be taken to be in $\mathbf{Q}$, hence (by multiplying by q^C) it may be taken to be 0. Thus, the two metrized line bundles are isomorphic, as desired. ○

Note, in particular, that if d is odd (respectively, even), then *there exists a section* s *of* $f_*\overline{\mathcal{L}}_{\mathrm{st}}$ *(respectively,* $f_*\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st}}$*) whose order* $\mathrm{ord}_{\mathfrak{p}}$ *at any prime* $\mathfrak{p} \in \mathbf{Q}/\mathbf{Z} \subseteq \mathbf{S^1}$ *is precisely* 0. Indeed, such an s is given by applying the isomorphism of Lemma 5.4 to the section given by the natural inclusion

$$\mathcal{O}_{C_d} \subseteq \mathcal{O}_{C_d}(\sum_{i=0}^{d-1} [i \cdot \alpha]) \quad (\text{respectively, } \mathcal{O}_{C_d} \subseteq \mathcal{O}_{C_d}(\sum_{i=0}^{d-1} [\beta + i \cdot \alpha]))$$

It is clear that this section is nonzero at all the nodes of C_d, hence at every prime $\mathfrak{p} \in \mathbf{Q}/\mathbf{Z}$, as desired.

Lemma 5.5. *Suppose that* $\overline{\mathcal{L}} = \mathcal{O}_{E_\infty}(d[e] + g)$*, where* $g = g_{\mathrm{st}} + \psi$ *(respectively,* $g = g^{\mathrm{ev}}_{\mathrm{st}} + \psi$*) if* d *is odd (respectively, even), and* $\psi \in \mathbf{Func}(\mathbf{S^1})$ *is* ≥ 0 *and* d*-invariant on* $\mathbf{S^1}$*. Suppose, moreover, that* ψ vanishes *at some point of* $\mathbf{Q}/\mathbf{Z}$*. Then the natural identification between* $f_*\overline{\mathcal{L}}$ *and* $f_*\overline{\mathcal{L}}_{\mathrm{st}}$ *(respectively,* $f_*\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st}}$*) over* U_{S_∞} *extends to an equality*

$$f_*\overline{\mathcal{L}} = f_*\overline{\mathcal{L}}_{\mathrm{st}} \quad (\text{resp } f_*\overline{\mathcal{L}} = f_*\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st}})$$

over S_∞*.*

Proof. To simplify notation, we assume in this proof that d is *odd.* (The even case is entirely similar.) Now note that because ψ is d-invariant, it follows that $\mathcal{G}_{\overline{\mathcal{L}}}$ acts on both $f_*\overline{\mathcal{L}}$ and $f_*\overline{\mathcal{L}}_{\mathrm{st}}$. Thus, (cf. Corollary 5.3) it follows that for some $r \in \mathbf{R}$, we have:

$$f_*\overline{\mathcal{L}} = q^{-r} \cdot f_*\overline{\mathcal{L}}_{\mathrm{st}}$$

(i.e., the metrics on the two sides differ by a factor of e^r). Since $\psi \geq 0$, it follows that $r \geq 0$. *Suppose that* $r > 0$. Then if s is any section of $f_*\overline{\mathcal{L}}_{\mathrm{st}}$, then there exists a rational number $\epsilon > 0$ such that $q^{-\epsilon} \cdot s$ forms an integral section of $f_*\overline{\mathcal{L}}$. Now suppose that ψ vanishes at the point $\mathfrak{p}$ of $\mathbf{Q}/\mathbf{Z}$. By the preceding discussion, there exists an s such that s does not vanish (as a section of $\overline{\mathcal{L}}_{\mathrm{st}}$) at $\mathfrak{p}$. Thus, $q^{-\epsilon} \cdot s$ *does not define an integral section of* $\overline{\mathcal{L}}_{\mathrm{st}}$ at $\mathfrak{p}$. On the other hand, since $\psi(\mathfrak{p}) = 0$, it follows that $\overline{\mathcal{L}}_{\mathrm{st}}$ and $\overline{\mathcal{L}}$ have the same integral structure at $\mathfrak{p}$. Thus, $q^{-\epsilon} \cdot s$ is not integral for $\overline{\mathcal{L}}$ at $\mathfrak{p}$ – a contradiction. This completes the proof. ○

Theorem 5.6. *Suppose that $\overline{\mathcal{L}} = \mathcal{O}_{E_\infty}(d[e] + g)$, where $g = g_{\mathrm{st}} + \psi$ (respectively, $g = g_{\mathrm{st}}^{\mathrm{ev}} + \psi$) if d is odd (respectively, even), and $\psi \in \mathbf{Func}(\mathbf{S^1})$ is d-invariant on $\mathbf{S^1}$. Let*

$$r_0 \stackrel{\mathrm{def}}{=} \inf_{\alpha \in \mathbf{S^1}} \ \psi(\alpha)$$

Then the natural identification between $f_*\overline{\mathcal{L}}$ and $f_*\overline{\mathcal{L}}_{\mathrm{st}}$ (respectively, $f_*\overline{\mathcal{L}}_{\mathrm{st}}^{\mathrm{ev}}$) over U_{S_∞} extends to an equality

$$f_*\overline{\mathcal{L}} = q^{-r_0} \cdot f_*\overline{\mathcal{L}}_{\mathrm{st}} \quad (\text{respectively, } f_*\overline{\mathcal{L}} = q^{-r_0} \cdot f_*\overline{\mathcal{L}}_{\mathrm{st}}^{\mathrm{ev}})$$

(i.e., the metric on $f_*\overline{\mathcal{L}}$ is e^{-r_0} times the metric on $f_*\overline{\mathcal{L}}_{\mathrm{st}}$ (respectively, $f_*\overline{\mathcal{L}}_{\mathrm{st}}^{\mathrm{ev}}$)) over S_∞.

Proof. For simplicity, we assume that d is *odd*. First, consider the case where $r_0 \in \mathbf{Q}$ and there exists an $\alpha \in \mathbf{Q}/\mathbf{Z}$ such that $\psi(\alpha) = r_0$. Then by multiplying through by q^{r_0} we see that we reduce to the case $r_0 = 0$. Moreover, the hypotheses of Lemma 5.5 are satisfied, so we see that the result follows from Lemma 5.5. This completes the case where $r_0 \in \mathbf{Q}$ and $\exists \alpha \in \mathbf{Q}/\mathbf{Z}$ such that $\psi(\alpha) = r_0$.

To handle the general case, one simply approximates ψ by piecewise linear functions ψ_i in $\mathbf{Func}(\mathbf{S^1})$ which satisfy the hypotheses of the preceding paragraph. As $\psi_i \to \psi$, it is clear that the metric on the resulting $f_*\overline{\mathcal{L}}_i$ converges to the metric on the original $f_*\overline{\mathcal{L}}$ (corresponding to ψ). This completes the proof. ○

Corollary 5.7. *Suppose that $\overline{\mathcal{L}} = \mathcal{O}_{E_\infty}(d[e] + d \cdot \phi_1)$. Then we have (on S_∞)*

$$f_*\overline{\mathcal{L}} = q^{\frac{1}{24d}} \cdot f_*\overline{\mathcal{L}}_{\mathrm{st}} \quad (\text{respectively, } f_*\overline{\mathcal{L}} = q^{\frac{1}{24d}} \cdot f_*\overline{\mathcal{L}}_{\mathrm{st}}^{\mathrm{ev}})$$

if d is odd (respectively, even).

Proof. For simplicity, we assume that d is *odd*. Then we have $\psi \stackrel{\mathrm{def}}{=} d \cdot \phi_1 - g_{\mathrm{st}} = \phi_d$. Moreover, by Corollary 4.6, the r_0 (as in Theorem 5.6) for this ψ is equal to $-\frac{1}{24d}$. Thus Corollary 5.7 follows from Theorem 5.6. ○

Next, we would like to switch gears, and consider the following situation. Suppose that we are given a *finite flat covering*

$$B \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$$

which is étale over $(\mathcal{M}_{1,1})_{\mathbf{Q}}$ (i.e., away from the divisor at infinity and finite primes). For simplicity, we will also assume that B is *regular*, and that the completion of B at the inverse image of the divisor at infinity of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ is a disjoint union of "S_N's" as in the above discussion (for some appropriate N and base ring $\mathcal{O}$). Let us write

$$\mathcal{I}_B$$

for the set of connected components of this union. Thus, for each $\iota \in \mathcal{I}_B$, we have a completion $B_\iota \cong S_{N_\iota}$, where S_{N_ι} is understood to be over a base ring $\mathcal{O}_\iota$. Let

$$E_B \subseteq C_B \to B$$

be the *tautological log elliptic curve* over B. Also, let us write $U_B \subseteq B$ for the complement of the divisor at infinity of B. By gluing together $E_B|_{U_B}$ and $(E_\infty)_\iota$ at each ι, we thus obtain a smooth group scheme

$$E_{\infty,B} \to B_\infty$$

over B_∞. (Thus, strictly speaking, B_∞ is an *algebraic stack in the finite, flat topology.*)

Then let us observe that one can glue together the theory discussed above over "S" with the usual theory of line bundles and vector bundles on E_B and B to obtain a theory of *metrized line bundles on* $E_{\infty,B}$ *and metrized vector bundles on* B_∞. Indeed, we shall not write out the routine details, but the point is that a metrized line bundle on $E_{\infty,B}$ is a line bundle on $E_{\infty,B}|_{U_B}$, together with a metric on the pull-back of this line bundle to each $E_{\infty,B}|_{U_{S_{\infty,\iota}}}$ (where we use the notation "$U_{S_{\infty,\iota}}$" relative to the understanding that each $B_\iota \cong S_{N_\iota}$ has its associated $U_{S_{\infty,\iota}} \subseteq S_{\infty,\iota} \to S_{N_\iota}$, etc.). Similarly, a metrized vector bundle on B_∞ is a vector bundle on U_B, together with a metric on the pull-backs of this vector bundle to each $U_{S_{\infty,\iota}}$.

Thus, if we are given, for each $\iota \in \mathcal{I}_B$, a function

$$g_\iota \in \mathbf{Func}(\mathbf{S}^1_\iota)$$

then we may form a *metrized line bundle*

$$\overline{\mathcal{L}}_B \stackrel{\text{def}}{=} \mathcal{O}_{E_{\infty,B}}(d[e] + \sum_\iota g_\iota)$$

in the obvious way. If we then push-forward this line bundle via $f_B : E_{\infty,B} \to B_\infty$, we obtain a *metrized vector bundle*

$$(f_B)_*\overline{\mathcal{L}}_B$$

whose local structure was studied in Corollary 5.3 and Theorem 5.6 above. In particular, let us note that it makes sense to speak of the *degree of a metrized vector bundle on* B_∞.

This degree will, in general, be a real number. There are many ways that one can define this degree. For instance, one may define it as the degree of the metrized line bundle which is the determinant of the given metrized vector bundle. Thus, it suffices to define the degree of a metrized line bundle on B_∞. But a metrized line bundle on B_∞ can easily be seen to be equivalent (up to torsion) to an element of $\mathrm{Pic}(B) \otimes_{\mathbf{Z}} \mathbf{R}$. Thus, the degree of a metrized line bundle may be defined by tensoring (over $\mathbf{Z}$ with $\mathbf{R}$) the usual degree map

$$\mathrm{Pic}(B) \to \mathrm{Pic}(B_{\mathbf{Q}}) \to \mathbf{Z}$$

on the smooth proper curve $B_{\mathbf{Q}}$. Here, because $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ is an algebraic stack, and B is a covering of this stack, it is each to get confused about "what units" the degree is measured in. *In this paper, we adopt the convention of expressing all degrees in "$\log(q)$" units.* In other words, in these units, the degree of the divisor at infinity of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ is equal to 1.

Theorem 5.8. *Let d be an odd (respectively, even) positive integer, and suppose that we are given functions*

$$g_\iota \in \mathbf{Func}(\mathbf{S}^{\mathbf{1}}_\iota)$$

for each $\iota \in \mathcal{I}_B$ such that $\psi_\iota \stackrel{\mathrm{def}}{=} g_\iota - (g_{\mathrm{st}})_\iota$ (respectively, $\psi_\iota \stackrel{\mathrm{def}}{=} g_\iota - (g^{\mathrm{ev}}_{\mathrm{st}})_\iota$) is d-invariant. Let

$$r_\iota \stackrel{\mathrm{def}}{=} \inf_{\alpha \in \mathbf{S}^{\mathbf{1}}_\iota} \ \psi_\iota(\alpha)$$

and

$$\overline{\mathcal{L}}_B \stackrel{\mathrm{def}}{=} \mathcal{O}_{E_{\infty,B}}(d[e] + \sum_\iota \ g_\iota);$$

$$(\overline{\mathcal{L}}_{\mathrm{st}})_B \stackrel{\mathrm{def}}{=} \mathcal{O}_{E_{\infty,B}}(d[e] + \sum_\iota \ (g_{\mathrm{st}})_\iota) \quad (\text{respectively, } (\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st}})_B \stackrel{\mathrm{def}}{=} \mathcal{O}_{E_{\infty,B}}(d[e] + \sum_\iota \ (g^{\mathrm{ev}}_{\mathrm{st}})_\iota))$$

Then the natural identification of push-forwards between $(f_B)_*\overline{\mathcal{L}}_B$ and $(f_B)_*(\overline{\mathcal{L}}_{\mathrm{st}})_B$ (respectively, $(f_B)_*(\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st}})_B$) over U_B extends to an equality

$$(f_B)_*\overline{\mathcal{L}}_B = (\prod_\iota \ q_\iota^{-r_\iota}) \cdot (f_B)_*(\overline{\mathcal{L}}_{\mathrm{st}})_B$$

$$\text{(respectively,} \quad (f_B)_*\overline{\mathcal{L}}_B = (\prod_{\iota} q_{\iota}^{-r_{\iota}}) \cdot (f_B)_*(\overline{\mathcal{L}}^{\text{ev}}_{\text{st}})_B \ )$$

(i.e., the metric on $(f_B)_*\overline{\mathcal{L}}_B$ *at* ι *is* $e^{-r_{\iota}}$ *times the metric on* $(f_B)_*(\overline{\mathcal{L}}_{\text{st}})_B$ *(respectively,* $(f_B)_*(\overline{\mathcal{L}}^{\text{ev}}_{\text{st}})_B$*)) over* S_∞*. Moreover, we have*

$$\deg((f_B)_*\overline{\mathcal{L}}_B) = -\frac{1}{24}(d-1) + d \cdot \sum_{\iota} r_{\iota}$$

where the degree is in "$\log(q)$*" units.*

Proof. The assertion concerning integral structures is a formal consequence of Theorem 5.6. Moreover, this assertion concerning integral structures allows one to immediately reduce the assertion concerning degrees to the "standard cases" of $(\overline{\mathcal{L}}_{\text{st}})_B$, $(\overline{\mathcal{L}}^{\text{ev}}_{\text{st}})_B$. In these cases, the assertion concerning the degree is an immediate consequence of Grothendieck-Riemann-Roch. We will carry out this computation in the following paragraph. Before doing this, however, we note that when working with metrized line bundles, one does not necessarily get the correct answer if one applies Grothendieck-Riemann-Roch to an arbitrary metrized line bundle, since for an arbitrary metrized line bundle, "$\mathbf{R}^1(f_B)_*$" is not well-defined (i.e., one might get various "analytic torsion effects"). This is why we reduced to the "standard cases" $(\overline{\mathcal{L}}_{\text{st}})_B$, $(\overline{\mathcal{L}}^{\text{ev}}_{\text{st}})_B$ which are (by Lemma 5.4), in fact, line bundles in the usual sense (whose $\mathbf{R}^1(f_B)_*$ is zero) over any covering of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ over which the d-torsion points of $E_{\infty,B}$ become rational.

Let us first treat the case of d *odd*. The case of d even is quite similar, and we will remark at the end of the proof what must be modified in this case.

By Grothendieck-Riemann-Roch, we have

$$\deg((f_B)_*\overline{\mathcal{L}}_B) = \deg(\mathbf{R}(f_{C_B})_*\mathcal{O}_{C_B}) + \frac{1}{2}[\overline{\mathcal{L}}_B] \cdot ([\overline{\mathcal{L}}_B] - [\omega_E])$$

(where $f_{C_B} : C_B \to B$ is the structure morphism; we take $\overline{\mathcal{L}}_B \stackrel{\text{def}}{=} (\overline{\mathcal{L}}_{\text{st}})_B$; and we use brackets "$[-]$" to denote the Chern class of a line bundle). Here, ω_E denotes the line bundle on B given by considering the invariant differentials on E_B. Then Serre duality gives:

$$\deg(\mathbf{R}(f_{C_B})_*\mathcal{O}_{C_B}) = \deg(\omega_E)$$

Next, we compute the various intersection numbers that appear by using the intersection theory of compactified divisors reviewed in §4. First of all, since ω_E is a line bundle on B (as opposed to C_B), and $\omega_E^{\otimes 12}$ is well-known to be isomorphic to the line bundle associated to the divisor at infinity, we have

$$[\overline{\mathcal{L}}_B] \cdot [\omega_E] = d \cdot \deg(\omega_E) = \frac{d}{12}$$

Next, we have

$$[\overline{\mathcal{L}}_B] \cdot [\overline{\mathcal{L}}_B] = d \cdot [\overline{\mathcal{L}}_B] \cdot [e] + \int_{\mathbf{S}^1} g_{\mathrm{st}} \cdot h_{\overline{\mathcal{L}}_{\mathrm{st}}}$$

On the other hand,

$$\begin{aligned}[\overline{\mathcal{L}}_B] \cdot [e] &= d \cdot [e]^2 + g_{\mathrm{st}}(0) = -d \cdot \deg(\omega_E) + d \cdot \phi_1(0) - \phi_d(0) \\ &= -d \cdot (\frac{1}{12}) + (d - \frac{1}{d}) \cdot \frac{1}{12} = -\frac{1}{12d}\end{aligned}$$

while

$$\int_{\mathbf{S}^1} g_{\mathrm{st}} \cdot h_{\overline{\mathcal{L}}_{\mathrm{st}}} = \sum_{i=0}^{d-1} g_{\mathrm{st}}(\frac{i}{d}) = d \cdot \phi_d(0) - d \cdot \phi_d(0) = 0$$

Thus, we obtain that $[\overline{\mathcal{L}}_B]^2 = -\frac{1}{12}$, so

$$\deg((f_B)_* \overline{\mathcal{L}}_B) = \frac{1}{2} \cdot (2 - 1 - d) \cdot \frac{1}{12} = -\frac{1}{24}(d-1)$$

as desired.

Finally, we consider the case of d *even*. In this case, we take $\overline{\mathcal{L}}_B \stackrel{\mathrm{def}}{=} (\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st}})_B$. It is then immediate that

$$[\overline{\mathcal{L}}_B] \cdot [\omega_E] = \frac{d}{12}$$

(just as in the odd case). Thus the only intersection number that remains to be computed is $[\overline{\mathcal{L}}_B]^2$.

We have

$$[\overline{\mathcal{L}}_B] \cdot [\overline{\mathcal{L}}_B] = d \cdot [\overline{\mathcal{L}}_B] \cdot [e] + \int_{\mathbf{S}^1} g^{\mathrm{ev}}_{\mathrm{st}} \cdot h_{\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st}}}$$

Moreover,

$$
\begin{aligned}
[\overline{\mathcal{L}}_B] \cdot [e] = d \cdot [e]^2 + g_{\mathrm{st}}^{\mathrm{ev}}(0) &= -d \cdot \deg(\omega_E) + d \cdot \phi_1(0) - \phi_d^{\mathrm{ev}}(0) \\
&= -d \cdot \deg(\omega_E) + d \cdot \phi_1(0) - \phi_d(\frac{1}{2d}) \\
&= -d \cdot (\frac{1}{12}) + (d + \frac{1}{2d}) \cdot \frac{1}{12} = \frac{1}{24d}
\end{aligned}
$$

while

$$
\begin{aligned}
\int_{\mathbf{S}^1} g_{\mathrm{st}}^{\mathrm{ev}} \cdot h_{\overline{\mathcal{L}}_{\mathrm{st}}^{\mathrm{ev}}} &= \sum_{i=0}^{d-1} g_{\mathrm{st}}^{\mathrm{ev}}(\frac{1}{2d} + \frac{i}{d}) \\
&= d \cdot \phi_d(\frac{1}{2d}) - d \cdot \phi_d(0) = \phi_1(\frac{1}{2}) - \phi_1(0) = -\frac{1}{24} - \frac{1}{12}
\end{aligned}
$$

Thus, $[\overline{\mathcal{L}}_B]^2 = d \cdot (\frac{1}{24d}) - \frac{1}{24} - \frac{1}{12} = -\frac{1}{12}$ (just as in the odd case), as desired. This completes the proof in the even case. $\bigcirc$

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Chapter V: The Evaluation Map

§0. Introduction

In this Chapter, we prepare for the proof of the *Comparison Isomorphism* (which is the main topic of this paper) in Chapter VI by studying the morphism which is to be the "comparison isomorphism." We refer to this morphism as the *evaluation map.* Our study of the evaluation map consists of several parts. First, we must set up the notation and define the evaluation map. This is done in §1, 2, 3. The evaluation map essentially consists of *restricting global sections of a certain natural metrized line bundle on the universal extension of an elliptic curve to the torsion points of the elliptic curve.* In §1, we discuss the definition of the natural metrized line bundles that we use, and in §2, we construct and study the elementary properties of the "evaluation map." In §3, we observe that the *"étale-integral structures"* defined in Chapter III, §6, are defined for arbitrary elliptic curves (i.e., not just for degenerating elliptic curves, as in the discussion of Chapter III, §6), and, moreover, that the functions in these étale-integral structures *assume integral values* at the torsion points appearing in the definition of the evaluation map. This integrality property will be important in the proof of the comparison isomorphism in Chapter VI.

After we have defined the evaluation map, we commence our study of *the extent to which it is an isomorphism.* We begin, in §4, by studying the extent to which a certain version of the evaluation map defined for degenerating elliptic curves is an isomorphism modulo various powers of the q-parameter. This essentially amounts to studying the extent to which *the twisted Schottky-Weierstrass zeta functions of Chapter IV, §3, satisfy linear relations modulo various powers of the q-parameter.* It turns out that the calculations of linear relations that we carry out in this § will yield the *key technical machinery* behind the main result of this paper. Next, in §5, we study *the determinant of the evaluation map* in the case when the base of the spectrum of an algebraically closed field. We then use the theory of §5 to determine, in §6, precisely when the evaluation map is an isomorphism for *"sufficiently generic"* elliptic curves.

§1. Construction of Certain Metrized Line Bundles

In this §, we continue the discussion of Chapter IV, §4,5, on metrized line bundles. In particular, *we introduce certain specific metrized line bundles* which we will need in order to construct the *evaluation maps* of §2 below. These metrized line bundles $\overline{\mathcal{L}}$ are *natural* in the sense that (it is not difficult to show that) they may be uniquely characterized (up to tensor product with a line bundle pulled back from the base) by the following two conditions:

(i.) They are *symmetric up to torsion* (i.e., the pull-back of $\overline{\mathcal{L}}$ via the "multiplication by -1" map of an elliptic curve differs from $\overline{\mathcal{L}}$ by a line bundle which defines a torsion element of the Picard group), of *relative degree d.*

(ii.) Their *curvatures* (at infinity) are invariant with respect to and "concentrated at" (in the sense of "delta functions") the d-torsion points to which we will restrict them when we consider "evaluation maps" in §2.

Condition (ii.) is natural considering that *the main purpose of constructing these line bundles is to restrict them (and their sections) to the subscheme of d-torsion points of an elliptic curve.* In fact, the "concentrated at" part of Condition (ii.) is inessential in the sense that, by Chapter IV, Theorem 5.6, even if the curvature is not concentrated at the d-torsion points, the *push-forward* of the metrized line bundle – which is what we are ultimately interested in – is the same as that of the "maximal metrized subsheaf inside the original line bundle among those subsheaves which *are* concentrated at the d-torsion points."

Let $m \geq 1$ be an integer. We would like to consider the *moduli* of pairs

$$(E \to S, \eta \in E(S))$$

where $E \to S$ is an elliptic curve, and $\eta \in E(S)$ *generates a cyclic subgroup of order* m in E. Over $\mathbf{Z}[\frac{1}{m}]$, there is no problem in defining an algebraic stack of such pairs. Moreover, it follows from the theory of [KM], Chapter 5 (cf. especially Theorem 5.1.1), that this stack extends (uniquely) to a finite, flat covering

$$B \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$$

of proper algebraic stacks over $\mathbf{Z}$ which is étale over $\mathbf{Z}[\frac{1}{m}]$ away from the divisor at infinity; near infinity, is obtained by adjoining some $q^{\frac{1}{r}}$, where r divides m to the ring of integers in some cyclotomic extension of $\mathbf{Q}$; and, at primes dividing m, parametrizes "points of exact order m." Moreover, the stack B is *regular.* In particular, *B satisfies the hypotheses of the discussion following Chapter IV, Corollary 5.7.*

Next, let d be a *positive integer.* Then let us recall the *metrized line bundles*

$$(\overline{\mathcal{L}}_{\mathrm{st}})_B \stackrel{\mathrm{def}}{=} \mathcal{O}_{E_{\infty,B}}(d[e] + \sum_{\iota} (g_{\mathrm{st}})_\iota)$$

$$(\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st}})_B \stackrel{\mathrm{def}}{=} \mathcal{O}_{E_{\infty,B}}(d[e] + \sum_{\iota} (g^{\mathrm{ev}}_{\mathrm{st}})_\iota)$$

(where e is the origin of the group object $f_B : E_{\infty,B} \to B_\infty$; ι ranges over the connected components of the divisor at infinity of B; and $g_{\mathrm{st}} = d \cdot \phi_1 - \phi_d$, $g^{\mathrm{ev}}_{\mathrm{st}} = d \cdot \phi_1 - \phi^{\mathrm{ev}}_d$ are the "standard Green's functions" studied in Chapter IV, §4,5) of Chapter IV, Theorem 5.8. Recall from the theory of Chapter IV, §4,5 (especially Chapter IV, Lemma 5.4), that on each $\mathbf{S}^{\mathbf{1}}_\iota$ (i.e., the "limit" of the set of irreducible components of the semi-stable models over the component at infinity ι), $(\overline{\mathcal{L}}_{\mathrm{st}})_B$ (respectively, $(\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st}})_B$) looks like the (line bundle associated to) the divisor

$$\sum_{i=0}^{d-1} [\frac{i}{d}] \quad \text{(respectively, } \sum_{i=0}^{d-1} [(\frac{1}{2d}) + (\frac{i}{d})] \text{)}$$

on $\mathbf{S}^{\mathbf{1}}_\iota$. Finally, in Chapter IV, Theorem 5.8, we showed that if d is *odd* (respectively, *even*), then

$$\deg((f_B)_*(\overline{\mathcal{L}}_{\mathrm{st}})_B) = -\frac{1}{24}(d-1) \quad \text{(respectively, } \deg((f_B)_*(\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st}})_B) = -\frac{1}{24}(d-1) \text{)}$$

in "$\log(q)$" units.

Next, let us observe that from the definition of the moduli stack B, we have a *tautological m-torsion point*

$$\eta \in E_{\infty,B}(B)$$

For each ι, one then has a divisor

$$\rho_\iota(\eta) \in \mathrm{Div}(\mathbf{S}^{\mathbf{1}}_\iota)$$

given by the unique point of $(\mathbf{Q}/\mathbf{Z})_\iota \subseteq \mathbf{S}^{\mathbf{1}}_\iota$ defined by η at ι. Let us denote this unique point by $\eta_\iota \in (\mathbf{Q}/\mathbf{Z})_\iota \subseteq \mathbf{S}^{\mathbf{1}}_\iota$. Thus, $\rho_\iota(\eta) = [\eta_\iota]$. Now, if d is *odd* (respectively, *even*), then let us consider the divisor

$$\Big(\sum_{i=0}^{d-1} [\eta_\iota + \frac{i}{d}]\Big) - \Big(\sum_{i=0}^{d-1} [\frac{i}{d}]\Big) \in \mathrm{Div}(\mathbf{S}^{\mathbf{1}}_\iota)$$

$$\text{(respectively, } \Big(\sum_{i=0}^{d-1} [\eta_\iota + \frac{1}{2d} + \frac{i}{d}]\Big) - \Big(\sum_{i=0}^{d-1} [\frac{i}{d}]\Big) \in \mathrm{Div}(\mathbf{S}^{\mathbf{1}}_\iota) \text{)}$$

Let us write ϕ_{η_ι} (respectively, $\phi^{\mathrm{ev}}_{\eta_\iota}$) for the *Green's function associated to this divisor* (cf. the discussion preceding Chapter IV, Proposition 4.4). Thus, $\Delta(\phi_{\eta_\iota}) = -\frac{\partial^2 \phi_{\eta_\iota}}{\partial \theta^2}$ (where

θ is the standard coordinate on $\mathbf{S}^1_\iota$) is equal to the delta distribution associated to this divisor. Moreover, by Chapter IV, Corollary 4.5, the function ϕ_{η_ι} is *piecewise linear* on $\mathbf{S}^1_\iota$, *d-invariant* (i.e., $\phi_{\eta_\iota}(\theta + \frac{1}{d}) = \phi_{\eta_\iota}(\theta)$), and attains its maximum at the points $\eta_\iota + \frac{i}{d}$ (respectively, $\eta_\iota + \frac{1}{2d} + \frac{i}{d}$), and its minimum at the points $\frac{i}{d}$, for $i = 0, \ldots, d-1$. Indeed, it follows immediately from the definitions that

$$\phi_{\eta_\iota}(\theta) = \frac{1}{d}\phi_{d\cdot\eta_\iota,0}(d\cdot\theta) \quad (\text{respectively, } \phi^{\text{ev}}_{\eta_\iota}(\theta) = \frac{1}{d}\phi_{d\cdot\eta_\iota+\frac{1}{2},0}(d\cdot\theta)\ )$$

where $\phi_{??,0}(\theta)$ is the function considered in Chapter IV, Corollary 4.5. Let us define

$$\psi_{\eta_\iota}(\theta) \stackrel{\text{def}}{=} \phi_{\eta_\iota}(\theta) - \phi_{\eta_\iota}(0) \quad (\text{respectively, } \psi^{\text{ev}}_{\eta_\iota}(\theta) \stackrel{\text{def}}{=} \phi^{\text{ev}}_{\eta_\iota}(\theta) - \phi^{\text{ev}}_{\eta_\iota}(0)\ )$$

Thus, $\psi_{\eta_\iota} \geq 0$ (respectively, $\psi^{\text{ev}}_{\eta_\iota} \geq 0$) on $\mathbf{S}^1_\iota$; $\psi_{\eta_\iota}(0) = 0$ (respectively, $\psi^{\text{ev}}_{\eta_\iota}(0) = 0$).

We are now ready to define the *metrized line bundles* which are the main topic of the present §. Let

$$(\overline{\mathcal{L}}_{\text{st},\eta})_B \stackrel{\text{def}}{=} \{\mathcal{T}^*_\eta((\overline{\mathcal{L}}_{\text{st}})_B)\} \otimes_{\mathcal{O}_{E_{\infty,B}}} \mathcal{O}_{E_{\infty,B}}(\sum_\iota \psi_{\eta_\iota})$$

$$(\text{respectively, } (\overline{\mathcal{L}}^{\text{ev}}_{\text{st},\eta})_B \stackrel{\text{def}}{=} \{\mathcal{T}^*_\eta((\overline{\mathcal{L}}^{\text{ev}}_{\text{st}})_B)\} \otimes_{\mathcal{O}_{E_{\infty,B}}} \mathcal{O}_{E_{\infty,B}}(\sum_\iota \psi^{\text{ev}}_{\eta_\iota})\ )$$

(where $\mathcal{T}_\eta : E_{\infty,B} \to E_{\infty,B}$ is the morphism given by translation by $\eta \in E_{\infty,B}(B)$). Note that the *curvature* of this metrized line bundle at ι is given by:

$$\delta_{\left(\sum_{i=0}^{d-1}[\eta_\iota+\frac{i}{d}]\right)} - \Delta(\psi_{\eta_\iota}) = \delta_{\left(\sum_{i=0}^{d-1}[\frac{i}{d}]\right)}$$

$$(\text{respectively, } \delta_{\left(\sum_{i=0}^{d-1}[\eta_\iota+\frac{1}{2d}+\frac{i}{d}]\right)} - \Delta(\psi^{\text{ev}}_{\eta_\iota}) = \delta_{\left(\sum_{i=0}^{d-1}[\frac{i}{d}]\right)}\ )$$

That is to say, *although, over U_S, $\overline{\mathcal{L}}_{\text{st},\eta}$ (respectively, $\overline{\mathcal{L}}^{\text{ev}}_{\text{st},\eta}$) is "twisted," i.e., differs from $(\overline{\mathcal{L}}_{\text{st}})_B$ (respectively, $(\overline{\mathcal{L}}^{\text{ev}}_{\text{st},\eta})_B$) by translation by η, "metrically speaking" (i.e., in the special fibers at infinity) it looks as though it was never twisted by translation by η (respectively, or even by the "$\frac{1}{2d}$'s" on the $\mathbf{S}^1_\iota$'s at infinity).*

Now Chapter IV, Theorems 5.6, 5.8, imply the following:

Proposition 1.1. *If d is odd (respectively, even), then the automorphism of $E_{\infty,B}$ given by translation by η induces a natural equality:*

$$(f_B)_*(\overline{\mathcal{L}}_{\mathrm{st},\eta})_B = (f_B)_*(\overline{\mathcal{L}}_{\mathrm{st}})_B \quad \text{(respectively, } (f_B)_*(\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta})_B = (f_B)_*(\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st}})_B \text{)}$$

of metrized vector bundles on B_∞*. In particular,* $\deg((f_B)_*(\overline{\mathcal{L}}_{\mathrm{st},\eta})_B) = -\frac{1}{24}(d-1)$ *(respectively,* $\deg((f_B)_*(\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta})_B) = -\frac{1}{24}(d-1)$*).*

Remark. We note here, for future reference, that Proposition 1.1 implies that

$$d \cdot \Big\{ -\frac{1}{12}(\sum_{i=0}^{d-1} i) + \deg((f_B)_*(\overline{\mathcal{L}}_{\mathrm{st},\eta})_B)\Big\} = -d \cdot \frac{1}{24}\{d(d-1) + (d-1)\} = -\frac{1}{24}d(d^2-1)$$

if d is *odd* (respectively,

$$d \cdot \Big\{ -\frac{1}{12}(\sum_{i=0}^{d-1} i) + \deg((f_B)_*(\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta})_B)\Big\} = -d \cdot \frac{1}{24}\{d(d-1) + (d-1)\} = -\frac{1}{24}d(d^2-1)$$

if d is *even*). This computation will be of fundamental importance in the proof of the *comparison isomorphism* in Chapter VI.

Note, moreover, that since the ψ_{η_ι}'s are *d-invariant*, one can glue together the theories of *theta groups* discussed in Chapter IV, §1, and Chapter IV, §5, to obtain a theta group $\mathcal{G}_{(\overline{\mathcal{L}}_{\mathrm{st}})_B} = \mathcal{G}_{(\overline{\mathcal{L}}_{\mathrm{st},\eta})_B}$ (if d is *odd*) which fits into an exact sequence

$$1 \to (\mathbf{G}_{\mathrm{m}})_B \to \mathcal{G}_{(\overline{\mathcal{L}}_{\mathrm{st}})_B} = \mathcal{G}_{(\overline{\mathcal{L}}_{\mathrm{st},\eta})_B} \to K_{(\overline{\mathcal{L}}_{\mathrm{st}})_B} = K_{(\overline{\mathcal{L}}_{\mathrm{st},\eta})_B} \to 1$$

where $K_{(\overline{\mathcal{L}}_{\mathrm{st}})_B} = K_{(\overline{\mathcal{L}}_{\mathrm{st},\eta})_B}$ is the kernel of multiplication by d on $E_{\infty,B}$. Naturally, the equality of metrized vector bundles in Proposition 1.1 is, in fact, an equality of modules over this group scheme $\mathcal{G}_{(\overline{\mathcal{L}}_{\mathrm{st}})_B} = \mathcal{G}_{(\overline{\mathcal{L}}_{\mathrm{st},\eta})_B}$. Similar statements hold for $(\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st}})_B$, $(\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta})_B$, if d is *even*.

Next, let us compute some *intersection numbers* (cf. the theory of Chapter IV, §4,5).

Proposition 1.2. *Suppose that* τ *is any* d-torsion point $\in E_{\infty,B}(T)$ *(i.e.,* $d \cdot \tau = 0$*) over some* B*-scheme* T*, where* $T \to B$ *is a finite, flat covering that satisfies the same hypotheses as* B *(i.e., the hypotheses in the discussion following Chapter IV, Corollary 5.7). Suppose that* d *is* odd*; then we have*

$$[\tau] \cdot (\overline{\mathcal{L}}_{\mathrm{st},\eta})_B = -\frac{1}{d} \sum_{\iota} \phi_1(-d \cdot \eta_\iota)$$

i.e.,

$$[\tau] \cdot (\overline{\mathcal{L}}_{\mathrm{st},\eta})_B = 0$$

if m does not *divide d;*

$$[\tau] \cdot (\overline{\mathcal{L}}_{\mathrm{st},\eta})_B = -\frac{1}{12d}$$

(in "$\log(q)$" units) if m divides d. Suppose that d is even*; then we have*

$$[\tau] \cdot (\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta})_B = -\frac{1}{d} \sum_{\iota} \phi_1(-d \cdot \eta_\iota + \frac{1}{2})$$

(in "$\log(q)$" units).

Proof. For simplicity, we shall assume that d is *odd*. The even case is entirely similar. The only difference is that, unlike in the odd case, the sum that one obtains in the even case cannot be described in "closed form" (i.e., "$= 0$ if m does not divide d, $= -\frac{1}{12d}$ if m divides d").

By the above discussion on theta groups, it follows that for some line bundle $\mathcal{N}$ on T such that $\mathcal{N}^{\otimes d} \cong \mathcal{O}_T$, we have:

$$\mathcal{T}^*_\tau((\overline{\mathcal{L}}_{\mathrm{st},\eta})_B) = ((\overline{\mathcal{L}}_{\mathrm{st},\eta})_B) \otimes_{\mathcal{O}_T} \mathcal{N}$$

Since $[\tau] \cdot [\mathcal{N}] = 0$, it thus suffices to consider the case $T = B$; $\tau = e$ (i.e., the origin of $E_{\infty,B}$). Since $\psi_{\eta_\iota}(0) = 0$, it follows that

$$\begin{aligned}
[e] \cdot [(\overline{\mathcal{L}}_{\mathrm{st},\eta})_B] &= [e] \cdot \mathcal{T}^*_\eta[(\overline{\mathcal{L}}_{\mathrm{st}})_B] = [-\eta] \cdot [(\overline{\mathcal{L}}_{\mathrm{st}})_B] \\
&= d \cdot [-\eta] \cdot [e] + \sum_{\iota} [-\eta_\iota] \cdot (d\phi_1 - \phi_d) \\
&= d \cdot \left\{ [-\eta] \cdot [e] + \sum_{\iota} [-\eta_\iota] \cdot \phi_1 \right\} - \frac{1}{d} \cdot \sum_{\iota} [-d \cdot \eta_\iota] \cdot \phi_1)
\end{aligned}$$

(where in the final equality, we use the fact that $\phi_d(\theta) = \frac{1}{d} \cdot \phi_1(d \cdot \theta)$ (cf. Chapter IV, Corollary 4.6)).

Next, let us observe that since the curvature at each ι of the metrized line bundle $\mathcal{O}_{E_{\infty,B}}(e + \phi_1)$ is (constant, hence) m-invariant, it follows (cf. Chapter IV, Proposition 5.1) that the m^2-th power of $\mathcal{O}_{E_{\infty,B}}(e + \phi_1)$ is *stabilized* by translation by η and $d \cdot \eta$ (cf. the discussion at the beginning of the proof involving the line bundle "$\mathcal{N}$"). On the other hand,

$$[\mathcal{O}_{E_{\infty,B}}(e + \phi_1)] \cdot [e] = [e]^2 + \phi_1(0) = -\frac{1}{12} + \phi_1(0) = 0$$

Thus, we obtain that

$$0 = [\mathcal{O}_{E_{\infty,B}}(e + \phi_1)] \cdot [-\eta] = [\mathcal{O}_{E_{\infty,B}}(e + \phi_1)] \cdot [-d \cdot \eta]$$

This implies, in particular, that the expression in large brackets in the above equalities concerning $[e] \cdot [(\overline{\mathcal{L}}_{\mathrm{st},\eta})_B]$ *vanishes.*

Thus, in summary, we have:

$$\begin{aligned} [e] \cdot [(\overline{\mathcal{L}}_{\mathrm{st},\eta})_B] &= -\frac{1}{d} \cdot \sum_{\iota} \ [-d \cdot \eta_\iota] \cdot \phi_1 \\ &= -\frac{1}{d} \cdot \sum_{\iota} \ \phi_1(-d \cdot \eta_\iota) \\ &= \frac{1}{d}[e] \cdot [-d \cdot \eta] \end{aligned}$$

But note that $-d \cdot \eta = e$ if and only if m divides d. Moreover, since $-d \cdot \eta$ is a *torsion point*, it follows that $[e] \cdot [-d \cdot \eta]$ is 0 if $-d \cdot \eta \neq e$, and $-\frac{1}{12}$ if $-d \cdot \eta = e$. This completes the proof. $\bigcirc$

Finally, we note the following consequence of the above discussion: Suppose that $m = 2d$. Write $\overline{\mathcal{L}}$ for $(\overline{\mathcal{L}}_{\mathrm{st},\eta})_B$ (respectively, $(\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta})_B$) if d is *odd* (respectively, *even*). Note that the fact that $m = 2d$ implies that $\overline{\mathcal{L}}$ is *symmetric.* Thus, we can use the action of the theta group $\mathcal{G}_{\overline{\mathcal{L}}}$ to construct *algebraic theta functions* (cf. Chapter IV, §1; Chapter IV, the discussion following Corollary 5.3), as follows. If $\alpha \in E_{\infty,B}(T)$ is any T-valued point for a B-scheme T as in Proposition 1.2, then we get an *isomorphism* (of metrized vector bundles on T_∞)

$$\overline{\mathcal{L}}|_{\mathcal{T}^*_\alpha K_{\overline{\mathcal{L}}}} \cong (\overline{\mathcal{L}}|_{\alpha^0}) \otimes_{\mathcal{O}_{K^0_{\overline{\mathcal{L}}}}} \mathcal{O}_{K_{\overline{\mathcal{L}}}}$$

(where $K^0_{\overline{\mathcal{L}}} \stackrel{\mathrm{def}}{=} K_{\overline{\mathcal{L}}}/2 \cdot K_{\overline{\mathcal{L}}}$, and $\overline{\mathcal{L}}|_{\alpha^0}$ is the metrized line bundle on $K^0_{\overline{\mathcal{L}}} \times_B T$ obtained by descending $\overline{\mathcal{L}}|_{\mathcal{T}^*_\alpha K_{\overline{\mathcal{L}}}}$). Thus, for instance, if d is odd, then we obtain a *trivialization* of the restriction of $\overline{\mathcal{L}}$ to the subscheme $\mathcal{T}_\alpha^* K_{\overline{\mathcal{L}}}$. In particular, *restriction to* $\mathcal{T}_\alpha^* K_{\overline{\mathcal{L}}}$ defines (by composing with the above isomorphism) a morphism of metrized vector bundles on T_∞ with $\mathcal{G}_{\overline{\mathcal{L}}}$-action:

$$(f_B)_* \overline{\mathcal{L}}|_T \to (\overline{\mathcal{L}}|_{\alpha^0}) \otimes_{\mathcal{O}_{K^0_{\overline{\mathcal{L}}}}} \mathcal{O}_{K_{\overline{\mathcal{L}}}}$$

For instance, if $T = B$, and $\alpha = e$, then we get a morphism

$$(f_B)_* \overline{\mathcal{L}} \to (\overline{\mathcal{L}}|_{e^0}) \otimes_{\mathcal{O}_{K^0_{\overline{\mathcal{L}}}}} \mathcal{O}_{K_{\overline{\mathcal{L}}}}$$

i.e., this morphism allows us to think of global sections of $\overline{\mathcal{L}}$ over $E_{\infty,B}$ as being (essentially) *functions* on the B-finite scheme $K_{\overline{\mathcal{L}}}$.

§2. The Definition of the Evaluation Map

In this §, we set up the morphism which will be the principal topic of the *comparison isomorphism* to be proven in Chapter VI. Roughly speaking, this morphism is the morphism given by *evaluating global sections of an ample line bundle on the universal extension of a (log) elliptic curve at the torsion points of the universal extension.*

Let $d \geq 1$ be an integer. Suppose that $S^{\log}$ is a fine noetherian scheme, and let

$$C^{\log} \to S^{\log}$$

be a *log elliptic curve* over $S^{\log}$ such that the "divisor at infinity" $D \subseteq S$ (i.e., the pull-back of the divisor at infinity of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ via the classifying morphism $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$) is a *Cartier divisor* (i.e., locally defined by a non-zero divisor) on S. Also, let us assume that étale locally on the completion of S along D, *the pull-back of the Tate parameter q to this completion admits a d-th root.*

Next, observe that by pulling back the corresponding objects in the universal case, we obtain

$$E \to S; \quad E_{\infty,S} \to S_\infty$$

(cf. Chapter IV, §4,5). By considering objects which are (étale locally on S) obtained by pulling back metrized line bundles on the tautological $E_{\infty,\overline{\mathcal{M}}}$ over $\overline{\mathcal{M}} \stackrel{\text{def}}{=} (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ (cf. the theory of Chapter IV, §4,5), we obtain a notion of "metrized line bundles on $E_{\infty,S}$." We leave the routine details of the precise formulation of this notion to the reader. Let

$$\overline{\mathcal{L}}$$

be a *metrized line bundle* on $E_{\infty,S}$ of relative degree d whose *curvatures* at all the components of D are *d-invariant* (cf. Chapter IV, Proposition 5.1). In fact, ultimately we will mainly be interested in the cases where S is a covering B (as in the discussion following Chapter IV, Corollary 5.7), or an object derived from such a B (i.e., a completion/localization of such a B; a point of such a B, etc.), and where $\overline{\mathcal{L}}$ is one of the bundles "$\overline{\mathcal{L}}_{\mathrm{st},\eta}$" or "$\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta}$" of §1 (or the pull-back of this bundle to an object derived from such a B).

Next, we introduce some new notation. Recall the *universal extension* $E^\dagger \to E$ of the log elliptic curve E (cf. Chapter III, Definition 1.2). Recall that the S-group scheme $E^\dagger$ is an extension of E by the affine group scheme W_E corresponding to the line bundle ω_E. Then for $n \geq 1$ an integer, let us denote by

$$E^\dagger_{[n]} \stackrel{\mathrm{def}}{=} \varinjlim \begin{pmatrix} W_E & \longrightarrow & E^\dagger \\ \big\downarrow n\cdot & & \\ W_E & & \end{pmatrix}$$

the W_E-torsor $E^\dagger_{[n]} \to E$ obtained by applying a "push-out" to this extension via the morphism $n\cdot : W_E \to W_E$ given by multiplication by n.

Note that it follows from the definitions that the morphism $d\cdot : E^\dagger \to E^\dagger$ ("multiplication by d") *factors through* $E^\dagger_{[d]}$. Thus, we get a *homomorphism*

$$E^\dagger_{[d]} \to E^\dagger$$

of group schemes over S which induces an *isomorphism* (since $d\cdot$ induces multiplication by d on W_E) between $W_E \subseteq E^\dagger_{[d]}$ and $W_E \subseteq E^\dagger$. Put another way, this morphism exhibits the W_E-torsor $E^\dagger_{[d]}$ as the pull-back to E via the multiplication by d map on E of the W_E-torsor $E^\dagger$. Moreover, since $E^\dagger$ extends naturally to a W_E-torsor $E^\dagger_C \to C$ (Chapter III, Corollary 4.3), and the morphism "multiplication by d" on E extends naturally to a morphism $C_d \to C$ (i.e., C_d is the semi-stable model of E over S with d irreducible components in the special fiber – cf. the notation of Chapter IV, §4), we thus obtain (by pulling back $E^\dagger_C \to C$ via $C_d \to C$) a *natural W_E-torsor*

$$E^\dagger_{C_d,[d]} \to C_d$$

extending the W_E-torsor $E^\dagger_{[d]} \to E$ over $E \subseteq C_d$. Note that the restriction

$$E^{\dagger}_{E_d,[d]} \to E_d$$

of $E^{\dagger}_{C_d,[d]}$ to $E_d \subseteq C_d$ (i.e., the complement of the nodes of C_d – cf. the notation of Chapter IV, §4) has a natural structure of *group scheme* over S with respect to which the homomorphism $E^{\dagger}_{[d]} \to E^{\dagger}$ considered above extends to a homomorphism $E^{\dagger}_{E_d,[d]} \to E^{\dagger}$. On the other hand, since $E_{\infty,S}$ is obtained by removing the nodes of an object $C_{\infty,S}$, which, in turn, is obtained by blowing up C_d, we thus see that the W_E-torsor $E^{\dagger}_{C_d,[d]} \to C_d$ restricts to an object $E^{\dagger}_{C_{\infty,S},[d]} \to C_{\infty,S}$, which, in turn, restricts to an object

$$E^{\dagger}_{\infty,[d]} \stackrel{\text{def}}{=} E^{\dagger}_{E_{\infty,S},[d]} \to E_{\infty,S}$$

(which extends $(E^{\dagger}_{E_d,[d]} \to E_d) \times_S S_{\infty}$).

Let us denote the kernel of the homomorphism $E_d \to E$ (i.e., the open version of the morphism $C_d \to C$ considered above) by

$${}_dE \subseteq E_d$$

It follows from the assumption concerning the existence of roots of the Tate parameter that ${}_dE$ forms a *finite, flat group scheme* over S.

Lemma 2.1. *There is a natural subgroup scheme*

$${}_dE^{\dagger} \subseteq E^{\dagger}_{C_d,[d]}$$

which maps isomorphically to ${}_dE \subseteq C_d$.

Proof. Indeed, observe that the homomorphism $E^{\dagger}_{E_d,[d]} \to E^{\dagger}$ fits into the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_E & \longrightarrow & E^{\dagger}_{E_d,[d]} & \longrightarrow & E_d & \longrightarrow & 0 \\
 & & \Big\downarrow \mathrm{id} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & W_E & \longrightarrow & E^{\dagger} & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

where the vertical arrow on the left is the identity. Thus, we conclude that $E^{\dagger}_{E_d,[d]} \to E^{\dagger}$ is finite and flat, and that its kernel maps isomorphically down to the kernel of $E_d \to E$, which is simply ${}_dE$ (cf. the discussion at the beginning of Chapter IV, §3). ○

We are now ready to define the *evaluation map*, which is the main topic of this §, and, indeed, of this paper. By abuse of notation, let us write f_S for all *structure morphisms of objects over S_∞ down to S_∞*. If $n \geq 1$ is an integer, let us denote by

$$(f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<n}$$

the subsheaf of $(f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})$ consisting of those sections whose *torsorial degree* $< n$ (cf. Chapter III, Definition 2.2). Thus, $(f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<n}$ forms a *metrized vector bundle on S_∞ of rank $d \cdot n$* (cf. the theory of Chapter IV, §5). Moreover, this metrized bundle admits a *natural action of the theta group $\mathcal{G}_{\overline{\mathcal{L}}}$ associated to $\overline{\mathcal{L}}$.*

On the other hand, for $\alpha \in E^\dagger(S) \subseteq E^\dagger_{\infty,[d]}(S_\infty)$, let us consider the push-forward to S_∞ of the restriction of the metrized line bundle $\overline{\mathcal{L}}$ on $E_{\infty,S}$ to the (result of base-changing by $S_\infty \to S$ the) subscheme $(E^\dagger_{E_d,[d]} \supseteq)\ \mathcal{T}^*_\alpha({}_dE^\dagger) \cong \mathcal{T}^*_\alpha({}_dE) \subseteq E_d$:

$$(f_S)_*(\overline{\mathcal{L}}|_{\mathcal{T}^*_\alpha({}_dE^\dagger_\infty)})$$

(where ${}_dE^\dagger_\infty \stackrel{\text{def}}{=} {}_dE^\dagger \times_S S_\infty$; "$\mathcal{T}_{??}$" denotes "translation by ??"). One sees easily that $\overline{\mathcal{L}}|_{\mathcal{T}^*_\alpha({}_dE^\dagger_\infty)}$ forms a *metrized vector bundle on S_∞ of rank d^2*. Moreover, since the action of $\mathcal{G}_{\overline{\mathcal{L}}}$ on $\overline{\mathcal{L}}$ covers the action of $K_{\overline{\mathcal{L}}} = {}_dE$ on $\mathcal{T}^*_\alpha({}_dE)$, we thus see that this metrized vector bundle admits a *natural action of the theta group $\mathcal{G}_{\overline{\mathcal{L}}}$*.

Proposition 2.2. *Under the above conditions, restriction of sections over $E^\dagger_{\infty,[d]}$ to the subscheme $\mathcal{T}^*_\alpha({}_dE^\dagger_\infty) \subseteq E^\dagger_{\infty,[d]}$ defines a natural morphism*

$$\Xi_{\overline{\mathcal{L}},d,\alpha} : (f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d} \to (f_S)_*(\overline{\mathcal{L}}|_{\mathcal{T}^*_\alpha({}_dE^\dagger_\infty)})$$

of metrized vector bundles of rank d^2 on S_∞ which is compatible with the actions of the theta group $\mathcal{G}_{\overline{\mathcal{L}}}$ on both sides. (When the various subscripts of Ξ are fixed in a discussion, they will often be omitted, especially in the case where α is taken to be the origin.)

> *The main purpose of this paper is to investigate the extent to which this restriction morphism $\Xi_{\overline{\mathcal{L}},d,\alpha}$ is an isomorphism.*

Next, suppose that

$$H \subseteq \mathcal{G}_{\overline{\mathcal{L}}}$$

is a *Lagrangian subgroup scheme* (cf. Chapter IV, Definition 1.3 – note that the theory of Lagrangian subgroups reviewed in Chapter IV, §1, extends immediately to the metrized case). Let $K_H \subseteq K_{\overline{\mathcal{L}}} \cong {}_dE \subseteq E_d$ be the image of H in $K_{\overline{\mathcal{L}}}$. (Thus, $H \cong K_H$.) Let

$$E_{\infty_H,S} \stackrel{\text{def}}{=} E_{\infty,S}/(K_H)_{S_\infty}$$

(where $(K_H)_{S_\infty} \stackrel{\text{def}}{=} K_H \times_S S_\infty$). Note that the immediate extension of Chapter IV, Theorem 1.4, to the metrized case shows that there exists a metrized line bundle

$$\overline{\mathcal{L}}_H$$

(of relative degree 1) on $E_{\infty_H,S}$ (defined by $H \subseteq \mathcal{G}_{\overline{\mathcal{L}}}$). Write

$${}_dE_H \stackrel{\text{def}}{=} {}_dE/K_H$$

(so ${}_dE_H \times_S S_\infty \subseteq E_{\infty_H,S}$). Thus, ${}_dE_H$ is a finite flat group scheme over S. Next, observe that since $K_H \subseteq {}_dE$ is annihilated by d, it follows that the finite morphism $C_d \to C$ considered above (which compactifies multiplication by d) factors (uniquely) as a composite

$$C_d \to C_H \to C$$

of finite morphisms, where C_H is S-flat, and, after base-change via $S_\infty \to S$ and restriction to $U_S \stackrel{\text{def}}{=} S - D$, the morphism $C_d \to C_H$ may be identified with the morphism $E_{\infty,S} \to E_{\infty_H,S}$. Thus, since $E^{\dagger}_{\infty,[d]} \to E_{\infty,S}$ is obtained from $E^{\dagger}_{C_d,[d]} \to C_d$ which, in turn, is obtained by pulling back $E^{\dagger}_C \to C$ via $C_d \to C$, it follows that we obtain natural W_E-torsors

$$E^{\dagger}_{C_H,[d]} \to C_H; \quad E^{\dagger}_{\infty_H,[d]} \to E_{\infty_H,S}$$

whose pull-backs to C_d, $E_{\infty,S}$ may be identified with $E^{\dagger}_{C_d,[d]} \to C_d$, $E^{\dagger}_{\infty,[d]} \to E_{\infty,S}$. Note, moreover, that (it is immediate from the definitions that) the lifting ${}_dE^{\dagger}$ of ${}_dE$ defines a lifting

$${}_dE^{\dagger}_{\infty_H} \subseteq E^{\dagger}_{\infty_H,[d]}$$

of ${}_dE_H \times_S S_\infty \subseteq E_{\infty_H,S}$. Finally, let us write α_H for the image of α in $E^{\dagger}_{\infty_H,[d]}(S_\infty)$.

Thus, by taking the H-invariants of the domain and range of $\Xi_{\overline{\mathcal{L}},d,\alpha}$ in Proposition 2.2 (and applying the immediate extension of Chapter IV, Theorem 1.4, to the metrized case), we obtain the following:

Proposition 2.3. *Under the above conditions, restriction of sections over $E^{\dagger}_{\infty_H,[d]}$ to the subscheme $\mathcal{T}^*_{\alpha}({}_dE^{\dagger}_{\infty_H}) \subseteq E^{\dagger}_{\infty_H,[d]}$ defines a natural morphism*

$$\Xi^H_{\overline{\mathcal{L}},d,\alpha} : (f_S)_*(\overline{\mathcal{L}}_H|_{E^{\dagger}_{\infty_H,[d]}})^{<d} \to (f_S)_*(\overline{\mathcal{L}}_H|_{\mathcal{T}^*_{\alpha_H}({}_dE^{\dagger}_{\infty_H})})$$

of metrized vector bundles of rank d on S_∞ which may be identified with the morphism obtained from the restriction map $\Xi_{\overline{\mathcal{L}},d,\alpha}$ of Proposition 2.2 by taking H-invariants.

Since, in this case, taking H-invariants is a faithful operation (cf. Chapter IV, Theorem 1.4), it follows that, for instance, *the bijectivity of $\Xi_{\overline{\mathcal{L}},d,\alpha}$ is equivalent to that of $\Xi^H_{\overline{\mathcal{L}},d,\alpha}$*. Often, it will be technically easier to analyze the restriction morphism Ξ after taking H-invariants.

Finally, to prepare for §5, 6 below, we consider the *dependence of Ξ on α*. Let w be a section of W_E. Then note that translation by w induces natural identifications

$$\mathcal{T}^*_w(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}}) = \overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}}; \quad \mathcal{T}^*_w(\overline{\mathcal{L}}_H|_{E^{\dagger}_{\infty_H,[d]}}) = \overline{\mathcal{L}}_H|_{E^{\dagger}_{\infty_H,[d]}}$$

Thus, translation by w effects automorphisms $\mathcal{T}^*_w$ of the metrized vector bundles

$$(f_S)_*(\overline{\mathcal{L}}_{E^{\dagger}_{\infty,[d]}})^{<d}; \quad (f_S)_*(\overline{\mathcal{L}}_H|_{E^{\dagger}_{\infty_H,[d]}})^{<d}$$

which are *unipotent* with respect to the filtration (on $(f_S)_*(-)^{<d}$) by *torsorial degree*. In other words, these automorphisms $\mathcal{T}^*_w$ induce the identity on each of the subquotients $(f_S)_*(-)^{<i}/(f_S)_*(-)^{<i-1}$ (for $i \in \mathbf{Z}$). Indeed, this follows immediately from the definition of a W_E-torsor. Thus, since both the domain and range of the morphisms of Proposition 2.2 (respectively, 2.3) are the *same* (i.e., not just isomorphic) for α and $\alpha + w$, it makes sense to compare these two morphisms, and, in particular, to compare their *determinants*. Note, moreover, that the morphisms of Propositions 2.2 (respectively, 2.3) for α, $\alpha + w$ differ precisely by composition with the automorphism $\mathcal{T}^*_w$. Thus, we obtain the following:

Lemma 2.4. *For any section w of W_E, we have:*

$$\det(\Xi_{\overline{\mathcal{L}},d,\alpha}) = \det(\Xi_{\overline{\mathcal{L}},d,\alpha+w}); \quad \det(\Xi^H_{\overline{\mathcal{L}},d,\alpha}) = \det(\Xi^H_{\overline{\mathcal{L}},d,\alpha+w})$$

(where "det$(-)$" *is a section of the metrized line bundle* det(range) $\otimes$ det(domain)$^{-1}$ *on* S_∞*).*

§3. Extension of the Étale-Integral Structure

In this §, we show that the *étale-integral structures* "$\mathcal{R}^{\mathrm{et}}$" discussed in Chapter III, §6, are defined for *arbitrary elliptic curves* (i.e., not just for degenerating elliptic curves, as in the discussion of Chapter III, §6), and, moreover, are that the functions in these étale-integral structures *assume integral values* at the torsion points appearing in the definition of the evaluation map. It will be relative to this étale-integral structure (defined for arbitrary elliptic curves) that we will prove the *comparison isomorphism* in Chapter VI.

We work with the notation of Chapter III, §5, 6. Thus, let $\mathcal{O}$ be a Zariski localization of $\mathcal{O}_K$, where K is a finite extension of $\mathbf{Q}$. Let

$$A \stackrel{\mathrm{def}}{=} \mathcal{O}[[q]]; \quad S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A)$$

Then we have a one-dimensional semi-abelian scheme

$$E \to S$$

over S. Roughly speaking, one may think of E as being "$\mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$." More rigorously, E may be compactified to a log elliptic curve $C^{\log} \to S^{\log}$. Moreover, $C_{\widehat{S}}$ (the result of base changing C to the q-adic completion $\widehat{S}$ of S) may be written as

$$C_{\widehat{S}} = C^{\infty}_{\widehat{S}}/\mathbf{Z}_{\mathrm{et}}$$

Recall that in Chapter III, §6, we used *the natural isomorphism* of Chapter III, Theorem 5.6, Corollary 5.9:

$$E^{\dagger}_C|_{C^{\infty}_{\widehat{S}}} \cong (W_E)_{\widehat{S}} \times_{\widehat{S}} C^{\infty}_{\widehat{S}}$$

to define a *new integral structure*

$$\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_C}$$

on $\mathcal{R}_{E^{\dagger}_C} \otimes \mathbf{Q}$ (cf. Chapter III, Proposition 6.1). If one trivializes ω_E by means of the section $d\log(U)$, then we may write

$$W_E = \mathrm{Spec}(\mathcal{O}_S[T])$$

(where T is the indeterminate corresponding to the chosen trivialization of ω_E). The new integral structure was obtained by essentially adjoining the polynomials

$$T^{[n]} \stackrel{\mathrm{def}}{=} \frac{1}{n!}T(T-1)(T-2)\cdot \ldots \cdot (T-(n-1))$$

(for $n \in \mathbf{Z}_{\geq 0}$) to the original integral structure.

Now let us write (for $d \geq 1$ an integer)

$$(\widetilde{C}_d)_{\widehat{S}} \stackrel{\mathrm{def}}{=} C^{\infty}_{\widehat{S}}/(d \cdot \mathbf{Z}_{\mathrm{et}})$$

Since $(\widetilde{C}_d)_{\widehat{S}} \to \widehat{S}$ is *proper*, it algebrizes to some $\widetilde{C}_d \to S$. Let $\widetilde{E} \to S$ be the open subscheme of $\widetilde{C}_d$ obtained by removing from $\widetilde{C}_d$ the irreducible components of the special fiber (i.e., the fiber over $q = 0$) that do not contain the image of the identity element $e_{C^{\infty}_{\widehat{S}}} \in C^{\infty}_{\widehat{S}}(\widehat{S})$ of $C^{\infty}_{\widehat{S}}$. Let $\widetilde{E}_d \to S$ denote the complement of the nodes of the special fiber of $\widetilde{C}_d$. Thus, $\widetilde{E}_d$ and $\widetilde{E}$ have a natural structure of group scheme over S, and $\widetilde{E}$ is a semi-abelian scheme over S. Moreover, let us observe that we have a natural *finite étale covering*

$$\widetilde{E}_d \to E$$

of degree d which compactifies to a finite morphism $\widetilde{C}_d \to C$. Relative to the notation of §2, if one takes "E" in *loc. cit.* to be $\widetilde{E}$ (in the notation of the present discussion), then the resulting "E_d" (respectively, "C_d") in the notation of *loc. cit.* corresponds to $\widetilde{E}_d$ (respectively, $\widetilde{C}_d$) in the present discussion. Moreover, I *claim* that:

> The resulting "$E^{\dagger}_{[d]}$," "$E^{\dagger}_{C_d,[d]}$" of *loc. cit.* correspond to $E^{\dagger}_{C}|_{\widetilde{C}_d}$, $E^{\dagger}_{C}|_{\widetilde{E}_d}$ in the notation of the present discussion.

Indeed, to prove this, one reasons as follows: The multiplication by d morphism $\widetilde{E}_d \to \widetilde{E}$ factors as the composite of the finite étale isogeny $\widetilde{E}_d \to E$ under consideration with some other finite isogeny $E \to \widetilde{E}$ (of degree d, whose kernel may be naturally identified with $\boldsymbol{\mu}_d$). Moreover, the *push-forward morphism* associated to the isogeny $E \to \widetilde{E}$ induces a natural isomorphism between the universal extensions of E and $\widetilde{E}$ (cf. the discussion at the beginning of Chapter IV, §3). Thus, it follows that the pull-back of the universal extension of $\widetilde{E}$ to $\widetilde{E}_d$ via the multiplication by d map may be identified with the pull-back of the universal extension of E to $\widetilde{E}_d$ via the finite étale isogeny $E \to \widetilde{E}$ under consideration.

On the other hand, the former pull-back is, by the *definition* of "$E^{\dagger}_{[d]}$" given in §2, none other than the "$E^{\dagger}_{[d]}$" associated to $\widetilde{E}_d$. (One concludes the corresponding statement for "$E^{\dagger}_{C_d,[d]}$" by using the fact that the complement of $\widetilde{E}_d$ in (the *normal* scheme) $\widetilde{C}_d$ is of codimension 2.) This completes the proof of the claim. Thus, in the following, we will write

$$E^{\dagger}_{\widetilde{C}_d,[d]}$$

for the $W_{\widetilde{E}}$-torsor over $\widetilde{C}_d$ which is the "$E^{\dagger}_{C_d,[d]}$" (in the notation of §2) for the case where one takes "E" (in §2) to be $\widetilde{E}$ (in the notation of the present discussion).

In particular, we see that by pulling back the new integral structure of Chapter III, Proposition 6.1, on $\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_C}$ via $\widetilde{C}_d \to C$, we obtain a new integral structure

$$\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{\widetilde{C}_d,[d]}} \subseteq \mathcal{R}_{E^{\dagger}_{\widetilde{C}_d,[d]}} \otimes \mathbf{Q}$$

Moreover, this integral structure has the property that:

> *If f is any section of $\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{\widetilde{C}_d,[d]}}$ (over, say, $\widetilde{E}_d$), then the restriction of f to any d-torsion point $\tau \in \widetilde{E}_d(S)$ defines an* integral *section of $\mathcal{O}_S$.*

Indeed, pulling everything back to $C^{\infty}_{\widehat{S}}$, we see this amounts to the observation that the pull-backs to $C^{\infty}_{\widehat{S}}$ of the d-torsion points of $\widetilde{E}_d$ have coordinates, relative to the decomposition $E^{\dagger}_C|_{C^{\infty}_{\widehat{S}}} \cong (W_E)_{\widehat{S}} \times_{\widehat{S}} C^{\infty}_{\widehat{S}}$, given by

$$(\beta, \alpha \cdot q^{\beta})$$

where $\beta \in \mathbf{Z}$, and α is a d-th root of unity. (Indeed, this follows from Chapter III, Corollary 5.9, and the fact that the "q-parameter" of $\widetilde{E}$ is given by q^d.) Thus, the above property follows from the fact that any $T^{[n]}$ has integral values at $\beta \in \mathbf{Z}$. (In fact, for $\beta \in \mathbf{Z}_{\geq 0}$, the value of $T^{[n]}$ at $T = \beta$ is 0 (if $\beta < n$), 1 (if $\beta = n$), $\binom{\beta}{n}$ (i.e., binomial coefficients, if $\beta > n$).)

Next, let us observe that the semi-abelian scheme $\widetilde{E} \to S$ is *universal* among degenerating one-dimensional semi-abelian schemes whose q-parameters admit a d-th root. Thus, what we have done above already defines a new integral structure on "$\mathcal{R}_{E^{\dagger}_{C_d,[d]}}$" for any degenerating one-dimensional semi-abelian scheme $E \to S$ (for which the base S is $\mathbf{Z}$-flat) whose q-parameter admits a d-th root. In fact, we would like to *extend* this new integral structure on $\mathcal{R}_{E^\dagger_{C_d,[d]}}$ from a formal neighborhood of the divisor at infinity (where the new integral structure has already been defined) to the whole of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$. That is to say, in the following discussion, we propose to do the following:

> *We would like to* "p-adically analytically continue" *the new integral structure on "$\mathcal{R}_{E^\dagger_{C_d,[d]}} \otimes \mathbf{Q}$" from a formal neighborhood of the divisor at infinity to the whole of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$.*

More precisely, given any one-dimensional semi-abelian scheme $E_T \to T$, where T is $\mathbf{Z}$-flat, and the q-parameters of $E_T \to T$ are non-zero divisors that admit d-th roots, we would like to define a new integral structure on the associated $\mathcal{R}_{E^\dagger_{(C_T)_d,[d]}} \otimes \mathbf{Q}$. Clearly, since for a fixed n, the new integral structure on $F^n(\mathcal{R}_{E^\dagger_{(C_T)_d,[d]}} \otimes \mathbf{Q})$ differs from the integral structure defined by $F^n(\mathcal{R}_{E^\dagger_{(C_T)_d,[d]}})$ at only finitely many primes, it suffices to define these new integral structure locally, *p-adically*, at each rational prime p. Thus, we may assume that T is a $\mathbf{Z}_p$-scheme.

Next, let us observe that in the discussion of Chapter III, §6, of the new integral structure in the case of a degenerating $E_T \to T$ (i.e., Chapter III, Proposition 6.1), the key object that we needed in order to define the new integral structure was the canonical decomposition $E^\dagger_C|_{C^\infty_{\widehat{S}}} \cong (W_E)_{\widehat{S}} \times_{\widehat{S}} C^\infty_{\widehat{S}}$, i.e., the *canonical section* κ (cf. Chapter III, Theorem 2.1). It is precisely because this canonical section is only defined in a formal neighborhood of the divisor at infinity that the discussion of Chapter III, §6 (i.e., Chapter III, Proposition 6.1) could only be carried out over such a formal neighborhood. In fact, however, careful inspection reveals that really, *for a fixed* n, in order to define $F^n(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{(C_T)_d,[d]}})$ *at a fixed prime* p, it suffices to have κ only *modulo* p^N, for some sufficiently large N.

Moreover, in the notation of the above discussion in the degenerating case, if we let

$$\widetilde{G} \subseteq C^\infty_{\widehat{S}}/(p^N \cdot d \cdot \mathbf{Z}_{\mathrm{et}})$$

be the group scheme over S obtained by removing the nodes from $C^\infty_{\widehat{S}}/(p^N \cdot d \cdot \mathbf{Z}_{\mathrm{et}})$, then we get a finite étale covering $\widetilde{G} \to \widetilde{E}$ (of degree p^N) with the property that κ is *already defined modulo* p^N after pull-back via $\widetilde{G} \to \widetilde{E}$ (or over the compactification $C^\infty_{\widehat{S}}/(p^N \cdot d \cdot \mathbf{Z}_{\mathrm{et}}) \to \widetilde{C}_d$). Indeed, this follows from the fact that $1_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ acts on κ by $\kappa \mapsto \kappa + 1$ (cf. Chapter III, Theorem 5.6), so *elements of* $p^N \cdot d \cdot \mathbf{Z}_{\mathrm{et}}$ *preserve* κ *modulo* p^N. Note, moreover, that since the isogeny $\widetilde{G} \to \widetilde{E}$ is of degree p^N, there exists an isogeny of group schemes $\widetilde{H} \to \widetilde{G}$ which is finite of degree p^N such that the composite

$$\widetilde{H} \to \widetilde{E}$$

is isomorphic to the "multiplication by p^N" map on $\widetilde{E}$ over $U_S = \mathrm{Spec}(A[q^{-1}])$ ($\subseteq S$). Thus, in summary,

> *The canonical section κ is defined modulo p^N after pull-back from $\widetilde{E}$ to $\widetilde{H}$.*

With these preparatory remarks behind us, we are now ready to begin the "analytic continuation" argument:

Analytic Continuation Argument:

Let $E_T \to T$ be a family of elliptic curves, where T is a finite, flat (hence, in particular, affine) scheme over $(\mathcal{M}_{1,0})_{\mathbf{Z}_p}$ (i.e., the complement of the divisor at infinity of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}_p}$). To simplify the discussion, we shall also assume that T is *normal.* Write

$$H_T \to E_T$$

for the isogeny given by multiplication by p^N. (Thus, H_T is abstractly isomorphic as a T-scheme to E_T.)

Consider the torsor $E^{\dagger}_{T,[d]} \to E_T$ (i.e., the "$E^{\dagger}_{[d]}$" associated to E_T). This torsor defines a class

$$\eta \in H^1(E_T, \omega_{E_T}|_{E_T}) \cong \mathcal{O}_T$$

(where the "$\cong$" follows from relative Serre duality for the morphism $E_T \to T$). If we pull this class back to H_T, then we obtain a class

$$\eta_{\widetilde{H}} \in \mathcal{E} \stackrel{\mathrm{def}}{=} H^1(H_T, \omega_{E_T}|_{H_T})$$

where $\mathcal{E}$ is a *line bundle on* T.

I *claim* that $\eta_{\widetilde{H}}$ is $\equiv 0$ modulo p^N. There are many ways to verify this claim. One way is the following: Since $\mathcal{E}$ is a *line bundle* on T, and T is *normal*, it suffices to verify that $\eta_{\widetilde{H}}$ vanishes modulo p^N after restricting $\eta_{\widetilde{H}}$ to the p-adic completion $\widehat{\mathcal{O}}_{T,\mathfrak{p}}$ of the localization of T at any generic point $\mathfrak{p}_T$ of $T \otimes \mathbf{F}_p$. On the other hand, the discrete valuation ring $\widehat{\mathcal{O}}_{T,\mathfrak{p}_T}$ is clearly dominated by a discrete valuation ring which arises as the p-adic completion $\widehat{\mathcal{O}}_{W,\mathfrak{p}_W}$ of some W at a generic point $\mathfrak{p}_W$ of $W \otimes \mathbf{F}_p$, where W is the base of some degenerating elliptic curve whose classifying morphism to $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ defines a *finite morphism* $W \to \mathrm{Spec}(\mathbf{Z}_p[[q]])$. On the other hand, for bases such as $\widehat{\mathcal{O}}_{W,\mathfrak{p}_W}$, the fact that $\eta_{\widetilde{H}}$ vanishes modulo p^N follows from the above discussion of the degenerating case (i.e., where we saw that the canonical section κ is defined modulo p^N after pull-back via "$\widetilde{H} \to \widetilde{E}$"). This concludes the proof of the claim.

Thus, the ω_E-torsor $E^{\dagger}_{T,[d]} \to E_T$ splits modulo p^N after pull-back via $H_T \to E_T$. We would like to show that this torsor not only splits modulo p^N, but that *it admits a splitting which extends the canonical splitting κ (modulo p^N) in the degenerating case*. To do this, we must characterize the canonical splitting κ in terms that make sense even for non-degenerating elliptic curves. We do this as follows: Note that since $E^{\dagger}_{T,[d]}$ has a structure of *abelian group scheme* over T, it admits an action by $\Sigma \stackrel{\mathrm{def}}{=} \{\pm 1\}$. Of course, H_T and E_T are also commutative group schemes over T, hence admit natural actions by $\Sigma = \{\pm 1\}$. Then let us observe that since the canonical splitting κ is a *group homomorphism* (cf. Chapter III, Theorem 2.1), it commutes with the respective actions of $\Sigma = \{\pm 1\}$. Let us regard the line bundle ω_{E_T} on T as an $\mathcal{O}_T$-module with Σ action, where Σ acts via the tautological character $\Sigma \to \{\pm 1\} \subseteq \mathcal{O}_T^{\times}$. Then it follows by general nonsense that the obstruction to defining a Σ-*equivariant* splitting of $E^{\dagger}_{T,[d]}$ over H_T modulo p^N is a class

$$\eta_{\Sigma} \in H^1(\Sigma, \omega_{E_T} \otimes (\mathbf{Z}/p^N)) = \{\omega_{E_T} \otimes (\mathbf{Z}/p^N)\} \otimes (\mathbf{Z}/2)$$

Moreover, if this class vanishes, then such a splitting will be unique modulo

$$H^0(\Sigma, \omega_{E_T} \otimes (\mathbf{Z}/p^N)) = \ker(2 : \omega_{E_T} \otimes (\mathbf{Z}/p^N))$$

(i.e., modulo the kernel of multiplication by 2 on $\omega_{E_T} \otimes (\mathbf{Z}/p^N)$).

Thus, we would like to see that η_{Σ} vanishes. Of course, this is only a problem when $p = 2$. In this case, since ω_{E_T} is a line bundle on T, one proves that η_{Σ} vanishes in precisely the same way as we proved above that $\eta_{\widetilde{H}}$ vanishes modulo p^N – i.e., by reducing to the degenerating case, where we know the result to be true. This completes the proof that *η_{Σ} is always zero*. As for uniqueness, in general, we only get uniqueness modulo p^{N-1} (of course, if p is odd, we get uniqueness modulo p^N). However, by taking N sufficiently large (e.g., one larger than the original N necessary to define the integral structure), this is not a problem. This uniqueness thus shows, by the same argument as that used above to prove that $\eta_{\widetilde{H}} \equiv 0$ modulo p^N – i.e., *by comparing with the degenerating case* – that:

> There exists a splitting of the torsor $E^{\dagger}_{T,[d]}$ over H_T modulo p^N which extends the canonical splitting κ considered in the degenerating case modulo p^{N-1}.

Thus, by taking N sufficiently large, we see that all the functions "$T^{[i]}$" (for $i <$ our fixed n) may be defined modulo a sufficiently large power of p so that we may form a new integral structure

$$F^n(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{T,[d]}}) \subseteq \mathcal{R}_{E^\dagger_{T,[d]}} \otimes \mathbf{Q}$$

which extends the integral structure of Chapter III, Proposition 6.1 in the degenerating case – *at least after pulling back from E_T to H_T*. But then, by using the "$\widehat{\mathcal{O}}_{T,\mathfrak{p}_T}$'s," reduction to the degenerating case, and finite flat descent (of coherent sheaves) for the morphism $H_T \to E_T$, it is easy to see that these integral structures *descend* back down to E_T, as desired. *This completes the "p-adic analytic continuation argument."*

In other words, we have proven the following:

Theorem 3.1. *Let $d \geq 1$ be an integer. Let*

$$C^{\log} \to S^{\log}$$

be any log elliptic curve as at the beginning of §2 – i.e., the associated q-parameter $\in \mathcal{O}_{\widehat{S}}$ (where $\widehat{S}$ is the completion of S along the divisor at infinity $D \subseteq S$ defined by $C^{\log} \to S^{\log}$) is a non-zero divisor which, étale locally, admits a d-th root. Suppose further that S is $\mathbf{Z}$-flat. Then the resulting $\mathcal{R}_{E^\dagger_{C_d,[d]}}$ (cf. §2, for the definition of $E^\dagger_{C_d,[d]}$) and its filtration admit natural integral structures

$$\ldots \subseteq F^n(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}) \subseteq \ldots \subseteq \mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}} \subseteq \mathcal{R}_{E^\dagger_{C_d,[d]}} \otimes \mathbf{Q}$$

where $F^n(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}})$ is a rank n vector bundle on C_d such that

$$(F^{n+1}/F^n)(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}) = \frac{1}{n!} \cdot \mathcal{O}_{C_d} \otimes_{\mathcal{O}_S} \tau_E^{\otimes n}$$

These integral structures are uniquely characterized by the properties that they are functorial in $C^{\log} \to S^{\log}$, and they extend the integral structures defined in the degenerating case in Chapter III, Proposition 6.1. Finally, the functions of $\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}$ assume integral values *at the d-torsion points of $E^\dagger_{C_d,[d]}(S)$.*

Proof. It remains only to remark that the final assertion concerning the integrality of the values at d-torsion points may be verified for normal T as in the analytic continuation argument above by checking integrality at the various $\widehat{\mathcal{O}}_{T,\mathfrak{p}_T}$ (where $\mathfrak{p}_T$ is a generic point of $T \otimes \mathbf{F}_p$). Moreover the integrality at $\widehat{\mathcal{O}}_{T,\mathfrak{p}_T}$ may be verified by reduction to the degenerating case, where it has already been verified (cf. the discussion following Chapter III, Proposition 6.1). $\bigcirc$

Definition 3.2. (Just as in Chapter III, Definition 6.2) the integral structures denoted by a superscript "et" will be referred to as *étale-integral, or et-integral, structures*. That is to say, "et" stands for "étale," and is the same as the "et" in $\mathbf{Z}_{\text{et}}$. In particular, the "étale-integral structure" may be thought of as the integral structure arising from thinking of algebraic functions on the universal extension as set-theoretic functions on $\mathbf{Z}_{\text{et}}$, i.e., on the fibers of the infinite étale covering $C^{\infty}_{\widehat{S}} \to C^{\infty}_{\widehat{S}}/\mathbf{Z}_{\text{et}} = C_{\widehat{S}}$.

The original integral structure on the various "$\mathcal{R}$'s" (i.e., the integral structures without a label) will be referred to as the *de Rham-integral, or DR-integral, structures*. This is because they arise from the original natural integral structures on the universal extension $E^{\dagger}$ as a sort of de Rham cohomology – i.e., "$H^1_{\text{DR}}(E, \mathcal{O}^{\times}_E)$" – associated to E.

Remark. In fact, even though in Theorem 3.1, we assumed that S was $\mathbf{Z}$-flat, by working in the universal case and then pulling back, one sees that the various sheaves we constructed with a superscript "et" are *defined for any* $C^{\log} \to S^{\log}$ as in Theorem 3.1 *even without the assumption of* $\mathbf{Z}$*-flatness*.

§4. Linear Relations Among Higher Schottky-Weierstrass Zeta Functions

In this §, we consider the extent to which the *twisted Schottky-Weierstrass zeta functions of Chapter IV, §3, satisfy linear relations modulo various powers of q over the base ring A* (notation as in Chapter IV, §3). Thus, throughout this §, we will use the notation and conventions of Chapter IV, §2,3. The calculations of linear relations that we carry out in this § will turn out to yield the key technical machinery behind the main result of this paper.

We begin by fixing our base ring $\mathcal{O}$ as in Chapter IV, §2,3). Thus, $\mathcal{O}$ is a Zariski localization of the ring of integers of a finite extension of $\mathbf{Q}$ such that $\text{Pic}(\mathcal{O}) = \{1\}$. We also fix a positive even integer $n = 2m$. Then over $S = \text{Spec}(A)$ (where $A = \mathcal{O}[[q]]$), we constructed in Chapter IV, §2, an *isogeny*

$$E \to \widetilde{E}$$

whose kernel is equal to $\boldsymbol{\mu}_n$. Over $\widetilde{E}$, we considered various line bundles such as:

$$\widetilde{\mathcal{L}}_{\widetilde{E}} \stackrel{\text{def}}{=} \mathcal{O}_{\widetilde{E}}(\widetilde{e}); \qquad \widetilde{\mathcal{M}}_{\widetilde{E}}$$

Here, $\widetilde{\mathcal{M}}_{\widetilde{E}}$ was defined as the restriction to $\widetilde{E} \subseteq \widetilde{C}$ of a line bundle $\widetilde{\mathcal{M}}$ on $\widetilde{C}$ obtained by descending the line bundle $\widetilde{\mathcal{L}}^{\otimes n}_{C^\infty_{\widehat{S}}}$ on $C^\infty_{\widehat{S}}$ via the "standard action" of $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ on $\widetilde{\mathcal{L}}^{\otimes n}_{C^\infty_{\widehat{S}}}$. Recall that the line bundle $\widetilde{\mathcal{L}}^{\otimes n}_{C^\infty_{\widehat{S}}}$ is the line bundle whose global sections over $C^\infty_{\widehat{S}}$ can be written as topological A-linear combinations of certain *special monomials*

$$q^{m \cdot k^2 + ik} \cdot U^{2mk+i} \cdot \theta^m$$

(where $k, i \in \mathbf{Z}$, $|i| \leq m$) discussed in Chapter IV, Proposition 2.2. Moreover, for any character $\chi \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$ (where $\Pi_n = (\mathbf{Z}_{\mathrm{et}}/n) \times \boldsymbol{\mu}_n$), we defined a line bundle

$$\widetilde{\mathcal{M}}^{\chi}_{\widetilde{E}}$$

and we saw in Chapter IV, Theorem 2.1 that for a certain particular character, which we denote by $\chi_\theta \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$, we have an isomorphism

$$\widetilde{\mathcal{L}}_{\widetilde{E}} \cong \mathcal{M}^{\chi_\theta}_{\widetilde{E}}$$

of degree 1 line bundles on $\widetilde{E}$.

In the following discussion, we would like to fix a character

$$\chi_{\mathcal{L}} \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$$

Let us write $\chi_{\mathcal{M}} \stackrel{\mathrm{def}}{=} \chi_{\mathcal{L}} \cdot \chi_\theta$ and denote the restrictions of $\chi_{\mathcal{M}}$ and $\chi_{\mathcal{L}}$ to $\boldsymbol{\mu}_n \subseteq (\mathbf{Z}_{\mathrm{et}}/n) \times \boldsymbol{\mu}_n = \Pi_n$ by $\chi_{\mathcal{M},\boldsymbol{\mu}} : \boldsymbol{\mu}_n \to \boldsymbol{\mu}_n$ and $\chi_{\mathcal{L},\boldsymbol{\mu}} : \boldsymbol{\mu}_n \to \boldsymbol{\mu}_n$, respectively. Thus, we have

$$\widetilde{\mathcal{L}}^{\chi_{\mathcal{L}}}_{\widetilde{E}} \cong \widetilde{\mathcal{M}}^{\chi_{\mathcal{M}}}_{\widetilde{E}}$$

Recall that there is a standard action of $\boldsymbol{\mu}_n$ on the special monomials

$$q^{m \cdot k^2 + ik} \cdot U^{2mk+i} \cdot \theta^m$$

(i.e., the action used to define $\widetilde{\mathcal{M}}$). Note that for this action, $\boldsymbol{\mu}_n$ acts *trivially* on θ^m. Let us refer to the special monomials on which $\boldsymbol{\mu}_n$ acts via $(\chi_{\mathcal{M},\boldsymbol{\mu}})^{-1}$ as $\chi_{\mathcal{L}}$*-special monomials*. For j a positive integer, let us write

$$c_j(\chi_{\mathcal{L}}) \in \mathbf{Q}$$

for the maximal nonzero rational number c such that modulo $q^{n\cdot c}$, the number of nonzero $\chi_{\mathcal{L}}$-special monomials is $< j$. Let us denote this set of $< j$ nonzero (modulo $q^{n\cdot c_j(\chi_{\mathcal{L}})}$) $\chi_{\mathcal{L}}$-special monomials by

$$C_j(\chi_{\mathcal{L}})$$

We remark that, here (and in the following discussion):

> *By "modulo" we mean, strictly speaking, when restricted to sections of* $\mathcal{L}^{\otimes n}_{C^{\infty}_{\widehat{S}}}|_{E_{\widehat{S}}}$ *over* $E_{\widehat{S}} = (\mathbf{G}_{\mathrm{m}})_{\widehat{S}}$.

When (as in the following discussion), the character $\chi_{\mathcal{L}}$ is fixed, we shall often just write c_j, C_j for $c_j(\chi_{\mathcal{L}})$ and $C_j(\chi_{\mathcal{L}})$, respectively.

Schola 4.1. Computation of Special Monomials Modulo Powers of q

Let us compute C_j and c_j. There are *three cases* to consider:

Case I: $\chi_{\mathcal{L},\boldsymbol{\mu}} = 1$. In this case, the $\chi_{\mathcal{L}}$-special monomials are those for which $i = \pm m$. Thus, the monomials for which the exponent $m \cdot k^2 + ik$ of q is 0 are precisely:

$$M \stackrel{\mathrm{def}}{=} U^m \cdot \theta^m, \quad M' \stackrel{\mathrm{def}}{=} \quad U^{-m} \cdot \theta^m = (-1)_{\mathrm{et}}(M)$$

The other $\chi_{\mathcal{L}}$-special monomials

$$k_{\mathrm{et}}(M) = q^{m\cdot k^2+mk} \cdot U^{2mk+m} \cdot \theta^m, \quad (-k)_{\mathrm{et}}(M') = q^{m\cdot k^2+mk} \cdot U^{-2mk-m} \cdot \theta^m$$

are given by applying $(\pm k)_{\mathrm{et}}$ (for $k \geq 0$) to these original two. In particular, there are precisely $2 + 2k$ monomials obtained by applying k'_{et} (for $|k'| \leq k$) to the original two monomials. Thus, for $k \geq 0$, we have

$$c_{2k+1} = c_{2k+2} = \frac{1}{n}(m \cdot k^2 + mk) = \frac{1}{2}(k^2 + k)$$

while the set $C_{2k+1} = C_{2k+2}$ is given by

$$\{k'_{\mathrm{et}}(M)\}_{0\leq k'<k} \quad \bigcup \quad \{k'_{\mathrm{et}}(M')\}_{-k<k'\leq 0}$$

Case II: $\chi_{\mathcal{L},\boldsymbol{\mu}} = \chi_\theta|_{\boldsymbol{\mu}_n}$. In this case, the $\chi_{\mathcal{L}}$-special monomials are those for which $i = 0$. Thus, the only monomial for which the exponent $m \cdot k^2 + ik$ of q is 0 is the monomial:

$$M \stackrel{\text{def}}{=} \theta^m$$

The other $\chi_{\mathcal{L}}$-special monomials

$$k_{\text{et}}(M) = q^{m \cdot k^2} \cdot U^{2mk} \cdot \theta^m, \quad (-k)_{\text{et}}(M) = q^{m \cdot k^2} \cdot U^{-2mk} \cdot \theta^m$$

are given by applying $(\pm k)_{\text{et}}$ (for $k \geq 0$) to this original monomial. In particular, there are precisely $1+2k$ monomials obtained by applying k'_{et} (for $|k'| \leq k$) to the original monomial. Thus, we have

$$c_1 = 0$$

and, for $k \geq 1$,

$$c_{2k} = c_{2k+1} = \frac{1}{n} \cdot m \cdot k^2 = \frac{1}{2} k^2$$

while $C_1 = \emptyset$; the set $C_{2k} = C_{2k+1}$ is given by

$$\{k'_{\text{et}}(M)\}_{0 \leq |k'| < k}$$

Case III: $\chi_{\mathcal{L},\boldsymbol{\mu}}{}^2 \neq 1$. In this case, the $\chi_{\mathcal{L}}$-special monomials are those for which $i = i_\chi$, for some fixed integer i_χ such that $0 < |i_\chi| < m$. Thus, the only monomial for which the exponent $m \cdot k^2 + ik$ of q is 0 is the monomial:

$$M \stackrel{\text{def}}{=} U^{i_\chi} \cdot \theta^m$$

The other $\chi_{\mathcal{L}}$-special monomials

$$k_{\text{et}}(M) = q^{m \cdot k^2 + i_\chi \cdot k} \cdot U^{2mk + i_\chi} \cdot \theta^m, \quad (-k)_{\text{et}}(M) = q^{m \cdot k^2 - i_\chi \cdot k} \cdot U^{-2mk + i_\chi} \cdot \theta^m$$

are given by applying $(\pm k)_{\text{et}}$ (for $k \geq 0$) to this original monomial. Note that (for $k \geq 1$)

$$m \cdot k^2 + |i_\chi| \cdot k > m \cdot k^2 - |i_\chi| \cdot k > m \cdot (k-1)^2 + |i_\chi| \cdot (k-1)$$

(Indeed, the second inequality follows from the fact that $2mk = m + m \cdot (2k-1) > m + |i_\chi| \cdot (2k-1)$.) In other words, *the exponent of q induces a total ordering on the set of $\chi_{\mathcal{L}}$-special monomials*. The first $2k$ monomials are given by applying k'_{et} (for integers

k' such that *either* $|k'| < k$ *or* $|k'| = k$, $k' \cdot i_\chi < 0$) to the original monomial, while the first $1 + 2k$ monomials are given by applying k'_{et} (for integers k' such that $|k'| \leq k$) to the original monomial. Thus, we have

$$c_1 = 0$$

and, for $k \geq 1$,

$$c_{2k} = \frac{1}{n} \cdot (m \cdot k^2 - |i_\chi| \cdot k); \quad c_{2k+1} = \frac{1}{n} \cdot (m \cdot k^2 + |i_\chi| \cdot k)$$

Moreover, $C_1 = \emptyset$; the set C_{2k} is given by

$$\{k'_{\text{et}}(M)\}_{0 \leq |k'| < k}$$

while the set C_{2k+1} is given by

$$\{k'_{\text{et}}(M)\}_{0 \leq |k'| < k} \quad \bigcup \quad \{k'_{\text{et}}(M)\}_{|k'| = k,\ k' \cdot i_\chi < 0}$$

This completes the computation of C_j and c_j.

Lemma 4.2. *For any choice of character* $\chi_{\mathcal{L}} \in \text{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$*, we have:*

$$\sum_{a=1}^{j} c_a = \frac{1}{24} j(j^2 - 1)$$

for any odd *integer* $j \geq 1$.

Proof. First, we observe that for any *even* integer $a = 2k \geq 2$, we have $c_a + c_{a+1} = k^2$. Indeed, in the cases where $\chi_{\mathcal{L},\boldsymbol{\mu}} \neq 1$ (i.e., Cases II and III), it is immediate from the formulas for c_a and c_{a+1} that we have $c_a + c_{a+1} = k^2$. On the other hand, in Case I, we have $c_a + c_{a+1} = \frac{1}{2}\{k^2 + k + (k-1)^2 + (k-1)\} = k^2$, as asserted. Thus, if we let $j' \stackrel{\text{def}}{=} \frac{1}{2}(j-1)$, then we have

$$\sum_{a=1}^{j} c_a = \sum_{b=1}^{j'} b^2 = \frac{1}{6} j'(j'+1)(2j'+1) = \frac{1}{24}(j-1)(j+1)j = \frac{1}{24} j(j^2 - 1)$$

as desired. ○

Note that in the above analysis, C_j was always obtained by applying various k_{et}'s to certain fixed monomials M, M' which were the unique $\chi_{\mathcal{L}}$-special monomials that were nonzero modulo q. Let us write (for $j \geq 3$ in Case I; for $j \geq 2$ in Cases II, III)

$$\mathrm{Max}(C_j) \quad (\text{respectively, } \mathrm{Min}(C_j))$$

for the maximum (respectively, minimum) integer k such that the *symbol* $k_{\text{et}}(M)$ or $k_{\text{et}}(M')$ appears in the lists of elements of C_j that were given in Schola 4.1. (Here, by "symbol," we simply mean that in Case I, we wish to distinguish the symbol M' from $(-1)_{\text{et}}(M)$.) Note that

$$\mathrm{Max}(C_j) \geq 0 \geq \mathrm{Min}(C_j)$$

Also, let us write

$$\mathrm{Span}(C_j) \stackrel{\text{def}}{=} \mathrm{Max}(C_j) - \mathrm{Min}(C_j)$$

Thus, in *Case I*, we have (for $k \geq 1$)

$$\mathrm{Max}(C_{2k+1} = C_{2k+2}) = k - 1; \quad \mathrm{Min}(C_{2k+1} = C_{2k+2}) = -(k-1)$$

$$\mathrm{Span}(C_{2k+1} = C_{2k+2}) = 2k - 2$$

In *Case II*, we have (for $k \geq 1$)

$$\mathrm{Max}(C_{2k} = C_{2k+1}) = k - 1; \quad \mathrm{Min}(C_{2k} = C_{2k+1}) = -(k-1)$$

$$\mathrm{Span}(C_{2k} = C_{2k+1}) = 2k - 2$$

In *Case III*, we have (for $k \geq 1$)

$$\mathrm{Max}(C_{2k}) = k - 1; \quad \mathrm{Min}(C_{2k}) = -(k-1); \quad \mathrm{Span}(C_{2k}) = 2k - 2$$

and

$$\mathrm{Max}(C_{2k+1}) = k - 1 + \epsilon_{\mathrm{Max}}; \quad \mathrm{Min}(C_{2k+1}) = -(k-1) - \epsilon_{\mathrm{Min}}$$

$$\mathrm{Span}(C_{2k+1}) = 2k - 1$$

where $\epsilon_{\rm Max} + \epsilon_{\rm Min} = 1$, and $\epsilon_{\rm Max}$ is 1 (respectively, 0) if $i_\chi < 0$ (respectively, $i_\chi > 0$). In particular, we observe that if $\mathrm{Min}(C_j) \leq a \leq \mathrm{Max}(C_j)$, then $a_{\rm et}(M) \in C_j$ (in all Cases); $a_{\rm et}(M') \in C_j$ (in Case I).

Now let a be an integer such that $\mathrm{Min}(C_j) \leq a \leq \mathrm{Max}(C_j)$, for some integer j such that $\mathrm{Max}(C_j)$, $\mathrm{Min}(C_j)$ are defined. Let $k_{\rm et}(M)$ be an arbitrary $\chi_{\mathcal{L}}$-special monomial (so k may be any integer). Then observe that:

> *If $(-a)_{\rm et}(k_{\rm et}(M)) = (-a+k)_{\rm et}(M)$ is nonzero modulo q, then it follows that $k_{\rm et}(M) \in C_j$.*

Indeed, if $(-a)_{\rm et}(k_{\rm et}(M))$ is nonzero modulo q, then (by the discussion in Schola 4.1) we deduce that $(-a)_{\rm et}(k_{\rm et}(M))$ is M or M' (in Case I); $(-a)_{\rm et}(k_{\rm et}(M)) = M$ (in Cases II, III). Thus, $k_{\rm et}(M)$ is $a_{\rm et}(M)$ or $a_{\rm et}(M')$ (in Case I); $k_{\rm et}(M) = a_{\rm et}(M)$ (in Cases II, III). In particular, $k_{\rm et}(M) \in C_j$, as desired. This observation implies the following:

Lemma 4.3. *Let j be an integer which is ≥ 3 (in Case I); ≥ 2 (in Cases II, III). If ψ is any topological A-linear combination of $\chi_{\mathcal{L}}$-special monomials which is $\equiv 0$ modulo $q^{n \cdot c_j}$, then $(-a)_{\rm et}(\psi)$ is $\equiv 0$ modulo q, for any a such that $\mathrm{Min}(C_j) \leq a \leq \mathrm{Max}(C_j)$.*

Proof. Indeed, such a ψ is a topological A-linear combination of $k_{\rm et}(M)$'s, where k ranges over all elements of $\mathbf{Z}$. If $k_{\rm et}(M) \notin C_j$, then it follows from the above observation that $(-a)_{\rm et}(k_{\rm et}(M)) \equiv 0$ modulo q. Thus, the only terms in the topological A-linear combination of $k_{\rm et}(M)$'s that might yield a nonzero contribution to $(-a)_{\rm et}(\psi)$ modulo q are those for which $k_{\rm et}(M) \in C_j$. But since $\psi \equiv 0$ modulo $q^{n \cdot c_j}$, it follows (from the definition of the set C_j!) that the coefficients of those terms must be $\equiv 0$ modulo q. Thus, $(-a)_{\rm et}(\psi) \equiv 0$ modulo q, as desired. (Note that in this argument, we make essential use of the fact that the elements of the set $\{k_{\rm et}(M)\}_{k \in \mathbf{Z}}$ are (topologically) *linearly independent* over A – a fact which follows immediately from the observation that the exponents of U which appear in distinct elements of this set are distinct.) ○

Now recall the operator δ^χ (i.e., given by $\delta^\chi(?) \stackrel{\rm def}{=} \{\chi(1_{\rm et}) \cdot 1_{\rm et}(?)\} -\, ?$, where here we take $\chi \stackrel{\rm def}{=} \chi_{\mathcal{M}}$) of Chapter IV, §3. Note that if, for instance, ψ is a topological A-linear combination of $\chi_{\mathcal{L}}$-special monomials which is zero modulo $q^{n \cdot c_j}$, then one can compute $\delta^\chi(\psi)$ modulo q as soon as one knows ψ and $1_{\rm et}(\psi)$ modulo q. By induction, one sees easily that one can compute the b-th iterate of this operator $(\delta^\chi)^b(\psi)$ (for some positive integer b) modulo q as soon as one knows $\psi, 1_{\rm et}(\psi), \ldots, b_{\rm et}(\psi)$ modulo q. Thus, Lemma 4.3 has the following consequence/generalization:

Corollary 4.4. *Let j be an integer which is ≥ 3 (in Case I); ≥ 2 (in Cases II, III). If ψ is any topological A-linear combination of $\chi_{\mathcal{L}}$-special monomials which is $\equiv 0$ modulo $q^{n \cdot c_j}$, then*

$$(-a)_{\mathrm{et}}((\delta^{\chi})^{b}(\psi)) \equiv 0 \quad (\text{modulo } q)$$

for any integers a*,* b *such that* $0 \leq b \leq \mathrm{Span}(C_j)$, $\ b + \mathrm{Min}(C_j) \leq a \leq \mathrm{Max}(C_j)$.

Next, let us recall the *twisted Schottky-Weierstrass zeta functions* ζ_i^{χ} (of Chapter IV, Theorem 3.1) associated to the character $\chi \stackrel{\mathrm{def}}{=} \chi_{\mathcal{M}}$. In the following discussion, we shall think of these functions as *sections of* $\mathcal{L}^{\otimes n}_{C^{\infty}_{\widehat{S}}}$ (i.e., *without the superscripted* χ – contrary to the notation of Chapter IV, Theorem 3.1) *on which* $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ *acts via* χ^{-1}. Note that (by Chapter IV, Theorem 3.1) the ζ_i^{χ} are all topological A-linear combinations of $\chi_{\mathcal{L}}$-special monomials, i.e., satisfy the assumptions on ψ in Corollary 4.4. Now let us suppose that for some integer $r \geq 0$, we are given an A-linear combination

$$\sum_{i=0}^{r} \gamma_i \cdot \zeta_i^{\chi}$$

(where the $\gamma_i \in A$) which is $\equiv 0$ modulo $q^{n \cdot c_j}$, for j as in Corollary 4.4. Let ϵ_{Case} be 1 if we are in Case I, and 0 otherwise. Suppose, moreover, that for some $s \leq \mathrm{Span}(C_j) + \epsilon_{\mathrm{Case}}$, we have:

$$\gamma_s \not\equiv 0; \quad \gamma_i \equiv 0 \text{ modulo } q$$

for all $i > s$. Then, by applying Corollary 4.4 to the above sum, we obtain that for some integer a:

$$0 \equiv a_{\mathrm{et}}\Big\{ \sum_{i=0}^{r} \gamma_i \cdot (\delta^{\chi})^{s-\epsilon_{\mathrm{Case}}}(\zeta_i^{\chi}) \Big\} \equiv a_{\mathrm{et}}\Big\{ \sum_{i=s-\epsilon_{\mathrm{Case}}}^{s} \gamma_i \cdot (\delta^{\chi})^{s-\epsilon_{\mathrm{Case}}}(\zeta_i^{\chi}) \Big\} \text{ modulo } q$$

i.e., the terms such that $i > s$ vanish modulo q because of the assumptions on the γ_i; the terms such that $i < s - \epsilon_{\mathrm{Case}}$ vanish because $(\delta^{\chi})^{s-\epsilon_{\mathrm{Case}}}(\zeta_i^{\chi}) = 0$ (by the formula for $\delta^{\chi}(\zeta_i^{\chi})$ given in Chapter IV, Theorem 3.1) for such i. If we are in Cases II or III, then since $(\delta^{\chi})^{s}(\zeta_s^{\chi}) = \pm\zeta_0^{\chi}$, and $\delta^{\chi}(\zeta_0^{\chi}) = \chi_{\mathcal{M}}(1) \cdot 1_{\mathrm{et}}(\zeta_0^{\chi}) - \zeta_0^{\chi} = 0$, we obtain that

$$0 \equiv \gamma_s \cdot a_{\mathrm{et}}(\zeta_0^{\chi}) = \gamma_s \cdot \chi_{\mathcal{M}}(a)^{-1} \cdot \zeta_0^{\chi} \text{ modulo } q$$

i.e., $\gamma_s \equiv 0$ modulo q, which is a *contradiction*. In Case I, we have $(\delta^{\chi})^{s-1}(\zeta_s^{\chi}) = \pm\zeta_1^{\chi} + \gamma \cdot \zeta_0^{\chi}$ (for some $\gamma \in A$), $(\delta^{\chi})^{s}(\zeta_s^{\chi}) = \pm\zeta_0^{\chi}$, so we obtain that

$$0 \equiv \gamma_{s-1} \cdot a_{\mathrm{et}}(\pm\zeta_1^{\chi} + \gamma \cdot \zeta_0^{\chi}) \pm \gamma_s \cdot a_{\mathrm{et}}(\zeta_0^{\chi}) \text{ modulo } q$$

(where the various "$\pm$'s" which appear are *not* necessarily "coordinated"). Moreover, $a_{\text{et}}(\zeta_1^\chi)$ is clearly equal to a root of unity times ζ_1^χ plus an A-multiple of ζ_0^χ, while $a_{\text{et}}(\zeta_0^\chi)$ is equal to a root of unity times ζ_0^χ. On the other hand, by Lemma 4.5 below, we know that in Case I, ζ_0^χ and ζ_1^χ are $\mathcal{O}$-linearly independent modulo q. Thus, we obtain that γ_{s-1} and γ_s are $\equiv 0$ modulo q, which is again a *contradiction*.

Lemma 4.5. *In Case I, the reductions modulo q of ζ_0^χ and ζ_1^χ are linearly independent over every residue field of $\mathcal{O}$.*

Proof. Write $\zeta_0^\chi = \sigma_0 \cdot \theta^m$, $\zeta_1^\chi = \sigma_1 \cdot \theta^m$. Then from $1_{\text{et}}(\theta^m) = q^m \cdot U^n \cdot \theta^m$, $1_{\text{et}}(\zeta_0^\chi) = \chi_{\mathcal{M}}(1)^{-1} \cdot \zeta_0^\chi$, we obtain that if we set

$$\zeta^\chi \stackrel{\text{def}}{=} \frac{1}{2} + \frac{U}{n \cdot \sigma_0} \cdot \frac{\partial \sigma_0}{\partial U}$$

then $1_{\text{et}}(\zeta^\chi) = \zeta^\chi - 1$ (cf. the case of "classical Schottky-Weierstrass zeta function ζ" discussed in Chapter III, §5 – especially Theorem 5.6). Note, here, that the operator $\frac{1}{2} + \frac{U}{n} \cdot \frac{\partial}{\partial U}$ is *integral* (even at primes of $\mathcal{O}$ which divide 2 or n) on $\chi_{\mathcal{L}}$-special monomials in Case I: Indeed, in the notation of Schola 4.1, above, this operator acts on $k_{\text{et}}(M)$ by multiplication by $\frac{1}{2} + \frac{1}{n} \cdot (2mk + m) = k + 1$, and on $(-k)_{\text{et}}(M')$ by multiplication by $\frac{1}{2} + \frac{1}{n}(-2mk - m) = -k$.

In other words, in the terminology of Chapter IV, §3, "$(\sigma_0 \cdot \theta^m) \cdot T + (\sigma_0 \cdot \zeta^\chi \cdot \theta^m)$" is an "extension polynomial on which $\mathbf{Z}_{\text{et}} \times \boldsymbol{\mu}_n$ acts via $\chi_{\mathcal{M}}$." Thus, we may take $\zeta_1^\chi = \sigma_0 \cdot \zeta^\chi \cdot \theta^m$.

On the other hand, ζ_0^χ modulo q may be computed as in the proof of Chapter IV, Theorem 2.1, to be equal to (an $A^\times$-multiple of)

$$(U^m + \gamma \cdot U^{-m}) \cdot \theta^m$$

(where γ is a root of unity). If we apply the operator $\frac{1}{2} + \frac{U}{n} \cdot \frac{\partial}{\partial U}$ to the coefficient of θ^m in this expression, we obtain

$$U^m \cdot \theta^m$$

Thus, since γ is a root of unity, hence $\in \mathcal{O}^\times$, we obtain that these two expressions are linearly independent over every residue field of $\mathcal{O}$, as desired. ○

Thus, we see that we have essentially proven the following result:

Theorem 4.6. *Let $\chi_{\mathcal{L}} \in \text{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$ be a character. Let $j \geq 2$ be an integer. Then the twisted Schottky-Weierstrass ζ-functions $\zeta_0^\chi, \ldots, \zeta_{j-1}^\chi$ (cf. Chapter IV, Theorem 3.1) associated to $\chi \stackrel{\text{def}}{=} \chi_{\mathcal{M}}$ satisfy the following:*

Case I: $\chi_{\mathcal{L},\boldsymbol{\mu}} = 1$. *If $j = 2$, then the reductions modulo q of ζ_0^χ and ζ_1^χ are linearly independent over $A/(q) = \mathcal{O}$. If $j \geq 3$, let $j_0 \stackrel{\text{def}}{=} 2k+1 \geq 3$ be any odd integer such that $j \geq j_0 \geq 3$. Let $\epsilon_{\text{Case}} \stackrel{\text{def}}{=} 1$.*

Case II: $\chi_{\mathcal{L},\boldsymbol{\mu}} = \chi_\theta|_{\boldsymbol{\mu}_n}$. *Let $j_0 \stackrel{\text{def}}{=} 2k \geq 2$ be any even integer such that $j \geq j_0 \geq 2$. Let $\epsilon_{\text{Case}} \stackrel{\text{def}}{=} 0$.*

Case III: $\chi_{\mathcal{L},\boldsymbol{\mu}}{}^2 \neq 1$. *Let $j_0 \geq 2$ be any integer such that $j \geq j_0 \geq 2$. Let $\epsilon_{\text{Case}} \stackrel{\text{def}}{=} 0$.*

In all the Cases, let $B \stackrel{\text{def}}{=} A/(q^{n\cdot c_{j_0}})$. Then there exists a B-submodule $R \subseteq B^j$ – i.e., a submodule of relations *– such that:*

(i) $\sum_{i=0}^{j-1} \gamma_i \cdot \zeta_i^\chi \equiv 0$ modulo $q^{n\cdot c_{j_0}}$ *for all $(\gamma_0, \ldots, \gamma_{j-1}) \in R$;*

(ii) the projection $R \to B^{j-j_0+1}$ onto the last $j - j_0 + 1$ factors is an isomorphism.

Finally, we have:

$(*)$ *Any $(\gamma_0, \ldots, \gamma_{j-1}) \in B^j$ such that $(\gamma_0, \ldots, \gamma_{j-1}) \notin q \cdot B^j$, and* $\sum_{i=0}^{j-1} \gamma_i \cdot \zeta_i^\chi \equiv 0$ modulo $q^{n\cdot c_{j_0}}$ *has at least one $i \geq$* $\mathrm{Span}(C_{j_0}) + 1 + \epsilon_{\text{Case}} = j_0 - 1 = |C_{j_0}|$ *(i.e., the cardinality of C_{j_0}) such that* $\gamma_i \not\equiv 0$ modulo q.

In fact, this statement $(*)$ holds "modulo $\mathfrak{p}$" for any maximal prime $\mathfrak{p}$ of $\mathcal{O}$, i.e.:

$(*_\mathfrak{p})$ *Any $(\gamma_0, \ldots, \gamma_{j-1}) \in B^j$ such that $(\gamma_0, \ldots, \gamma_{j-1}) \notin (q, \mathfrak{p}) \cdot B^j$, and* $\sum_{i=0}^{j-1} \gamma_i \cdot \zeta_i^\chi \equiv 0$ modulo $(q^{n\cdot c_{j_0}}, \mathfrak{p})$ *has at least one $i \geq$* $\mathrm{Span}(C_{j_0}) + 1 + \epsilon_{\text{Case}} = j_0 - 1 = |C_{j_0}|$ *(i.e., the cardinality of C_{j_0}) such that $\gamma_i \not\equiv$* 0 modulo $(q, \mathfrak{p})$.

Proof. Indeed, the statement $(*)$ was precisely what we proved above; its "modulo $\mathfrak{p}$" version $(*_\mathfrak{p})$ follows by precisely the same argument. When $j = 2$ in Case I, Theorem 4.6 follows from Lemma 4.5. Thus, let us assume that we are in Case I and $j \geq 3$, or that we are in Cases II or III and $j \geq 2$. To see that there exists a B-submodule R in each Case satisfying properties (i) and (ii), let us first observe that the map

$$(\gamma_0, \ldots, \gamma_{j-1}) \mapsto \sum_{i=0}^{j-1} \gamma_i \cdot \zeta_i^{\chi} \text{ modulo } q^{n \cdot c_{j_0}}$$

defines a B-linear morphism

$$\Xi : B^j \to \left\{ B\text{-linear combinations of the monomials } \in C_{j_0} \text{ modulo } q^{n \cdot c_{j_0}} \right\}$$

Note that because of the way in which j_0 was chosen in each of the three cases (and because of the way in which we defined C_{j_0}) it follows that C_{j_0} has cardinality $j_0 - 1$. We would like to show that the kernel of Ξ is isomorphic (as a B-module) to B^{j-j_0+1} (via the projection of (ii)). *Suppose that we can show this after base-changing from $\mathcal{O}$ to an arbitrary residue field of $\mathcal{O}$.* Then it will follow from elementary commutative algebra (by thinking about finitely generated modules over the principal ideal domain $\mathcal{O}$) that the kernel of Ξ is *abstractly* isomorphic to B^{j-j_0+1} *as an $\mathcal{O}$-module*; moreover, the projection of (ii) will then be a morphism between free $\mathcal{O}$-modules of the same rank which is bijective modulo all of the maximal primes of $\mathcal{O}$. Thus, it will follow that the projection of (ii) is bijective over $\mathcal{O}$, hence that $\mathrm{Ker}(\Xi)$ is isomorphic as a B-module to B^{j-j_0+1} via the projection of (ii), as desired.

Thus, it suffices to show (i) and (ii) after base-changing to an arbitrary residue field of $\mathcal{O}$. Let us denote the objects obtained by executing this base-change by means of a subscript "res." Then the range of Ξ_{res} is a B_{res}-module which is a quotient of $B_{\mathrm{res}}^{j_0-1}$. In particular, it follows from elementary commutative algebra (since B_{res} is a quotient of the principal ideal domain A_{res}) that there exists a B_{res}-submodule $R_{\mathrm{res}} \subseteq B_{\mathrm{res}}^j$ that satisfies (i) and which is $\cong B_{\mathrm{res}}^{j-j_0+1}$. The fact that such an R_{res} also satisfies (ii) is a formal consequence of $(*_{\mathfrak{p}})$. This completes the proof of the "res" case, and hence of the entire Theorem. ○

Remark. From a certain point of view, Theorem 4.6 is the *key technical result* behind the main theorems of this paper.

Corollary 4.7. *Let $\chi_{\mathcal{L}} \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$ be a character. Let $j \geq 2$ be an integer. Then the twisted Schottky-Weierstrass ζ-functions $\zeta_0^{\chi}, \ldots, \zeta_{j-1}^{\chi}$ (cf. Chapter IV, Theorem 3.1) satisfy the following property: Let j' be defined as follows:*

Case I: *Let j' be the smallest odd integer $\geq j+1$.*

Case II: *Let j' be the smallest even integer $\geq j+1$.*

Case III: *Let $j' \stackrel{\mathrm{def}}{=} j+1$.*

Let $\mathfrak{p}$ be a maximal prime of $\mathcal{O}$. Then if $\gamma_0, \ldots, \gamma_{j-1} \in A$ are such that

$$\sum_{i=0}^{j-1} \gamma_i \cdot \zeta_i^{\chi} \equiv 0 \text{ modulo } q^{n \cdot c_{j'}} \quad \text{(respectively, modulo } (q^{n \cdot c_{j'}}, \mathfrak{p}))$$

then all of the γ_i's are $\equiv 0$ modulo q (respectively, modulo $(q, \mathfrak{p})$). Finally, j' satisfies the property: $|C_{j'}| = j' - 1 \leq j + 1$; if j is odd, then $|C_{j'}|$ is $= j + 1$ (in Case I), $= j$ (in Cases II, III); if j is even, then $|C_{j'}|$ is $= j + 1$ (in Case II), $= j$ (in Cases I, III).

Proof. The final statement follows immediately from the definitions. Thus, let us prove the statement concerning linear independence. To simplify the notation, we prove the result in the "non-resp'd" case. (The proof in the "resp'd" case is entirely similar.) First, j' is defined as "the smallest j_0 (as in Theorem 4.6) which is $\geq j+1$." Then if we let $\gamma_i \stackrel{\text{def}}{=} 0$ for $i \geq j$, suppose that there exists some $\gamma_i \not\equiv 0$ modulo q, and apply the statement $(*)$ of Theorem 4.6, we obtain that there exists some $i \geq j' - 1 \geq j$ for which $\gamma_i \not\equiv 0$ modulo q. But in light of the definition of the γ_i, this is absurd. This contradiction completes the proof. ○

Finally, we close this § by observing that, although in the above discussion, we took an abstract point of view with respect to showing the existence of linear combinations of the ζ_i^{χ} satisfying certain congruence property modulo powers of q, in fact, it is possible to give an *explicit description* of the linear combinations of ζ_i^{χ}'s that arise. Indeed, one has the following result:

Theorem 4.8. (Congruence Canonical Schottky-Weierstrass Zeta Functions) *Let $\sigma_{\chi} \in \Gamma(C^{\infty}_{\widehat{S}}, (\mathcal{L}^{\otimes n}_{C^{\infty}_{\widehat{S}}})^{\chi})$ be as in Chapter IV, §3. Let $i_{\chi} \in \{-m, -m+1, \ldots, m-1\}$ be as in Chapter IV, Theorem 3.3. (Note that this notation is consistent with the notation of the present § for Case III.) Then, up to an $A^{\times}$-multiple, σ_{χ} is equal to the series*

$$\sum_{k \in \mathbf{Z}} q^{m \cdot k^2 + i_{\chi} \cdot k} \cdot U^{2mk + i_{\chi}} \cdot \theta^m \cdot \chi(k_{\text{et}}) = \sum_{k \in \mathbf{Z}} k_{\text{et}}(U^{i_{\chi}} \cdot \theta^m) \cdot \chi(k_{\text{et}})$$

Now if r is a nonnegative integer, let $L_r(T) \stackrel{\text{def}}{=} T + \lambda_r - \frac{i_{\chi}}{2m}$, where $\lambda_r \in \mathbf{Z}$ is defined as follows:

$$\lambda_r \stackrel{\text{def}}{=} \left[\frac{r}{2} + \frac{i_{\chi}}{2m}\right]$$

i.e., the greatest integer $\leq$ the number in brackets. Then if we set ζ_0^{CG} to be equal to the above series, and

$$\zeta_r^{\mathrm{CG}} \stackrel{\mathrm{def}}{=} \binom{L_r(\delta^*)}{r}(\zeta_0^{\mathrm{CG}}) = \sum_{k \in \mathbf{Z}} \binom{k+\lambda_r}{r} \cdot q^{m\cdot k^2 + i_\chi \cdot k} \cdot U^{2mk+i_\chi} \cdot \theta^m \cdot \chi(k_{\mathrm{et}})$$

$$\zeta_r^{\mathrm{CG}}[T] \stackrel{\mathrm{def}}{=} \binom{L_r(\delta^*+T)}{r}(\zeta_0^{\mathrm{CG}}) = \sum_{i=0}^{r} \zeta_i^{\mathrm{CG}} \cdot T^{[r-i]}$$

(where δ^*, T are as in Chapter IV, Theorem 3.3), then for any positive integer j, we have

$$\zeta_{j-1}^{\mathrm{CG}} \equiv 0 \text{ modulo } q^{n\cdot c_{j_0}}$$

(where j_0 is as in Theorem 4.6). Finally, the ζ_r^{CG} are integral over $\mathbf{Z}$, and form a collection of twisted higher Schottky-Weierstrass zeta functions as in Chapter IV, Theorem 3.1.

Proof. Note that the series given for (an $A^\times$-multiple of) σ_χ is $\mathbf{Z}_{\mathrm{et}}$-invariant. On the other hand, since the set of $\mathbf{Z}_{\mathrm{et}}$-invariant elements of $\Gamma(C^\infty_{\widehat{S}}, (\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}})^\chi)$ forms a free A-module of rank 1, it follows immediately that σ_χ is an $A^\times$-multiple of this series. The explicit series formula for ζ_r^{CG} then follows immediately. In particular, since $L_r(k+\frac{i_\chi}{2m})$ will always be $\in \mathbf{Z}$, it follows immediately that ζ_r^{CG} is integral over $\mathbf{Z}$. Since δ^*+T is $\mathbf{Z}_{\mathrm{et}}$-invariant, it follows immediately that the ζ_r^{CG} form a collection of twisted higher Schottky-Weierstrass zeta functions as in Chapter IV, Theorem 3.1. In fact, *one can even write the ζ_r^{CG} as explicit linear combinations of the $\zeta_i^{\mathrm{BI},\chi}$'s (of Chapter IV, Theorem 3.3) by using Chapter III, Lemma 7.5.*

Thus, it remains only to show that $\zeta_{j-1}^{\mathrm{CG}}$ has the desired *congruence property.* The point here is to observe that if we define a *filtration* $F^r(\mathbf{Z})$ (for $r \geq 0$ an integer) on $\mathbf{Z}$ by:

$$F^r(\mathbf{Z}) \stackrel{\mathrm{def}}{=} \{0-\lambda_r, 1-\lambda_r, \ldots, r-1-\lambda_r\}$$

then $F^{r+1}(\mathbf{Z})$ is obtained from $F^r(\mathbf{Z})$ by appending one more integer "$k[r]$" directly to the left/right of $F^r(\mathbf{Z})$ (where "left/right" depends only on the parity of r), and, moreover, *the exponent of q in the $\chi_{\mathcal{L}}$-special monomial corresponding to this "$k[r]$" is $\geq$ the exponents of q in the $\chi_{\mathcal{L}}$-special monomial corresponding to the integers "k" in $F^r(\mathbf{Z})$*. In particular, it follows that for each of the $\chi_{\mathcal{L}}$-special monomials

$$q^{m\cdot k^2+i_\chi\cdot k} \cdot U^{2mk+i_\chi} \cdot \theta^m$$

belonging to C_{j_0}, $L_{j-1}(k+\frac{i_\chi}{2m})$ is a *nonnegative integer* $\leq j-2$. Thus, $\binom{L_{j-1}(k+\frac{i_\chi}{2m})}{j-1} = 0$. That is to say, the coefficients (in the series defining $\zeta_{j-1}^{\mathrm{CG}}$) of all the $\chi_{\mathcal{L}}$-special monomials in C_{j_0} are *zero*. This proves the desired congruence property. ○

§5. The Determinant of the Evaluation Map

In this §, we prepare for the discussion of the following § by showing that (at least if the base is an algebraically closed field, and the elliptic curve in question is "sufficiently generic," then) the *determinants of the evaluation maps* of Propositions 2.2, 2.3, have a *particularly simple form*, so long as they are not identically zero (Theorem 5.6).

Here, we use the notation of §2, except that we also assume (unless specified otherwise) that $S = \mathrm{Spec}(k)$, where k is an *algebraically closed field*.

Remark. In fact, logically speaking, we shall only use the theory of this § in the case where $\mathrm{char}(k) = 0$. Thus, *the reader who is only interested in material that is logically necessary for the proofs of the main theorems of this paper may assume in this § that* $\mathrm{char}(k) = 0$.

In particular, in this §, $E \to S$ will be a (proper!) elliptic curve, and $S = S_\infty$. Thus, since we do not have to deal with log structures, metrized line bundles, etc., the discussion of §2 simplifies substantially. Also, in this §, we will assume that

$$\overline{\mathcal{L}} = \mathcal{L} = \mathcal{O}_E(d \cdot [e])$$

and that we are given a *symmetric* Lagrangian subgroup

$$H \subseteq \mathcal{G}_{\mathcal{L}}$$

i.e., a Lagrangian subgroup which is mapped to itself under the automorphism of $\mathcal{G}_{\mathcal{L}}$ induced by the automorphism $(-1) : E \to E$ ("multiplication by -1") of E. Also, we assume that H is *of multiplicative type*, i.e., that H is étale locally isomorphic to $\boldsymbol{\mu}_d$. Since, however, k is algebraically closed, this implies that $H \cong \boldsymbol{\mu}_d$. Let us fix an identification:

$$H = K_H = \boldsymbol{\mu}_d$$

Then

$$_dE_H = {_dE}/K_H \cong H^* = \mathbf{Z}/d\mathbf{Z}$$

(where the "$*$" denotes the Cartier dual). In particular, under these assumptions, if $\mathrm{char}(k) = p > 0$, then it follows that E is an *ordinary* elliptic curve (i.e., its Hasse invariant is nonzero).

Conversely, if $\mathrm{char}(k) = 0$ or E is *ordinary*, then I *claim* that there exists a symmetric Lagrangian subgroup $H \subseteq \mathcal{G}_{\mathcal{L}}$ of multiplicative type. Indeed, by the discussion preceding Chapter IV, Definition 1.3, it follows that the existence of such an H amounts to the issue of whether or not a certain exact sequence $\mathbf{Z}$-modules (character groups)

$$0 \to \mathbf{Z}/d\mathbf{Z} \to ?? \to \mathbf{Z} \to 0$$

equipped with a $\mathbf{Z}/2\mathbf{Z} = \{\pm 1\}$-action *splits.* If d is odd, then the existence of such a splitting follows from the fact that $H^1(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/d\mathbf{Z}) = 0$. If d is even, and k is of characteristic 0, the existence of such a splitting follows from "the existence of a symmetric θ-structure on a totally symmetric ample line bundle" [Mumf1], Remark 2, p. 318. If d is even, and k is of positive characteristic, then the existence of such a splitting follows from the fact that the relevant class in $H^1(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/d\mathbf{Z})$ is locally constant, hence may be computed after lifting the elliptic curve in question to characteristic 0. This completes the verification of the claim.

Thus, other than the condition that E be ordinary if $\mathrm{char}(k) > 0$, the assumption of the existence of an H as above does not result in any loss of generality.

The Lagrangian subgroup $H \subseteq \mathcal{G}_{\mathcal{L}}$ defines a line bundle $\mathcal{L}_H$ on $E_H = E/K_H$ which descends $\mathcal{L}$. Since $\mathcal{L}$ and $H \subseteq \mathcal{G}_{\mathcal{L}}$ are *symmetric*, it follows that $\mathcal{L}_H$ is also *symmetric.* Since $\mathcal{L}_H$ is of degree 1, it thus follows that

$$\mathcal{L}_H \cong \mathcal{O}_{E_H}(\epsilon_H)$$

for some unique $\epsilon_H \in E_H(k)$, which satisfies $2 \cdot \epsilon_H = 0$. If G is a finite, flat subgroup scheme of E or E_H, let us write $[G]$ for the divisor in E or E_H defined by G. We will also use this notation for translates of G. Before continuing our discussion, we need certain basic facts concerning the various line bundles that appear:

Lemma 5.1. *We have:*

$$[K_H] \sim (d-1) \cdot [e] + [\delta]$$

where "$\sim$" denotes linear equivalence of divisors on E; $\delta = e$ if d is odd; and $\delta = (\frac{1}{2})\boldsymbol{\mu} \in K_H(k) \subseteq E(k)$, i.e., the unique generator of the subgroup $\boldsymbol{\mu}_2(k) \subseteq \boldsymbol{\mu}_d(k) = K_H(k)$, if d is even.

Proof. Indeed, this sort of equality may be proven by lifting to characteristic 0. Thus, for the rest of the proof, we assume that $\mathrm{char}(k) = 0$. Then K_H is étale over k, so

$$[K_H] = \sum_{\alpha \in K_H(k)} [\alpha]$$

If $\alpha \in K_H(k)$ is such that $\alpha \neq -\alpha$, then both $[\alpha]$ and $[-\alpha]$ appear in the sum, and $[\alpha] + [-\alpha] \sim 2[e]$. Moreover, the only $[\alpha]$'s such that $\alpha = -\alpha$ are $[e]$ and $[\delta]$. Substituting these two observations into the above sum thus yields the result. ○

Lemma 5.2. *We have: $\epsilon_H = e_H$ (the origin of E_H) if d is odd; $\epsilon_H \in {}_dE_H(k)$ if d is even and* $\mathrm{char}(k) = 2$; *and $\epsilon_H \in E_H(k) \backslash {}_dE_H(k)$ is of order precisely* 2 *if d is even and* $\mathrm{char}(k) \neq 2$.

Proof. Suppose first that d is *odd*. Then since the line bundle $\mathcal{L} = \mathcal{O}_E(d \cdot [e])$ is not fixed by translation by any point of $E(k)$ of order precisely 2, and, moreover, the points of order 2 of $E(k)$ are in bijective correspondence with the points of order 2 of $E_H(k)$, the fact that $\epsilon_H = e_H$ follows from the fact that $\mathcal{L} \cong \mathcal{O}_E([K_H])$ (cf. Lemma 5.1).

Now suppose that d is *even* and $\mathrm{char}(k) = 2$. In this case, $\mathbf{Z}/2\mathbf{Z} \subseteq \mathbf{Z}/d\mathbf{Z} \cong {}_dE_H(k)$ is equal to the entire set of points in $E_H(k)$ annihilated by 2. Thus, $\epsilon_H \in {}_dE_H(k)$.

Finally, suppose that d is *even* and $\mathrm{char}(k) \neq 2$. Since we know that $2 \cdot \epsilon_H = 0$, it follows that it suffices to prove that $\epsilon_H \notin {}_dE_H(k)$. Next, observe that since there exists an isomorphism ${}_dE \cong H \times H^*$ such that the projection to the factor H^* coincides with the morphism ${}_dE \to {}_dE_H \cong H^*$, it follows that there exists an element $\widetilde{\epsilon} \in E(k)$ whose order is precisely 2 and which maps to the nonzero element of $\mathbf{Z}/2\mathbf{Z} \subseteq \mathbf{Z}/d\mathbf{Z} \cong {}_dE_H(k) \subseteq E_H(k)$. On the other hand, by Lemma 5.1, we have $\mathcal{O}_E([K_H]) = (d-1) \cdot [e] + [(\frac{1}{2})\boldsymbol{\mu}] \not\cong \mathcal{L}$, so $\epsilon_H \neq e_H$. Since $\mathcal{T}^*_{\widetilde{\epsilon}}(\mathcal{L}) \cong \mathcal{L}$ (here we use the fact that d is even), it thus follows that $\mathcal{O}_E([\widetilde{\epsilon} + K_H]) \not\cong \mathcal{L}$, which implies that ϵ_H is not equal to the image of $\widetilde{\epsilon}$ in $E_H(k)$. Thus, we conclude that $\epsilon_H \notin {}_dE_H(k)$, as desired. ○

Corollary 5.3. *Suppose that d is odd, or that d is even and* $\mathrm{char}(k) = 2$. *Then $[\epsilon_H + {}_dE_H] = [{}_dE_H]$ (an equality, not just a linear equivalence, of divisors on E_H).*

Let us write $\mathcal{M} \stackrel{\mathrm{def}}{=} \mathcal{O}_{E_H}([\epsilon_H + {}_dE_H])$. Note that there is an evident action of ${}_dE_H$ on $\mathcal{M}$ (resulting from the fact that ${}_dE_H$ stabilizes the divisor $[\epsilon_H + {}_dE_H]$). If one descends to $\mathcal{M}$ to $E = E_H/{}_dE_H$ via this action, then the resulting line bundle on E is simply

$$\mathcal{O}_{E_H}([\epsilon])$$

where $\epsilon \in E(k)$ is the image of ϵ_H.

Lemma 5.4. *Suppose that d is even, and* $\mathrm{char}(k) \neq 2$. *Let $\mathcal{L}^\chi$ be the line bundle on $E = E_H/{}_dE_H$ given by descending the line bundle $\mathcal{M}$ relative to the action of ${}_dE_H \cong \mathbf{Z}/d\mathbf{Z}$ on $\mathcal{M}$ given by twisting the evident action by the unique nontrivial character $\chi : {}_dE_H \cong \mathbf{Z}/d\mathbf{Z} \to \mu_2 = \{\pm 1\}$ of order* 2. *Then $\mathcal{L}^\chi \cong \mathcal{O}_E(e)$.*

Proof. Clearly, $\deg(\mathcal{L}^\chi) = 1$. Since $2 \cdot \epsilon_H = 0$, and $\chi^{\otimes 2} = 1$, it follows that $\mathcal{L}^\chi$ is *symmetric*. Thus, $\mathcal{L}^\chi \cong \mathcal{O}_E(\epsilon_\chi)$, for some unique $\epsilon_\chi \in E(k)$, which is necessarily of order 2. As observed above, if one descends $\mathcal{M}$ via the evident action of ${}_dE_H$ on $\mathcal{M}$, one gets $\mathcal{O}_{E_H}([\epsilon])$, where $\epsilon \in E(k)$ is the image of ϵ_H. Moreover, $\epsilon = (\frac{1}{2})\boldsymbol{\mu}$ (by Lemma 5.2). Since the kernel of the morphism $\mathrm{Pic}^0(E) \to \mathrm{Pic}^0(E_H)$ induced by $E_H \to E = E_H/{}_dE_H$ has a unique element of order precisely 2, it follows that ϵ_χ is the unique element of $E(k)$ of order ≤ 2 such that $\epsilon_\chi \neq \epsilon$, $\mathcal{O}_E(\epsilon_\chi)|_{E_H} \cong \mathcal{M}$. But observe that $\mathcal{O}_E(e)|_{E_H} = \mathcal{O}_{E_H}([{}_dE_H]) \cong \mathcal{M}$ (where the last isomorphism follows from the fact that $\mathcal{M}$ is of even degree, hence fixed by translation by ϵ_H, which is order 2). Thus, we conclude that $\epsilon_\chi = e$, as desired. ○

Corollary 5.5. *Suppose that $0 \neq s \in \Gamma(E_H, \mathcal{M})$ has the following property with respect to the evident action of ${}_dE_H$ on $\mathcal{M}$: s is fixed by this action if d is odd, or if d is even and* $\mathrm{char}(k) = 2$*; ${}_dE_H$ acts on s by the character χ (of Lemma 5.4) if d is even and* $\mathrm{char}(k) \neq 2$*. Then the inverse image in E via $E \to E_H = E/K_H$ of the zero locus $V(s)$ of s is equal to the divisor $[{}_dE]$.*

Proof. Indeed, if d is odd, or if d is even and $\mathrm{char}(k) = 2$, then s descends to a nonzero section $s' \in \Gamma(E_H, \mathcal{O}_E(\epsilon))$. Since $\mathcal{O}_E(\epsilon)$ is of degree 1, $V(s') = [\epsilon]$. Thus, $V(s) = [\epsilon_H + {}_dE_H] = [{}_dE_H]$ (by Corollary 5.3), so the inverse image in E of $V(s)$ is $[{}_dE]$, as desired. If d is even and $\mathrm{char}(k) \neq 2$, then (by Lemma 5.4) s descends to a nonzero section $s' \in \Gamma(E_H, \mathcal{O}_E(e))$. Since $\mathcal{O}_E(e)$ is of degree 1, $V(s') = [e]$, so the inverse image in E of $V(s)$ is $[{}_dE]$, as desired. ○

Now we are ready to return to our discussion of the *evaluation maps* of §2. In the present discussion, we shall omit the symbol "∞" from the various notations of §2, since here, $S = S_\infty$, $E = E_\infty$, etc. We would like to consider the morphism $\Xi^H_{\mathcal{L},d,\alpha}$ of Proposition 2.3, for a point $\alpha \in E^\dagger(k)$. Write

$$V \stackrel{\mathrm{def}}{=} \Gamma(E^\dagger_{H,[d]}, \mathcal{L}_H|_{E^\dagger_{H,[d]}})^{<d}$$

Also, let us denote the elements of ${}_dE_H = \mathbf{Z}/d\mathbf{Z}$ by $(\frac{i}{d})_H$, for $i = 0, \ldots, d-1$. Then the morphism of Proposition 2.3 becomes

$$\Xi^H_\alpha : V \to \bigoplus_{i=0}^{d-1} \left\{ \mathcal{O}_{E_H}([\epsilon_H + (\frac{i}{d})_H])|_\alpha \right\}$$

(where we identify $\mathcal{L}_H$ with $\mathcal{O}_{E_H}([\epsilon_H])$).

Now let $\beta \in {}_dE^\dagger_H(k) \cong {}_dE_H(k)$. We would like to consider the relationship between Ξ^H_α and $\Xi^H_{\alpha+\beta}$. Note that both maps involve restricting sections $\in V$ to the *same* points inside $E^\dagger_{H,[d]}$, except in a different order. Thus, one may think of $\Xi^H_{\alpha+\beta}$ as the result of composing Ξ^H_α with the permutation of the summands in the direct sum on the right (which are indexed by $i = 0, \ldots, d-1$) by the permutation $i \mapsto i + i_\beta$ modulo d (where

$(\frac{i_\beta}{d})_H \in {}_dE_H(k)$ is the element defined by β). Thus, if one takes determinants, one obtains that the determinant

$$\det(\Xi^H_{\alpha+\beta}) \in \det(V)^{-1} \otimes_k \mathcal{O}_{E_H}([\epsilon_H + {}_dE_H])|_\alpha$$

is equal to

$$(-1)^{\text{sgn}(\beta)} \cdot \det(\Xi^H_\alpha) \in \det(V)^{-1} \otimes_k \mathcal{O}_{E_H}([\epsilon_H + {}_dE_H])|_\alpha$$

where $\text{sgn}(\beta)$ is the sign of the permutation $i \mapsto i + i_\beta$ modulo d. Note that if d is *odd*, or if d is even and $\text{char}(k) = 2$, then $(-1)^{\text{sgn}(\beta)}$ is always 1. If d is *even* and $\text{char}(k) \neq 2$, then $(-1)^{\text{sgn}(\beta)} = \chi(\beta)$, where χ is the character of Lemma 5.4. Thus, if one lets α be the *tautological point* $\in E^\dagger(E^\dagger)$, one gets a section

$$\det(\Xi^H_\alpha) \in \det(V)^{-1} \otimes_k \Gamma(E^\dagger, \mathcal{O}_{E_H}([\epsilon_H + {}_dE_H])|_\alpha)$$

which, by Lemma 2.4, descends to E. Moreover, clearly this determinant of Ξ^H_α only depends on the image of α in E_H. Thus, since this image is just the tautological point $\in E_H(E_H)$, we get a section

$$\det(\Xi^H_\alpha) \in \det(V)^{-1} \otimes_k \Gamma(E_H, \mathcal{O}_{E_H}([\epsilon_H + {}_dE_H])) = \det(V)^{-1} \otimes_k \Gamma(E_H, \mathcal{M})$$

which, by the above discussion, *satisfies the hypotheses of Corollary 5.5* (as soon as it is nonzero). In other words, we see that the zero locus of $\det(\Xi^H_\alpha)$ on E is equal either to all of E or to $[{}_dE]$. That is to say, we have proven the following:

Theorem 5.6. *Let E be an elliptic curve over an algebraically closed field k. Suppose that E is* ordinary *if* $\text{char}(k) > 0$. *Let $d \geq 1$ be an integer; $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(d \cdot [e])$; and $\alpha \in E^\dagger(k)$. Then if the morphism*

$$\Xi_{\mathcal{L},d,\alpha} : \Gamma(E^\dagger_{[d]}, \mathcal{L}|_{E^\dagger_{[d]}})^{<d} \to \mathcal{L}|_{\mathcal{T}^*_\alpha({}_dE^\dagger)}$$

is an isomorphism for any single $\alpha \in E^\dagger(k)$, then it follows that it is an isomorphism for all $\alpha \in E^\dagger(k)$ except for those which map to ${}_dE(k) \subseteq E(k)$. Moreover, if this morphism is an isomorphism for any single $\alpha \in E^\dagger(k)$, then the scheme-theoretic zero locus of the determinant of this morphism in the case where α is the tautological point $\in E^\dagger(E^\dagger)$ is precisely d times *the divisor $E^\dagger \times_E ({}_dE) \subseteq E^\dagger$. Finally, the same (except with "d times" in the preceding sentence deleted) assertions hold for the morphism*

$$\Xi^H_{\mathcal{L},d,\alpha} : \Gamma(E^{\dagger}_{H,[d]}, \mathcal{L}_H|_{E^{\dagger}_{H,[d]}})^{<d} \to \mathcal{L}_H|_{\mathcal{T}^*_\alpha({}_dE^{\dagger}_H)}$$

where $H \subseteq \mathcal{G}_{\mathcal{L}}$ is any Lagrangian subgroup (cf. Chapter IV, Definition 1.3).

Proof. It remains only to note that, in the statement of Theorem 5.6, it is not necessary to assume that the Lagrangian subgroup is symmetric or of multiplicative type. Indeed, the assertions in the statement of Theorem 5.6 concerning the Lagrangian subgroup follow immediately from the general theory of §2, especially Proposition 2.3; Chapter IV, Theorem 1.4. Note that we need the "d times" in the "non-Lagrangian case" because the matrices appearing in the non-Lagrangian case amount essentially to d copies (arranged diagonally) of the matrices appearing in the Lagrangian case. Thus, the determinants must be raised to the d-th power. ○

§6. The Generic Case

In this §, we give an explicit computation of a certain *evaluation map* (i.e., a map like that defined in §2) defined for certain degenerating elliptic curves in mixed characteristic. In particular, we will show that the determinant of this map is nonzero in mixed characteristic. This result, combined with the results of §5, will allow us to conclude *the generic bijectivity of certain evaluation maps in mixed characteristic*. Moreover, this sort of generic bijectivity result will be the crucial technical ingredient in the proof of the main results of Chapter VI.

In this §, we use the notation of the first portion of §3, i.e., the discussion of the degenerating case (preceding the "Analytic Continuation Argument"). Let $N \geq 1$ be an integer which is prime to d and *invertible* in the base ring $\mathcal{O}$ (a Zariski localization of the ring of integers of a number field). Let p be a prime number that appears as the characteristic of a residue field of $\mathcal{O}$. We would like to concentrate our attention, in this §, on *integral structures at the prime p*. Let

$$A_N \stackrel{\text{def}}{=} \mathcal{O}[[q^{\frac{1}{N\cdot d}}]]; \quad S_N \stackrel{\text{def}}{=} \text{Spec}(\mathcal{O}[[q^{\frac{1}{N\cdot d}}]]) = \text{Spec}(A_N)$$

Thus, we have a natural finite morphism $S_N \to S$, which we regard as a "base extension" of the base $S = \text{Spec}(\mathcal{O}[[q]])$ of (the first portion of) §3. Note that if one thinks of $\widetilde{E}$ as being "$\mathbf{G}_{\mathrm{m}}/q^{d\cdot\mathbf{Z}}$" then $q^{d/N} \in \mathbf{G}_{\mathrm{m}}(U_{S_N})$ (where $U_{S_N} \stackrel{\text{def}}{=} \text{Spec}(A_N[q^{-1}])$ ($\subseteq S_N$)) defines a *rational point*

$$\eta \in \widetilde{C}_d(S_N)$$

which, relative to the group structure on $\widetilde{C}_d$ over U_S, is annihilated by N. Similarly, $q \in \mathbf{G}_{\mathrm{m}}(U_{S_N})$ defines a *rational point*

$$\tau \in \widetilde{E}_d(S) \subseteq \widetilde{C}_d(S_N)$$

which is a torsion point of order d. In particular, by using the group scheme structure on $\widetilde{E}_d$ (together with the fact that the action of $\widetilde{E}_d$ on itself extends to an action of $\widetilde{E}_d$ on $\widetilde{C}_d$), we may also form the points

$$\eta \cdot \tau^{\beta} \in \widetilde{C}_d(S_N)$$

(for $\beta \in \mathbf{Z}$). (Here, we use the letter "β" so that our notation will be consistent with the notation in the discussion of d-torsion points in §3.) Note that, relative to the theory of Chapter IV, §4, the point of the "$\mathbf{S^1}$" associated to $\widetilde{E}$ defined by τ (respectively, $\eta \cdot \tau^{\beta}$) is equal to $\frac{1}{d}$ (respectively, $\frac{1}{N} + \frac{\beta}{d}$).

Next, let us write $e_{\widetilde{E}_d} \subseteq \widetilde{E}(S)$ for the identity element of the group scheme $\widetilde{E}$. Write

$$\widetilde{\mathcal{L}} \stackrel{\mathrm{def}}{=} \mathcal{O}_{\widetilde{C}_d}(e_{\widetilde{E}_d})$$

for the corresponding line bundle on $\widetilde{C}_d$. Note that since the points $\eta \cdot \tau^{\beta}$ do not intersect $e_{\widetilde{E}_d}$, it follows that the restriction of the line bundle $\widetilde{\mathcal{L}}$ to any of these points $\eta \cdot \tau^{\beta}$ admits a *natural trivialization*. In this §, we would like to consider the following *evaluation map* (cf. the evaluation maps defined in §2):

$$\Xi_{\eta} : \Gamma(\widetilde{C}_d, \widetilde{\mathcal{L}} \otimes_{\mathcal{O}_{\widetilde{C}_d}} F^d(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{\widetilde{C}_d,[d]}})) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_N} \longrightarrow \bigoplus_{\beta=0}^{d-1} (\widetilde{\mathcal{L}}|_{\eta \cdot \tau^{\beta}}) = \bigoplus_{\beta=0}^{d-1} \mathcal{O}_{S_N}$$

given by restriction. Note that here, to define the evaluation map Ξ_{η}, we use the facts that (i) $\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{\widetilde{C}_d,[d]}} \otimes \mathbf{Q} = \mathcal{R}_{E^{\dagger}_{\widetilde{C}_d,[d]}} \otimes \mathbf{Q}$; (ii) in characteristic zero, the torsion points $\eta, \tau \in \widetilde{C}_d(S_N)$ lift naturally to torsion points of $E^{\dagger}_{\widetilde{C}_d,[d]}(S_N \otimes \mathbf{Q})$ (where "torsion" refers to the group structure on $E^{\dagger}_{\widetilde{C}_d,[d]}|_{U_{S_N}}$). This much gives us a map $\Xi_{\eta} \otimes \mathbf{Q}$; to see that Ξ_{η} is, in fact, defined without tensoring with $\mathbf{Q}$, we need an integrality statement like that at the end of Theorem 3.1. The reason we cannot apply the integrality statement at the end of Theorem 3.1 directly is that instead of restricting to d-torsion points (i.e., τ^{β}'s), we are restricting to d-torsion points *shifted* by η. This is not a problem, however, since, just as in the proof of the integrality statement at the end of Theorem 3.1, we see that it suffices to prove that the "$T^{[n]}$'s" (notation of the first part of §3) take on integral values for $T = \frac{d}{N} + \beta$. But since *we assumed that* $N \in \mathcal{O}^\times$, this is an easy exercise. (For instance, it may be proven by observing that it suffices to prove it locally at an arbitrary prime number l not dividing N; but then $\frac{d}{N}$ may be l-adically approximated by an integer, and it is well-known that $T^{[n]}$ takes integral values at integers; thus, the integrality of $T^{[n]}$ for $T = \frac{d}{N} + \beta$ at the prime l follows from the fact that $\mathbf{Z}_l$ is l-adically closed in $\mathbf{Q}_l$.) This completes the definition of Ξ_η.

In this §, *we would like to study the extent to which* Ξ_η *is an isomorphism.* To do this, we would like to *compute the determinant of* Ξ_η. To compute this determinant explicitly, we must introduce various explicit bases of the domain and range of Ξ_η. Note that the *range of* Ξ_η already has a natural basis (over $\mathcal{O}_{S_N}$) given by the fact that it is naturally a direct sum of copies of $\mathcal{O}_{S_N}$. Next, we consider the *domain of* Ξ_η. To simplify notation, let us write

$$V \stackrel{\mathrm{def}}{=} \Gamma(\widetilde{C}_d, \widetilde{\mathcal{L}} \otimes_{\mathcal{O}_{\widetilde{C}_d}} F^d(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{\widetilde{C}_d,[d]}})) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_N}$$

Note that V has a *natural filtration* $F^n(V)$, induced by the filtration on $\mathcal{R}^{\mathrm{et}}_{E^\dagger_{\widetilde{C}_d,[d]}}$ (cf. Theorem 3.1). For $0 \leq n \leq d-1$, the subquotient $(F^{n+1}/F^n)(V)$ is a (locally) free $\mathcal{O}_{S_N}$-module of rank 1. By further localizing $\mathcal{O}$, we may assume that these modules are, in fact, free, hence generated (over $\mathcal{O}_{S_N}$) by some $\phi_n \in F^{n+1}(V)$, where $0 \leq n \leq d-1$. Thus,

$$\phi_0, \ldots, \phi_{d-1}$$

form a basis of V *over* $\mathcal{O}_{S_N}$.

Next, let us recall the *infinite étale covering* $C^\infty_{\widehat{S}} \to C^\infty_{\widehat{S}}/(d \cdot \mathbf{Z}_{\mathrm{et}}) = (\widetilde{C}_d)_{\widehat{S}}$. We would like to pull-back the various sections ϕ_i (for $i = 0, \ldots, d-1$) over $(\widetilde{C}_d)_{S_N} \stackrel{\mathrm{def}}{=} \widetilde{C}_d \times_S S_N$ to $C^\infty_{\widehat{S}_N} \stackrel{\mathrm{def}}{=} C^\infty_{\widehat{S}} \times_{\widehat{S}} \widehat{S}_N$. If we do this, then it follows from the definition of the "$\mathcal{R}^{\mathrm{et}}$" in §3 (and Chapter III, §6) that we can write

$$\phi_i = \sum_{j=0}^{i} \zeta_{i,j} \cdot T^{[j]}$$

where $\zeta_{i,j} \in \Gamma(C^\infty_{\widehat{S}_N}, \widetilde{\mathcal{L}}|_{C^\infty_{\widehat{S}_N}})$. Note, moreover, that for each $i = 0, \ldots, d-1$, the section $\zeta_{i,i}$ *descends to* $(\widetilde{C}_d)_{S_N}$ (cf. the theory of Chapter III, §6). Since ϕ_i generates $(F^{i+1}/F^i)(V)$, and restriction of sections of $\Gamma((\widetilde{C}_d)_{S_N}, \widetilde{\mathcal{L}}_{(\widetilde{C}_d)_{S_N}})$ to η clearly defines an isomorphism

$$\Gamma((\widetilde{C}_d)_{S_N}, \widetilde{\mathcal{L}}_{(\widetilde{C}_d)_{S_N}}) \cong \mathcal{O}_{S_N}$$

it follows that $\zeta_{i,i}(\eta) \in \mathcal{O}_{S_N}^{\times}$. Thus, by replacing ϕ_i by an appropriate $\mathcal{O}_{S_N}^{\times}$-multiple of ϕ_i, we may assume that the ϕ_i's have been normalized so that, for $i = 0, \ldots, d-1$, we have

$$\zeta_{i,i}(\eta) = 1$$

(cf. the theory of Chapter III, §6).

Now we are ready to compute *the determinant of* Ξ_η. Relative to the bases chosen above, this amounts to computing the determinant of the matrix

$$\mathbf{M} = \{M_{i,\beta}\}$$

where $i, \beta = 0, \ldots, d-1$, and

$$M_{i,\beta} \stackrel{\text{def}}{=} \phi_i(\eta_\infty \cdot \tau_\infty^\beta) = \sum_{j=0}^{i} \zeta_{i,j}(\eta_\infty \cdot \tau_\infty^\beta) \cdot (T^{[j]}|_{T=\frac{d}{N}+\beta})$$

Here, $\eta_\infty \in C^\infty_{\widehat{S}_N}(\widehat{S}_N)$, $\tau_\infty \in C^\infty_{\widehat{S}_N}(\widehat{S}_N)$ are the points defined by $q^{\frac{d}{N}}, q \in \mathbf{G}_\mathrm{m}(U_{S_N})$, respectively, by thinking of $C^\infty_{\widehat{S}_N}$ as (the q-adic completion of) a sort of *Néron model* for $\mathbf{G}_\mathrm{m}$ over U_{S_N} (cf. the discussion at the beginning of Chapter III, §5). Thus, η_∞, τ_∞ project to η, τ, respectively.

We would like to show that the determinant in question is *invertible* as a section of $\mathcal{O}_{S_N}$. Since sections of $\mathcal{O}_{S_N}$ are invertible if and only if they are invertible modulo $q^{\frac{1}{N}}$, it thus follows that it suffices to show that $\det(\mathbf{M})$ is invertible modulo $q^{\frac{1}{N}}$. Thus, in the following computations, *we will work modulo* $q^{\frac{1}{N}}$.

Next, let us observe that, *modulo* $q^{\frac{1}{N}}$, the rational points $\eta_\infty \cdot \tau_\infty^\beta$ (for $\beta = 0, \ldots, d-1$) of $C^\infty_{\widehat{S}_N}$ over $\widehat{S}_N$ all lie in the irreducible components of the special fiber of $C^\infty_{\widehat{S}_N}$ labeled $1, \ldots, d-1$ (cf. the discussion at the beginning of Chapter III, §5). Indeed, these $d-1$ irreducible components include a total of precisely d nodes, and it is these d nodes which are images of the $\eta_\infty \cdot \tau_\infty^\beta$ (for $\beta = 0, \ldots, d-1$) *modulo* $q^{\frac{1}{N}}$. On the other hand, the restriction of the line bundle $\widetilde{\mathcal{L}}$ to the union of these $d-1$ irreducible components is clearly equal to the *trivial bundle*. Let us write $\mathcal{U}$ for the *union* of these $d-1$ components. Then $\mathcal{U}$ is a chain of $d-1$ copies of $\mathbf{P}^1$ over $\mathcal{O}$. In particular, $\mathcal{U}$ is reduced and connected, so it follows that

$$\Gamma(\mathcal{U}, \widetilde{\mathcal{L}}|_\mathcal{U}) = \mathcal{O}$$

i.e., sections of $\widetilde{\mathcal{L}}$ are constant over $\mathcal{U}$, hence, in particular, *assume the same values* at the points $\eta_\infty \cdot \tau_\infty^\beta$ (for $\beta = 0, \ldots, d-1$) that we are interested in.

Thus, *modulo* $q^{\frac{1}{N}}$, we have:

$$M_{i,\beta} \equiv \sum_{j=0}^{i} \zeta_{i,j}(\eta_\infty) \cdot (T^{[j]}|_{T=\frac{d}{N}+\beta})$$

Now we would like to perform various *elementary row operations* on this matrix. Here, we think of the matrix entries (i, β) where i is *fixed* and β *varies* as a *row*. Thus, we will speak of "the row $(i, \sim)$." Note first of all that the row $(0, \sim)$ consists of 1's, i.e.,

$$(0, \beta) \equiv \zeta_{0,0}(\eta_\infty) \cdot (T^{[0]}|_{T=\frac{d}{N}+\beta}) = 1$$

(where "$\equiv$" in the following discussion will always mean "modulo $q^{\frac{1}{N}}$"). Now I *claim* that by performing elementary row operations on the matrix $\mathbf{M}$, one may transform this matrix (modulo $q^{\frac{1}{N}}$) into a matrix $\mathbf{M}' = \{M'_{i,\beta}\}$, where

$$M'_{i,\beta} \equiv \zeta_{i,i}(\eta_\infty) \cdot (T^{[i]}|_{T=\frac{d}{N}+\beta}) = T^{[i]}|_{T=\frac{d}{N}+\beta}$$

Indeed, we prove this for each row $(i, \sim)$ by induction on i. First of all, it is clearly already true for $i = 0$. The induction step is then proven by observing that the difference between the original row $(i, \sim)$ of $\mathbf{M}$ and the row $(i, \sim)$ of $\mathbf{M}'$ is

$$\equiv \sum_{j=0}^{i-1} \zeta_{i,j}(\eta_\infty) \cdot (T^{[j]}|_{T=\frac{d}{N}+\beta})$$

i.e., the "difference row" may be thought of as the sum (for $j = 0, \ldots, i-1$) of rows of the form: "the constant $\zeta_{i,j}(\eta_\infty)$ times the row $(j, \sim)$ of $\mathbf{M}'$." Since for $j < i$, we have already converted the row $(j, \sim)$ of $\mathbf{M}$ into the row $(j, \sim)$ of $\mathbf{M}'$ by means of elementary row operations (by the induction hypothesis), we thus see that further elementary row operations will allow us to convert the row $(i, \sim)$ of $\mathbf{M}$ into the row $(i, \sim)$ of $\mathbf{M}'$, as desired. This completes the proof of the claim.

Thus, it suffices to prove that $\det(\mathbf{M}')$ *is invertible.* Since

$$M'_{i,\beta} \equiv T^{[i]}|_{T=\frac{d}{N}+\beta}$$

this amounts to an elementary and explicit calculation. Of course, one may perform this calculation explicitly, but one indirect way to show that this determinant is invertible is the following: It suffices to show that this determinant is invertible at every prime l that does not divide N. At such an l, $\frac{d}{N}$ may be l-adically approximated by *integers.* Moreover, since $\mathbf{Z}_l^\times$ is an l-adically closed subset of $\mathbf{Q}_l$, it thus suffices to prove that the determinant is invertible when $\frac{d}{N}$ is replaced by an arbitrary $I \in \mathbf{Z}$. Thus, it suffices to show the invertibility of the matrix $\mathbf{M}''[I] = \{M''_{i,\beta}[I]\}$, where

$$M''_{i,\beta}[I] = T^{[i]}|_{T=I+\beta}$$

But now let us note that by repeated application of the relation $\delta(T^{[i]}) = T^{[i-1]}$, one verifies easily that $\mathbf{M}''[I]$ may be converted into $\mathbf{M}''[I+1]$ by means of elementary row operations. In particular, $\det(\mathbf{M}''[I])$ is *independent of* I, so *it suffices to show that* $\det(\mathbf{M}''[0])$ *is invertible.* But

$$M''_{i,\beta}[0] = T^{[i]}|_{T=\beta}$$

is 0 if $\beta < i$, and 1 if $\beta = i$. That is to say, the matrix $\mathbf{M}''[0]$ is *upper triangular*, with 1's along the diagonal. Thus,

$$\det(\mathbf{M}''[0]) = 1$$

as desired. In other words, we have proven the following result:

Lemma 6.1. *The evaluation map*

$$\Xi_\eta : \Gamma(\widetilde{C}_d, \widetilde{\mathcal{L}} \otimes_{\mathcal{O}_{\widetilde{C}_d}} F^d(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{\widetilde{C}_d,[d]}})) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_N} \longrightarrow \bigoplus_{\beta=0}^{d-1} (\widetilde{\mathcal{L}}|_{\eta\cdot\tau^\beta}) = \bigoplus_{\beta=0}^{d-1} \mathcal{O}_{S_N}$$

is an isomorphism.

Now we would like to *consider the consequences of Lemma 6.1, in light of the theory of* §*5.* To do this, let $\mathfrak{p}$ be a generic point of $S_N \otimes \mathbf{F}_p$. Write

$$W \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\widehat{\mathcal{O}}_{S_N,\mathfrak{p}})$$

Thus, W is a *trait*, i.e., the spectrum of a discrete valuation ring (of mixed characteristic). Let us write $\mathfrak{p}_W$ for the closed point of W. In the following, we would like to work over W. To do this, we first pull-back the various objects that we have been working with over S_N to W. Thus, we obtain a torsor

$$\widetilde{E}^{\dagger}_{W,[d]} \to \widetilde{E}_W$$

together with a line bundle $\widetilde{\mathcal{L}}_W$ on $\widetilde{E}_W$, and a d-torsion point $\tau_W \in \widetilde{E}_W(W)$. In the following discussion, we will often wish to base-change from W to the bases $X \stackrel{\mathrm{def}}{=} \widetilde{E}_W$,

$Y \stackrel{\text{def}}{=} \widetilde{E}^{\dagger}_{W,[d]}$. When we wish to think of these objects not as group schemes over W, but rather as *base extensions* of W, we will use the notation "X," "Y" for $\widetilde{E}_W$, $\widetilde{E}^{\dagger}_{W,[d]}$. We will denote the result of base-changing objects over W (which have a subscript "W") to objects over X and Y by means of a subscript X or Y. Finally, let us write

$$\delta_X \in \widetilde{E}_X(X) = \widetilde{E}_W(X); \quad \delta_Y \stackrel{\text{def}}{=} \delta_X|_Y \in \widetilde{E}_W(Y); \quad \delta^{\dagger}_Y \in \widetilde{E}^{\dagger}_{Y,[d]}(Y) = \widetilde{E}^{\dagger}_{W,[d]}(Y)$$

for the tautological points given by the respective identity maps.

Next, we would like to consider the following *evaluation map* (where the requisite integrality follows from the statement at the end of Theorem 3.1):

$$\Xi^{\mathcal{T}}_X: \quad V^{\mathcal{T}}_X \stackrel{\text{def}}{=} \Gamma(\widetilde{E}_X, (\mathcal{T}^*_{\delta_X}\widetilde{\mathcal{L}}_X) \otimes_{\mathcal{O}_{\widetilde{E}_X}} F^d(\mathcal{R}^{\text{et}}_{\widetilde{E}^{\dagger}_{X,[d]}})) \longrightarrow \bigoplus_{\beta=0}^{d-1} (\mathcal{T}^*_{\delta_X}\widetilde{\mathcal{L}}_X)|_{\tau^{\beta}_X}$$

(where $\mathcal{T}_{\delta_X}: \widetilde{E}_X \to \widetilde{E}_X$ is the morphism given by translation by the point $\delta_X \in \widetilde{E}_X(X)$) of locally free $\mathcal{O}_X$-modules of rank d. Write η_W for the pull-back of the torsion point η of Lemma 6.1 to a point of $X(W) = \widetilde{E}_W(W)$. Note that since $N \in \mathcal{O}^{\times}_W$, and η_W is an N-torsion point, it follows that η_W lifts naturally to a unique N-torsion point $\eta^{\dagger}_W \in \widetilde{E}^{\dagger}_{W,[d]}(W)$ (cf. Chapter III, Corollary 5.9). Moreover, since $\frac{d}{N}$ is a p-adic integer, translation by $\frac{d}{N}$ preserves the "$T^{[n]}$" (over S_N or over W). Thus, we see that the morphism $\mathcal{T}_{\eta^{\dagger}_W}$ (translation by $\eta^{\dagger}_W$) on $\widetilde{E}^{\dagger}_{W,[d]}$ induces a filtration-preserving *automorphism*

$$\mathcal{T}^*_{\eta^{\dagger}_W}[\mathcal{R}^{\text{et}}]: \quad \mathcal{R}^{\text{et}}_{\widetilde{E}^{\dagger}_{W,[d]}} \to \mathcal{R}^{\text{et}}_{\widetilde{E}^{\dagger}_{W,[d]}}$$

Thus, we get an isomorphism

$$\mathcal{T}^*_{\eta^{\dagger}_W}[V]: \quad \Gamma(\widetilde{E}_W, \widetilde{\mathcal{L}}_W \otimes_{\mathcal{O}_{\widetilde{E}_W}} F^d(\mathcal{R}^{\text{et}}_{\widetilde{E}^{\dagger}_{W,[d]}})) \cong \Gamma(\widetilde{E}_W, (\mathcal{T}^*_{\eta_W}\widetilde{\mathcal{L}}_W) \otimes_{\mathcal{O}_{\widetilde{E}_W}} F^d(\mathcal{R}^{\text{et}}_{\widetilde{E}^{\dagger}_{W,[d]}}))$$

But now observe that the *domain* of $\mathcal{T}^*_{\eta^{\dagger}_W}[V]$ is simply V_W (i.e., the pull-back to W of the $\mathcal{O}_{S_N}$-module "V" that we considered earlier), while the *range* of $\mathcal{T}^*_{\eta^{\dagger}_W}[V]$ is $\eta^*_W(V^{\mathcal{T}}_X)$, i.e., the restriction of the $\mathcal{O}_X$-module $V^{\mathcal{T}}_X$ considered above to the point $\eta_W \in X(W)$. Relative to these observations, we thus see that the restriction to W of the morphism Ξ_η of Lemma 6.1 may be thought of as:

$$\Xi_\eta|_W = \eta_W^*(\Xi_X^{\mathcal{T}}) \circ \mathcal{T}^*_{\eta_W^\dagger}[V]$$

In particular, it follows from Lemma 6.1 that $\det(\Xi_X^{\mathcal{T}})$ (which is naturally a section of a certain line bundle on the scheme X) *is not identically zero on (the irreducible scheme)* $X \otimes (\mathcal{O}_W/\mathfrak{p}_W)$. In other words, *the vanishing locus of* $\det(\Xi_X^{\mathcal{T}})$ *is finite and flat over* W*, hence is determined by its restriction to the generic fiber of* $X = \widetilde{E}_W \to W$.

Next, we would like to work over the base $Y = \widetilde{E}^\dagger_{W,[d]}$. Note that over Y, translation by $\delta_Y^\dagger$ defines an isomorphism

$$\mathcal{T}^*_{\delta_Y^\dagger}[V]: \quad \Gamma(\widetilde{E}_Y, \widetilde{\mathcal{L}}_Y \otimes_{\mathcal{O}_{\widetilde{E}_Y}} F^d(\mathcal{R}^{\mathrm{et}}_{\widetilde{E}^\dagger_{Y,[d]}})) \otimes \mathbf{Q} \;\cong\; \Gamma(\widetilde{E}_Y, (\mathcal{T}^*_{\delta_Y}\widetilde{\mathcal{L}}_Y) \otimes_{\mathcal{O}_{\widetilde{E}_Y}} F^d(\mathcal{R}^{\mathrm{et}}_{\widetilde{E}^\dagger_{Y,[d]}})) \otimes \mathbf{Q}$$

Note that here, we must tensor with $\mathbf{Q}$ since (one may easily check that) $\delta_Y^\dagger$ does not preserve the "et-integral structure $\mathcal{R}^{\mathrm{et}}$" of §3. Next, let us consider the "composite evaluation map":

$$\Xi_X^{\mathcal{T}}|_Y \circ \mathcal{T}^*_{\delta_Y^\dagger}[V]: \quad \Gamma(\widetilde{E}_Y, \widetilde{\mathcal{L}}_Y \otimes_{\mathcal{O}_{\widetilde{E}_Y}} F^d(\mathcal{R}^{\mathrm{et}}_{\widetilde{E}^\dagger_{Y,[d]}})) \otimes \mathbf{Q} \;\longrightarrow\; \bigoplus_{\beta=0}^{d-1} (\mathcal{T}^*_{\delta_Y}\widetilde{\mathcal{L}}_Y)|_{\tau_Y^\beta} \otimes \mathbf{Q}$$

Now, sorting through the notation, one sees that this composite evaluation map is (up to translation by the torsion point "ϵ_H" of §5) the same as the evaluation map denoted "Ξ_α^H" in §5. Since the determinant of this evaluation map is not identically zero (by what we did in the preceding paragraph), it thus follows from Theorem 5.6, that (if we denote by $\epsilon_W \in \widetilde{E}_W(W) = X(W)$ the point denoted by "ϵ_H" in §5), we thus obtain that *the vanishing locus of the determinant of this composite evaluation map is given by the divisor* $\sum_{\beta=0}^{d-1} [\epsilon_W \cdot \tau_W^\beta]$ *on* $\widetilde{E}_W \otimes \mathbf{Q}$. Combining this with what we did in the preceding paragraph, we thus conclude that:

> The vanishing locus of $\det(\Xi_X^{\mathcal{T}})$ is given by the divisor $\sum_{\beta=0}^{d-1} [\epsilon_W \cdot \tau_W^\beta]$ on $X = \widetilde{E}_W$.

Now recall the semi-abelian scheme $E \to S$ considered at the beginning of §3. One may think of $\widetilde{E}$ as being the quotient $E \to \widetilde{E}$ of E by a subgroup scheme of E which is naturally isomorphic to $\boldsymbol{\mu}_d$. Alternatively, relative to Schottky uniformizations, one may think of the isogeny $E \to \widetilde{E}$ as corresponding to the morphism $\mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}} \to \mathbf{G}_{\mathrm{m}}/q^{d\cdot\mathbf{Z}}$ induced by the morphism "raising to the d-th power" on $\mathbf{G}_{\mathrm{m}}$. If we pull-back the isogeny $E \to \widetilde{E}$ to W, we get an isogeny

$$E_W \to \widetilde{E}_W$$

When we wish to think of $E_W \to W$ as a base extension of W, we will denote E_W by $Z \stackrel{\mathrm{def}}{=} E_W$, and write $\gamma_Z \in E_W(Z)$ for the identity morphism. Finally, we will denote the pull-backs of the torsor $E^{\dagger}_{[d]} \to E$ to W, Z by $E^{\dagger}_{W,[d]} \to E_W$, $E^{\dagger}_{Z,[d]} \to E_Z$, respectively.

If we let $\mathcal{L}_W \stackrel{\mathrm{def}}{=} \mathcal{O}_{E_W}(e_{E_W})$ (where e_{E_W} is the origin of E_W), then we have an *evaluation map* for E_Z (where the requisite integrality follows from the statement at the end of Theorem 3.1):

$$\Pi^{\mathcal{T}}_Z : \quad \Gamma(E_Z, (\mathcal{T}^*_{\gamma_Z}\mathcal{L}^{\otimes d}_Z) \otimes_{\mathcal{O}_{E_Z}} F^d(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{Z,[d]}})) \quad \longrightarrow \quad (\mathcal{T}^*_{\gamma_Z}\mathcal{L}^{\otimes d}_Z)|_{({}_dE_Z)}$$

(Here ${}_dE_Z \subseteq E_Z$ is the subgroup scheme of d-torsion points in E_Z.) Thus, one observes, relative to the theory of §5, that for an appropriate choice of "Lagrangian subgroup H" (which lifts $\boldsymbol{\mu}_d \subseteq E$), the evaluation maps $\Pi^{\mathcal{T}}_Z$ and $\Xi^{\mathcal{T}}_X$ correspond to one another (up to translation by ϵ_W) with respect to the operation of "taking invariants with respect to the Lagrangian subgroup" (cf. the discussion of §5). Thus, we obtain the following result, which is the *main result of this* §:

Theorem 6.2. *For any integer $d \geq 1$, the scheme-theoretic zero locus of the determinant of the evaluation map*

$$\Pi^{\mathcal{T}}_Z : \quad \Gamma(E_Z, (\mathcal{T}^*_{\gamma_Z}\mathcal{L}^{\otimes d}_Z) \otimes_{\mathcal{O}_{E_Z}} F^d(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{Z,[d]}})) \quad \longrightarrow \quad (\mathcal{T}^*_{\gamma_Z}\mathcal{L}^{\otimes d}_Z)|_{({}_dE_Z)}$$

on $Z = E_W$ is given precisely by $d \cdot [{}_dE_Z] \subseteq E_Z$, where $[{}_dE_Z]$ is the divisor defined by the kernel of multiplication by d on E_Z.

Here, W is a trait of mixed characteristic $(0,p)$; $E_W \to W$ is an elliptic curve with the property that the moduli of the special fiber over W are generic in characteristic p; $\mathcal{L}_W$ is the line bundle defined by the identity element on E_W; $E_Z \stackrel{\mathrm{def}}{=} E_W \times_W Z$, $\mathcal{L}_Z = \mathcal{L}_W|_Z$; $\gamma_Z \in E_Z(Z) = E_W(Z)$ is the point defined by the identity morphism; $\mathcal{T}_{\gamma_Z}$ is translation by the point γ_Z; $E^{\dagger}_{Z,[d]}$ is the torsor "$E^{\dagger}_{[d]}$" (cf. §2) for the elliptic curve E_Z; and $\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{Z,[d]}}$ is the "et-integral structure" (cf. Definition 3.2) on the filtered sheaf of functions $\mathcal{R}_{E^{\dagger}_{Z,[d]}}$ on $E^{\dagger}_{Z,[d]}$.

Note that in the above proof, we made essential use of the $\boldsymbol{\mu}_d$*-isogeny* (i.e., isogeny whose kernel may be identified with $\boldsymbol{\mu}_d$)

$$E \to \widetilde{E}$$

That is to say, we proved Theorem 6.2, which essentially concerns objects over E, by *descending* objects on E to $\widetilde{E}$ via this isogeny, and then *analyzing* the resulting objects *over* $\widetilde{E}$ (cf. the discussion preceding Lemma 6.1). In fact, however, one can also give a proof of Theorem 6.2 by descending with respect to the $\mathbf{Z}/d\mathbf{Z}$*-isogeny*

$$\widetilde{E}_d \to E$$

(cf. §3 for an explanation of the notation – roughly speaking, this is the isogeny that looks like the isogeny

$$\text{“}\mathbf{G}_{\mathrm{m}}/q^{d\mathbf{Z}} \to \mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}\text{”}$$

arising from the *identity* $\mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ on $\mathbf{G}_{\mathrm{m}}$), and then *applying the theory developed in §4* to the descended objects on E. The advantage of the $\boldsymbol{\mu}_d$-isogeny proof given above is that it allows one to calculate the relevant matrix *very explicitly* (up to the elementary row operations) without much preparation, and is *independent of the rather technical theory discussed in §4*. On the other hand, once one assumes the theory of §4, the proof of Theorem 6.2 is relatively short and proceeds as follows:

Alternate Proof of Theorem 6.2:

We omit various “general nonsense” details concerning “descent with respect to the isogeny in question” and “base-change to W” that were discussed extensively in the preceding proof of Theorem 6.2 and which are the same in both proofs.

Recall from the discussion of §3 that $E^{\dagger}_{\widetilde{C}_d,[d]}$, i.e., the “$E^{\dagger}_{[d]}$” for $\widetilde{C}_d$, is precisely the pull back via the compactified isogeny $\widetilde{C}_d \to C$ of the usual universal extension torsor $E^{\dagger}_C \to C$ over C ($\supseteq E$). Moreover, it also follows from the discussion of §3 that the étale-integral structure on $E^{\dagger}_{\widetilde{C}_d,[d]}$ is precisely the pull-back of the étale-integral structure defined on $E^{\dagger}_C$ in Chapter III, Proposition 6.1. Thus, in summary, although we are ultimately interested in proving the result (i.e., bijectivity) for the evaluation map for functions in $\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{\widetilde{C}_d,[d]}}$, it suffices (assuming that we can descend the relevant line bundle on $\widetilde{C}_d$ down to C – cf. the following paragraph) to prove the result for the evaluation map for functions in $\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_C}$.

The final detail that we must take care of before we discuss the relevant evaluation map on C is the specification of the relevant line bundle on $\widetilde{C}_d$. More precisely, this "line bundle on $\widetilde{C}_d$" will be a metrized line bundle on $\widetilde{E}_{\infty,S}$. Moreover, we want this metrized line bundle to descend to a metrized line bundle on $E_{\infty,S}$ via the $\mathbf{Z}/d\mathbf{Z}$-isogeny $\widetilde{E}_d \to E$. We take this metrized line bundle on $\widetilde{E}_{\infty,S}$ to be

$$\widetilde{\overline{\mathcal{L}}}_{\mathrm{st},\widetilde{\eta}} \quad (\text{respectively, } \widetilde{\overline{\mathcal{L}}}^{\mathrm{ev}}_{\mathrm{st},\widetilde{\eta}})$$

if d is *odd* (respectively, *even*) – i.e., the metrized line bundles "$\overline{\mathcal{L}}_{\mathrm{st},\eta}$," "$\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta}$" of §1 for $\widetilde{E}_{\infty,S}$ (as opposed to $E_{\infty,S}$). Here, $\widetilde{\eta}$ is the torsion point of $\widetilde{E}_{\infty,S}(S_N)$ defined by $q^N \in \mathbf{G}_{\mathrm{m}}(U_{S_N})$ (where we think of $\widetilde{E}_{\infty,S}$ as "$\mathbf{G}_{\mathrm{m}}/q^{d\mathbf{Z}}$). Thus, it remains only to see that the metrized line bundle that we have chosen descends via $\widetilde{E}_d \to E$. Going back to the definition of $\widetilde{\overline{\mathcal{L}}}_{\mathrm{st},\widetilde{\eta}}$, $\widetilde{\overline{\mathcal{L}}}^{\mathrm{ev}}_{\mathrm{st},\widetilde{\eta}}$ in §1, we see that these metrized line bundles are obtained by translating "$\widetilde{\overline{\mathcal{L}}}_{\mathrm{st}}$," "$\widetilde{\overline{\mathcal{L}}}^{\mathrm{ev}}_{\mathrm{st}}$" by $\widetilde{\eta}$ and then adding various d-invariant ψ's. Since it is clear that the d-invariant ψ's descend, it thus suffices to see that $\widetilde{\overline{\mathcal{L}}}_{\mathrm{st}}$, $\widetilde{\overline{\mathcal{L}}}^{\mathrm{ev}}_{\mathrm{st}}$ descend. But this is clear from the explicit forms of these line bundles given in Chapter IV, Lemma 5.4. Let us denote the resulting descended metrized line bundle on $E_{\infty,S}$ by

$$\overline{\mathcal{L}}$$

Note that $\overline{\mathcal{L}}$ has *relative degree* 1 over the base. Moreover, the curvature of $\overline{\mathcal{L}}$ is the same same as the curvature of $\mathcal{O}_C(e_C)$ (where e_C is the identity element $\in C(S)$ of C). Thus, one may think of $\overline{\mathcal{L}}$ as a sort of *twisted form* of $\mathcal{O}_C(e_C)$. This "twist" may be resolved by pulling back to an appropriate covering of C (cf. Chapter IV, §2,3; §4 of the present Chapter). Since this twist is defined by η ($\stackrel{\mathrm{def}}{=}$ the image of $\widetilde{\eta}$ in $E_{\infty,S}$), and η does not lie in the identity component of the special fiber of $E_{\infty,S}$ (a consequence of the fact that N is assumed to be prime to d), it follows that the twist applied to $\mathcal{O}_C(e_C)$ to obtain $\overline{\mathcal{L}}$ is of the sort effected by a character "$\chi_{\mathcal{L}}$" which falls under *Case III* (relative to the terminology of §4).

Thus, to summarize, the above discussion shows that it suffices to consider the evaluation map given by evaluating sections of $\overline{\mathcal{L}} \otimes_{\mathcal{O}_C} F^d(\mathcal{R}^{\mathrm{et}}_{E^\dagger_C})$ (over $E_{\infty,S}$) on $\boldsymbol{\mu}_d \subseteq (\mathbf{G}_{\mathrm{m}})_{\widehat{S}} = E_{\widehat{S}} \hookrightarrow E^\dagger{}_{\widehat{S}}$ (where $E_{\widehat{S}} \hookrightarrow E^\dagger{}_{\widehat{S}}$ is the canonical section κ of Chapter III, Theorem 2.1). Since this evaluation map is between vector bundles of rank d on S_∞, it follows that it suffices to prove that it is *injective* modulo any prime $\mathfrak{p}$ of $\mathcal{O}$, at least after one inverts q. But *this injectivity follows immediately from Corollary 4.7* (applied in the case $j \stackrel{\mathrm{def}}{=} d$). Indeed, to see this, we break up the evaluation map of sections of $\overline{\mathcal{L}} \otimes_{\mathcal{O}_C} F^d(\mathcal{R}^{\mathrm{et}}_{E^\dagger_C})$ on $\boldsymbol{\mu}_d \subseteq \mathbf{G}_{\mathrm{m}} = E_{\widehat{S}} \subseteq E^\dagger{}_{\widehat{S}}$ into *two steps*:

(1) First, we restrict sections of $\overline{\mathcal{L}} \otimes_{\mathcal{O}_C} F^d(\mathcal{R}^{\mathrm{et}}_{E^\dagger_C})$ to $E_{\widehat{S}} = (\mathbf{G}_{\mathrm{m}})_{\widehat{S}} \subseteq E^\dagger{}_{\widehat{S}}$.

(2) Then, we strict functions on $E_{\widehat{S}} = (\mathbf{G}_{\mathrm{m}})_{\widehat{S}}$ to $\boldsymbol{\mu}_d \subseteq (\mathbf{G}_{\mathrm{m}})_{\widehat{S}} = E_{\widehat{S}}$.

Strictly speaking, the linear independence assertion of Corollary 4.7 implies that the *first restriction (1)* is *injective*. (That is, if it were not, then there would be some A-linear relation among $\zeta_0^\chi, \ldots, \zeta_{d-1}^\chi$ with at least one coefficient which is nonzero modulo $(q, \mathfrak{p})$. Thus, in particular, we would get a linear relation modulo $(q^{n \cdot c_{j'}}, \mathfrak{p})$ (with at least one coefficient which is nonzero modulo $(q, \mathfrak{p})$), but this *contradicts* Corollary 4.7.)

Thus, it remains to see that the *second restriction (2)* is injective. To see this, first observe that since we are in Case III, the number "$|C_{j'}|$" (notation of Corollary 4.7) is equal to $j = d$, i.e., "modulo $q^{n \cdot c_{j'}}$" (notation of Corollary 4.7) the only $\chi_{\mathcal{L}}$-special monomials that may occur (i.e., are nonzero) are $\mathcal{O}_S$-multiples of

$$\text{“}U^{I+i \cdot n}\text{”}$$

(where U, n are as in Corollary 4.7; $i = 0, \ldots, d-1$; and I is some fixed constant, independent of i) – cf. Schola 4.1. Moreover, "U^n" (notation of §4) corresponds precisely to the standard multiplicative coordinate on $E_{\widehat{S}} = (\mathbf{G}_{\mathrm{m}})_{\widehat{S}}$ (notation of the present discussion). Let us denote this standard coordinate by V. Thus, we see that the assertion that the second restriction (2) is injective amounts to the following fact:

> For any prime $\mathfrak{p}$ of $\mathcal{O}$, every polynomial in V of degree $\leq d-1$ which vanishes modulo $\mathfrak{p}$ on $\boldsymbol{\mu}_d \subseteq (\mathbf{G}_{\mathrm{m}})_{\widehat{S}} = E_{\widehat{S}}$ is identically 0 modulo $\mathfrak{p}$.

But this follows immediately from elementary algebraic geometry as follows: If we compactify $\mathbf{G}_{\mathrm{m}}$ by $\mathbf{P}^1$, then we see that we are reduced to verifying the assertion that *every global section over $\mathbf{P}^1$ of the line bundle*

$$\mathcal{O}_{\mathbf{P}^1}(-[\boldsymbol{\mu}_d] + (d-1)[\infty])$$

(where "$[\boldsymbol{\mu}_d]$" is the degree d divisor in $\mathbf{G}_{\mathrm{m}} \subseteq \mathbf{P}^1$ defined by $\boldsymbol{\mu}_d$, and "$[\infty]$" is the (degree 1) divisor on $\mathbf{P}^1$ at "infinity") *is identically zero* (in every characteristic). But since this line bundle has degree -1, this assertion is an immediate consequence of elementary algebraic geometry.

This completes the proof of the various injectivity assertions mentioned above, and hence of the entire alternate proof of Theorem 6.2. ○

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Chapter VI: The Scheme-Theoretic Comparison Theorem

§0. Introduction

In this Chapter, we prove the *Scheme-Theoretic Comparison Isomorphism*. This result essentially asserts that:

> *There is a natural bijection between certain natural types of algebraic functions on the universal extension of an elliptic curve and the "set-theoretic functions" on the torsion points of the elliptic curve.*

This natural bijection is given by restricting the algebraic functions on the universal extension to the torsion points of the universal extension. This restriction morphism is referred to as the *evaluation map* (and is the main topic of Chapter V). Ultimately, the goal of this paper is to show that the evaluation map is not only a bijection, say, in characteristic zero (cf. Theorem 3.1), but that it *preserves natural integral structures* on both sides at all the primes of the rational number field. In the present Chapter, we address the *scheme-theoretic* portion of this goal, namely, we show that we essentially get an isomorphism schematically over $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ (cf. Theorem 4.1). In other words, we show that the integral structures at all of the *finite primes* (essentially) coincide.

In §1, we define the *algebraic functions* on the universal extension that we are interested in. The sheaves of such functions are called the "$\{\mathbf{v}, \mathrm{et}\}$-push forwards" of a metrized line bundle. In §2, we study the various basic properties of these push-forwards, with an eye to proving (in §4) that they are compatible with base-change. In §3, we prove the comparison isomorphism in characteristic zero, i.e., that the evaluation map is bijective in characteristic zero (Theorem 3.1). Finally, in §4, we complete the proof of compatibility with base-change, and show that the evaluation map also (essentially) preserves integral structures on both sides at all *finite* primes (Theorem 4.1).

§1. Definition of a New Integral Structure at Infinity

In this §, we prepare for the *comparison isomorphism* results in the remainder of this Chapter by introducing *a new integral structure in a neighborhood of the divisor at infinity* (i.e., the locus on the base where the log elliptic curve in question degenerates). It will be relative to this new integral structure that the evaluation map of Chapter V, §2, will turn out to be an isomorphism.

In this §, we use the notations of Chapter V, §2. Let us first recall the W_E-torsor

$$E^{\dagger}_{C_d,[d]} \to C_d$$

considered in Chapter V, §2. Note that since $E^{\dagger}_{C_d,[d]}$ extends the torsor $E^{\dagger}_{[d]} \to E$, which is obtained from $E^{\dagger} \to E$ by "pushing out" with respect to multiplication by d on W_E, it follows that the *canonical splitting* κ (of Chapter III, Theorem 2.1) of $E^{\dagger} \to E$ over $E_{\widehat{S}}$ defines a canonical splitting

$$\kappa_{E^{\dagger}_{[d]}} : E_{\widehat{S}} \to (E^{\dagger}_{[d]})_{\widehat{S}}$$

Note that since the *étale-integral structure* of Chapter V, Theorem 3.1, is defined using this canonical splitting (cf. Chapter V, §3; Chapter III, §6), it follows that $\kappa_{E^{\dagger}_{[d]}}$ defines, for any integer $i \geq 0$, a *direct sum decomposition*

$$F^i(\mathcal{R}^{\mathrm{et}}_{(E^{\dagger}_{[d]})_{\widehat{S}}}) \quad \cong \quad \bigoplus_{j=0}^{i-1} \quad \frac{1}{j!} \cdot \tau_E^{\otimes j}|_{E_{\widehat{S}}}$$

(cf. Chapter V, Theorem 3.1). Similarly, one sees easily that by *translation by d-torsion points*, we get analogous direct sum decompositions of $F^i(\mathcal{R}^{\mathrm{et}}_{(E^{\dagger}_{E_d,[d]})_{\widehat{S}}})$ over the q-adic completions of E_d at the other $d-1$ connected components of the special fiber of E_d. *Note*, however, that these translated decompositions *do not glue together at the nodes*: indeed, if they did, then we would obtain that $\kappa_{E^{\dagger}_{[d]}}$ extends over C_d, which is absurd (cf. Chapter III, Theorem 5.6).

Now let us suppose that we are given a *"valuation"*

$$\mathbf{v} \in \mathbf{Q}_{\geq 0} \cdot \log(q)$$

(where $\mathbf{Q}_{\geq 0}$ is the set of nonnegative rational numbers, and "$\log(q)$" is a formal symbol). In the following, we will write

$$\exp(\mathbf{v}) \stackrel{\mathrm{def}}{=} q^{\mathbf{v}/\log(q)}$$

We would like to construct, for each integer $i \geq 0$, an *S-flat coherent sheaf*

$$F^i(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C_d,[d]}}\{\mathbf{v}\})$$

on C_d *associated to* $\mathbf{v}$, as follows. First of all, over $U_S = S - D$, we have

$$F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}\})|_{U_S} \stackrel{\mathrm{def}}{=} F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}|_{U_S})$$

Next, let us define $F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}\})$ over $E_{\widehat{S}}$ by using the direct sum decomposition given above in the following way: We give $F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}\})|_{E_{\widehat{S}}}$ the integral structure (relative to this decomposition) defined by:

$$F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}\})|_{E_{\widehat{S}}} \stackrel{\mathrm{def}}{=} \mathcal{O}_{E_{\widehat{S}}} \oplus \left(\bigoplus_{j=1}^{i-1} \frac{1}{j!} \cdot \exp(-\mathbf{v}) \cdot \tau_E^{\otimes j}|_{E_{\widehat{S}}} \right)$$

We then define $F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}\})$ over the completions of E_d at the other $d-1$ connected components of E_d in a similar way, using the *translated direct sum decompositions* discussed above.

Now observe that we have defined $F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}\})$ over all of E_d. Thus, it remains to extend the resulting vector bundle over E_d over C_d (in some sort of natural fashion). But this is just a matter of elementary commutative algebra, which we review in Lemma 1.1 below. In fact, Lemma 1.1 will also show that there is a *unique* extension of this vector bundle on E_d over C_d which is S-flat, coherent, and satisfies the property that any length 2 regular sequence of local sections of $\mathcal{O}_{C_d}$ is also regular on this extension. Relative to the application of Lemma 1.1, we remark that:

(1) All of the notation used in Lemma 1.1 is *strictly internal* to Lemma 1.1 and has nothing to do with the notation in the remainder of the discussion of this §.

(2) Of course, in general, C_d need not be regular, but it suffices to prove extendability in the universal case (e.g., the "$E_N \to S_N$" of Chapter IV, §4, for "N" taken to be d), in which case C_d *is* regular, and the nodes of C_d (with the reduced induced scheme structure) are regular of dimension $\dim(C_d) - 2$.

This completes the definition of the S-flat coherent sheaf

$$F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}\})$$

on C_d.

Lemma 1.1. *Let A be a regular local ring of dimension ≥ 2. Let $B \stackrel{\mathrm{def}}{=} A/(s,t)$ be a quotient which is regular local ring of dimension* $\dim(A) - 2$. *Let $\mathcal{F}$ be a vector bundle on $U \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A)\backslash\mathrm{Spec}(B) \subseteq X \stackrel{\mathrm{def}}{=} \mathrm{Spec}(B)$. Then $i_*\mathcal{F}$ is a coherent sheaf on X with the property that any A-regular sequence of length 2 in A is also $(i_*\mathcal{F})$-regular. Thus, in particular, if* $\dim(A) = 2$, *then $i_*\mathcal{F}$ is a vector bundle on X.*

Proof. It is well-known that in this situation, $i_*\mathcal{F}$ is a coherent sheaf (cf., e.g., [SGA2], p. 97, Proposition 3.2). As for the assertions concerning depth, we reason as follows: Let $a, b \in A$ be regular on A. Then b acts injectively on any localization of $A/(a)$. Thus, the long exact sequence obtained by applying the derived functors of i_* to

$$0 \longrightarrow \mathcal{F} \xrightarrow{a} \mathcal{F} \longrightarrow \mathcal{F}/a \cdot \mathcal{F} \longrightarrow 0$$

shows that a acts injectively on $i_*\mathcal{F}$, and that we have an inclusion $(i_*\mathcal{F})/a \hookrightarrow i_*(\mathcal{F}/a \cdot \mathcal{F})$. On the other hand, $\mathcal{F}/a \cdot \mathcal{F}$ is just a vector bundle over some open subscheme of $\mathrm{Spec}(A/(a))$, so multiplication by b is injective on $\mathcal{F}/a \cdot \mathcal{F}$, hence also on $i_*(\mathcal{F}/a \cdot \mathcal{F})$, as well as $(i_*\mathcal{F})/a$. This shows that a, b is regular on $i_*\mathcal{F}$, as desired. If $\dim(A) = 2$, then this shows that $i_*\mathcal{F}$ has depth 2 as an A-module, hence (by the Auslander-Buchsbaum formula – see, e.g., [Mats]) is a vector bundle. This completes the proof. ○

As remarked above, $F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}\})$ has a *filtration*

$$0 \subseteq \ldots \subseteq F^j(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}\}) \subseteq \ldots \subseteq F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}\})$$

(where $1 \leq j \leq i$) whose subquotients admit natural inclusions

$$(F^{j+1}/F^j)(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}\}) \quad \subseteq \quad \frac{1}{j!} \cdot \exp(-\mathbf{v}) \cdot \tau_E^{\otimes j}|_{C_d}$$

Moreover, if $\mathbf{v}_1 \leq \mathbf{v}_2$, then we have a natural inclusion

$$F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}_1\}) \quad \subseteq \quad F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\mathbf{v}_2\})$$

We denote the union of all of these sheaves (as $\mathbf{v}$ ranges over all elements of $\mathbf{Q}_{\geq 0} \cdot \log(q)$) by

$$F^i(\mathcal{R}^{\mathrm{et}}_{E^\dagger_{C_d,[d]}}\{\infty\})$$

We summarize the properties of these sheaves as follows:

Proposition 1.2. *The quasi-coherent sheaf of $\mathcal{O}_{C_d}$-modules*

$$\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C_d,[d]}}\{\mathbf{v}\} \stackrel{\mathrm{def}}{=} \bigcup_{i\geq 0} F^i(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C_d,[d]}}\{\mathbf{v}\})$$

comes equipped with an exhaustive filtration

$$F^i(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C_d,[d]}}\{\mathbf{v}\})$$

*(for $i \geq 0$) whose i-th member is an S-*flat coherent sheaf *(of generic rank i) on C_d. Moreover, this quasi-coherent sheaf restricts (as a filtered object) to $\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C_d,[d]}}$ (as in Chapter V, Theorem 3.1) away from the divisor at infinity. Finally, the natural morphisms*

$$(F^{i+1}/F^i)(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C_d,[d]}}\{\mathbf{v}\}) \rightarrow \frac{1}{i!}\cdot \exp(-\mathbf{v})\cdot \mathcal{O}_{C_d} \otimes_{\mathcal{O}_S} \tau_E^{\otimes i}$$

(if $i \geq 1$) and

$$F^1(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C_d,[d]}}\{\mathbf{v}\}) \rightarrow \mathcal{O}_{C_d}$$

arising from Chapter V, Theorem 3.1, are isomorphisms *over E_d, i.e., away from the nodes of C_d.*

By pulling back these various objects from C_d to $C_{\infty,S}$ and $E_{\infty,S}$, we obtain corresponding sheaves over $C_{\infty,S}$ and $E_{\infty,S}$ (cf. the construction of $E^{\dagger}_{C_{\infty,S},[d]}$, $E^{\dagger}_{\infty,[d]} = E^{\dagger}_{E_{\infty,S},[d]} \rightarrow E_{\infty,S}$ in Chapter V, §2). Naturally, if we are given different sequences $\mathbf{v}_\iota$ at the various connected components ι of the divisor at infinity of S, then we may adjust the integral structure according to the sequence $\mathbf{v}_\iota$ at ι, and thus obtain global objects suffixed by "$\{\mathbf{v}\}$," where $\mathbf{v} \stackrel{\mathrm{def}}{=} \{\mathbf{v}_\iota\}_\iota$. (We leave the formal details to the reader.)

In particular, if $\overline{\mathcal{L}}$ is a metrized line bundle on $E_{\infty,S}$ of relative degree d and whose curvatures are d-invariant (i.e., a metrized line bundle of the sort considered in Chapter V, §2), then by taking global sections over $E_{\infty,S}$ of

$$\overline{\mathcal{L}} \otimes_{\mathcal{O}_{E_{\infty,S}}} F^i(\mathcal{R}^{\text{et}}_{E^{\dagger}_{\infty,[d]}}\{\mathbf{v}\})$$

(for $i \geq 0$) we obtain a natural *metrized vector bundle (of rank $d \cdot i$) with $\mathcal{G}_{\overline{\mathcal{L}}}$-action*

$$(f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<i}\{\mathbf{v}, \text{et}\} \quad (\supseteq (f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<i})$$

on S_∞.

Definition 1.3. In the following, we shall refer to these push-forwards as $\{\mathbf{v}, \text{et}\}$-*push forwards.*

§2. Compatibility with Base-Change

In this §, we study the push-forwards defined in §1 and show, in particular, that these push-forwards are compatible with base-change if and only if they are compatible with base-change "modulo p." We will then show, in certain special cases, compatibility with base-change modulo p in §4 below, and thus conclude (in §4) that (at least in these cases) *the various push-forwards defined in §1 are compatible with base-change.*

In this §, we continue to use the notation of §1. Let us suppose that we are also given an *m-torsion point*

$$\eta \in E_{\infty,S}(S_\infty)$$

(where $m \geq 1$ is an integer) as in Chapter V, §1. Then, by pulling back the metrized line bundles constructed in the universal case in Chapter V, §1, we obtain metrized line bundles

$$\overline{\mathcal{L}}_{\text{st},\eta}; \quad \overline{\mathcal{L}}^{\text{ev}}_{\text{st},\eta}$$

on $E_{\infty,S}$. Let us take the metrized line bundle $\overline{\mathcal{L}}$ (in the discussion at the end of §1) to be $\overline{\mathcal{L}}_{\text{st},\eta}$ (respectively, $\overline{\mathcal{L}}^{\text{ev}}_{\text{st},\eta}$) if d is *odd* (respectively, *even*). Then, in this §, we would like to study the $\{\mathbf{v}, \text{et}\}$-*push forwards*

$$(f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<j}\{\mathbf{v}, \text{et}\}$$

(for $j \geq 1$) on S_∞. In particular, we would like to study the extent to which these push-forwards are *compatible with base-change*, among bases $S^{\log}$ as in §1 (=as in Chapter V, §2), i.e., bases such that the pull-back of divisor at infinity of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ is a Cartier divisor on S and for which the q-parameter locally admits a d-th root.

First observe that over $U_S \stackrel{\text{def}}{=} S - D$, *compatibility with base-change follows immediately from elementary algebraic geometry* (cf., e.g., [Harts], Chapter III, Corollary 12.9, Theorem 12.11), since: (i) the sheaf $\mathcal{R}^{\text{et}}$ admits a filtration with subquotients which are (locally on S) isomorphic to the structure sheaf (Proposition 1.2); (ii) the higher direct image sheaves of the line bundle $\overline{\mathcal{L}}|_{U_S}$ with respect to the morphism $E|_{U_S} \to U_S$ vanish. Thus, it suffices to localize S to a neighborhood of an arbitrary point of D. In the following discussion, we will often carry out such localizations without further justification. In particular, we may assume that "there is only one ι," i.e., that D is irreducible.

Let us consider the line bundle $\overline{\mathcal{L}}$. Without loss of generality, we assume in the case of d *even* that the q-parameter has a $2d$-th root on S. (This may always be achieved by replacing S by a finite, flat covering of S; since this covering will be flat, it will not effect compatibility with base-change issues.) If d is even, then we let $\beta \in E_{\infty,S}(S)$ be a $2d$-torsion point as in Chapter IV, Lemma 5.4. Now, we let

$$\overline{\mathcal{M}}$$

be the metrized line bundle $\overline{\mathcal{L}}_{\text{st}}$ (respectively, $\mathcal{T}_\beta^* \overline{\mathcal{L}}_{\text{st}}^{\text{ev}}$) if d is *odd* (respectively, *even*). (Here, "$\mathcal{T}_\beta$" is the automorphism of $E_{\infty,S}$ given by translation by β.) Note that the *curvature* of $\overline{\mathcal{M}}$ is the same as that of $\overline{\mathcal{L}}$ (cf. Chapter V, the discussion preceding Proposition 1.1, for a computation of the curvature of $\overline{\mathcal{L}}$; Chapter IV, Lemma 5.4, for a computation of the curvature of $\overline{\mathcal{M}}$). Moreover, it is clear from the definitions of $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ on $E_{\infty,S}|_{U_S}$ (namely, both are obtained by translating a multiple of the divisor defined by the identity element by some torsion point) that there exists an integer $N \geq 1$ such that (after possibly localizing S further) we obtain an *isomorphism of metrized line bundles*:

$$\overline{\mathcal{L}}^{\otimes N} \cong \overline{\mathcal{M}}^{\otimes N}$$

In particular, it follows that we may write

$$\overline{\mathcal{L}} = \overline{\mathcal{M}} \otimes \overline{\mathcal{Q}}$$

where $\overline{\mathcal{Q}}$ is a metrized line bundle on $E_{\infty,S}$ such that $\overline{\mathcal{Q}}^{\otimes N}$ is *trivial.* Note, moreover, that $\overline{\mathcal{M}}$ is represented by a *line bundle* on C_d in the usual sense (cf. Chapter IV, Lemma 5.4). The point here (cf. the proof of Chapter IV, Theorem 5.8) is that we would like to reduce compatibility with base-change assertions concerning the *metrized line bundle* $\overline{\mathcal{L}}$ (which is a delicate issue) to compatibility with base-change assertions for the *"line bundle in the usual sense* $\overline{\mathcal{M}}$*"* (an issue which is well-known from elementary algebraic geometry).

Thus, $\overline{\mathcal{M}}$ is a line bundle in the usual sense on C_d. Let us assume (without loss of generality) that the q-parameter on $\widehat{S}$ admits an $2N \cdot d$-th root. Then it follows that there exists a finite isogeny $E'_d \to E_d$ (of degree N) whose kernel is isomorphic to $\boldsymbol{\mu}_N$. Moreover, this isogeny compactifies to a diagram

$$\begin{array}{ccc} E'_d & \subseteq & C'_d \\ \downarrow & & \downarrow \\ E_d & \subseteq & C_d \end{array}$$

The purpose of introducing E'_d is that *the pull-back of the metrized line bundle $\mathcal{Q}$ to $E'_{\infty,S}$ is a line bundle in the usual sense.* Indeed, to see this, one reasons as follows: First of all, observe (from the general theory of abelian schemes) that the kernel of the pull-back morphism on Picard groups induced by the isogeny $E'_{\infty,S}|_{U_S} \to E_{\infty,S}|_{U_S}$ is the Cartier dual of $\mathrm{Ker}(E'_{\infty,S}|_{U_S} \to E_{\infty,S}|_{U_S}) = \boldsymbol{\mu}_N$. Thus, the kernel of the pull-back morphism is $\mathbf{Z}/N\mathbf{Z}$ (cf. the discussion in Chapter IV, §2, of the isogenies "$E^{[n]} \to E \to \widetilde{E}$" and line bundles on $\widetilde{E}$ which become trivial when pulled back to $E^{[n]}$). In other words, if one thinks of the subscheme of N-torsion points of $E_{\infty,S}|_{U_S}$ as $\boldsymbol{\mu}_N \times (\mathbf{Z}/N\mathbf{Z})$, then this kernel is generated by the line bundle corresponding to a divisor of the form $\tau - e_{E_{\infty,S}}$, where $e_{E_{\infty,S}}$ is the identity element of $E_{\infty,S}$, and τ is an N-torsion point whose projection to $\mathbf{Z}/N\mathbf{Z}$ generates $\mathbf{Z}/N\mathbf{Z}$. In particular, this means that the pull-back of $\mathcal{Q}|_{U_S}$ to $E'_{\infty,S}|_{U_S}$ is isomorphic to the pull-back to $E'_{\infty,S}|_{U_S}$ of the line bundle associated to a divisor of the form $\tau - e_{E_{\infty,S}}$, where $\tau \in \boldsymbol{\mu}_N(S) \subseteq E(S) \subseteq E_{\infty,S}(S)$. On the other hand, note that for such a τ, the *line bundle* (in the usual sense) $\mathcal{O}_{C_d}(\tau - e_{C_d})$ already forms a metrized line bundle on $E_{\infty,S}$ whose curvature is zero. Thus, in summary, $\mathcal{Q}|_{E'_{\infty,S}}$ and $\mathcal{O}_{C_d}(\tau - e_{C_d})|_{E'_{\infty,S}}$ have isomorphic restrictions to $E'_{\infty,S}|_{U_S}$ and, moreover, both have curvature zero. In particular, by Chapter IV, Proposition 4.3, it follows that these two metrized line bundles on $E'_{\infty,S}$ are isomorphic, i.e., that the metrized line bundle $\mathcal{Q}|_{E'_{\infty,S}}$ is represented by a line bundle in the usual sense on C'_d, as desired.

Thus, to summarize, we have $\overline{\mathcal{L}} = \overline{\mathcal{M}} \otimes \overline{\mathcal{Q}}$, where $\overline{\mathcal{M}}$ arises from a line bundle in the usual sense on C_d; and $\overline{\mathcal{Q}}|_{E'_{\infty,S}}$ arises from a line bundle in the usual sense on C'_d. In particular, we conclude that:

> The metrized line bundle $\overline{\mathcal{L}}|_{E'_{\infty,S}}$ arises from a line bundle $\mathcal{P}$ in the usual sense on C'_d. Moreover, the restriction of $\mathcal{P}$ to each to the irreducible components of the special fiber of C'_d is of degree N (since $E'_d \to E_d$ is of degree N).

Now fix an integer $j \geq 1$, and let

$$\mathcal{F} \stackrel{\mathrm{def}}{=} F^j(\mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{C_d,[d]}}\{\mathbf{v}\})|_{C'_d}$$

Thus, $\mathcal{F}$ is an S-flat coherent sheaf on C'_d whose restriction to E'_d is a vector bundle of rank j which admits a filtration whose subquotients are isomorphic to $\mathcal{O}_{E'_d}$. Moreover, the push-forwards

$$(f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<j}\{\mathbf{v}, \mathrm{et}\}$$

that we are interested in are simply the $\boldsymbol{\mu}_N$-invariant portions of

$$f'_*(\mathcal{F} \otimes_{\mathcal{O}_{C'_d}} \mathcal{P})$$

(where, in the following, we shall write f'_* for the push-forwards to S of objects on C'_d or E'_d) with respect to the evident natural $\boldsymbol{\mu}_N$-action. Thus, since the group scheme $\boldsymbol{\mu}_N$ is of multiplicative type, hence *reductive* (so the operation of "taking the $\boldsymbol{\mu}_N$-invariant part" commutes with base-change), it follows that:

> *The issue of the compatibility of the push-forward*
>
> $$(f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<j}\{\mathbf{v}, \mathrm{et}\}$$
>
> *with base-change may be reduced to the issue of the compatibility of the push-forward* $f'_*(\mathcal{F} \otimes_{\mathcal{O}_{C'_d}} \mathcal{P}) = f'_*\{(\mathcal{F} \otimes_{\mathcal{O}_{C'_d}} \mathcal{P})|_{E'_d}\}$ *with base-change.*

At this point, before we continue to study compatibility with base-change issues further, we pause to note the following important lemma:

Lemma 2.1. *The metrized vector bundle*

$$(f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<j}\{\mathbf{v}, \mathrm{et}\}$$

(which a priori *– cf. Chapter IV, Definition 5.2 – is just a quasi-coherent sheaf on S_∞) is, in fact, a coherent sheaf on S_∞. If, moreover, the classifying morphism $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ is* flat*, then this coherent sheaf is, in fact, a vector bundle on S_∞.*

Proof. Indeed, the above discussion shows that this metrized vector bundle is just the $\boldsymbol{\mu}_N$-invariant portion of the push-forward

$$f'_*(\mathcal{F} \otimes_{\mathcal{O}_{C'_d}} \mathcal{P})$$

On the other hand, this latter push-forward is clearly a *coherent sheaf* of $\mathcal{O}_{S_\infty}$-modules. Thus, the metrized vector bundle $(f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<j}\{\mathbf{v},\mathrm{et}\}$ is, in fact, a *coherent sheaf* of $\mathcal{O}_{S_\infty}$-modules, as desired.

If the classifying morphism is flat, then formation of push-forwards is compatible with base-change, so we reduce immediately to the "universal case" – i.e., where S is, say, of the form $\mathrm{Spec}(\mathcal{O}[[q^{\frac{1}{2N\cdot d}}]])$, and $\mathcal{O}$ is the Zariski localization of the ring of integers of a number field. In this case, S_∞ is a projective limit of regular schemes of dimension 2, so we conclude from from Lemma 1.1 that "any metrized vector bundle on S_∞ which is a coherent sheaf of $\mathcal{O}_{S_\infty}$-modules is, in fact, a vector bundle on S_∞," as desired. ○

Next, we consider base-change issues. The following result follows from elementary homological algebra:

Lemma 2.2. *Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra. Then we have an exact sequence:*

$$\begin{aligned} f'_*\{(\mathcal{F}\otimes_{\mathcal{O}_{C'_d}}\mathcal{P})|_{E'_d}\}\otimes_{\mathcal{O}_S}\mathcal{A} &\to f'_*\{(\mathcal{F}\otimes_{\mathcal{O}_{C'_d}}\mathcal{P})|_{E'_d}\otimes_{\mathcal{O}_S}\mathcal{A}\} \\ &\to Tor_1^{\mathcal{O}_S}(\mathbf{R}^1 f'_*\{(\mathcal{F}\otimes_{\mathcal{O}_{C'_d}}\mathcal{P})|_{E'_d}\},\mathcal{A}) \to 0 \end{aligned}$$

Moreover, when $j=1$, i.e., $\mathcal{F}=\mathcal{O}_{C'_d}$, we have $\mathbf{R}^1 f'_*(\mathcal{F}\otimes_{\mathcal{O}_{C'_d}}\mathcal{P})=0$, and $f'_*(\mathcal{F}\otimes_{\mathcal{O}_{C'_d}}\mathcal{P})\otimes_{\mathcal{O}_S}\mathcal{A}=f'_*(\mathcal{F}\otimes_{\mathcal{O}_{C'_d}}\mathcal{P}\otimes_{\mathcal{O}_S}\mathcal{A})$.

Proof. Indeed, the exact sequence follows by considering the usual spectral sequence relating $H^*(\mathcal{C}^\cdot)\otimes_{\mathcal{O}_S}\mathcal{A}$ and $H^*(\mathcal{C}^\cdot\otimes_{\mathcal{O}_S}\mathcal{A})$, where $\mathcal{C}^\cdot$ is a complex of flat, coherent $\mathcal{O}_S$-modules. In this case, since the push-forward functor f'_* has cohomological dimension ≤ 1, the complex $\mathcal{C}^\cdot$ may, in fact, be taken to consist of just two terms, i.e., $\mathcal{C}^\cdot$ may be written $\mathcal{C}^0\to\mathcal{C}^1$. The case $j=1$ follows by considering the fact that $\mathcal{P}$ is a line bundle whose restriction to each of the irreducible components of the special fiber of C'_d is of degree $N\geq 1$, and applying [Harts], Chapter III, Corollary 12.9, Theorem 12.11. ○

Now let us suppose that S is the spectrum of a *regular local ring of dimension* 2. Suppose that α,β are a system of regular parameters for this regular local ring, and that $V(\beta)\subseteq S$ is set-theoretically equal to the divisor at infinity $D\subseteq S$. Suppose, moreover, that we know that the morphism

$$f'_*\{(\mathcal{F}\otimes_{\mathcal{O}_{C'_d}}\mathcal{P})|_{E'_d}\}\otimes_{\mathcal{O}_S}\mathcal{A}\to f'_*\{(\mathcal{F}\otimes_{\mathcal{O}_{C'_d}}\mathcal{P})|_{E'_d}\otimes_{\mathcal{O}_S}\mathcal{A}\}$$

is *surjective* when $\mathcal{A}$ is taken to be $\mathcal{O}_S/(\alpha)$. Then I *claim* that it follows that this morphism is surjective for all $\mathcal{A}$ on which β acts injectively. Indeed, our assumption concerning the case when $\mathcal{A}=\mathcal{O}_S/(\alpha)$ implies that

$$Tor_1^{\mathcal{O}_S}(\mathbf{R}^1 f'_*\{(\mathcal{F} \otimes_{\mathcal{O}_{C'_d}} \mathcal{P})|_{E'_d}\}, \mathcal{O}_S/(\alpha)) = 0$$

i.e., that α acts injectively on the coherent sheaf $\mathcal{B} \stackrel{\text{def}}{=} \mathbf{R}^1 f'_*\{(\mathcal{F} \otimes_{\mathcal{O}_{C'_d}} \mathcal{P})|_{E'_d}\}$. Note that although $\mathcal{B}$ itself might not be finitely generated over $\mathcal{O}_S$, it may be written as a direct limit of finitely generated $\mathcal{O}_S$-submodules $\mathcal{B}' \subseteq \mathcal{B}$. Note that since α acts injectively on $\mathcal{B}$, it also acts injectively on $\mathcal{B}'$. Similarly, since $\mathcal{B}$ is set-theoretically supported on D, it follows that $\mathcal{B}'$ is also set-theoretically supported on D. But this implies that the $\mathcal{O}_S$-module $\mathcal{B}'$ has depth ≥ 1, hence (by the "Auslander-Buchsbaum Formula" – cf., e.g., [Mats]) that the $\mathcal{O}_S$-module $\mathcal{B}$ has cohomological dimension ≤ 1. Since, moreover, $\mathcal{B}'$ is (set-theoretically) supported on D, we thus conclude that there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_S^r \stackrel{\phi}{\longrightarrow} \mathcal{O}_S^r \longrightarrow \mathcal{B}' \longrightarrow 0$$

(where r is an integer). Moreover, since the zero locus of $\det(\phi) \in \mathcal{O}_S$ is set-theoretically equal to D, we conclude that there exists an $\mathcal{O}_S$-linear morphism $\psi : \mathcal{O}_S^r \to \mathcal{O}_S^r$ such that $\psi \circ \phi = \beta^a$, for some integer $a \geq 0$. In particular, we conclude that the above exact sequence remains exact after tensoring with any $\mathcal{A}$ on which β acts injectively. But this implies that for such an $\mathcal{A}$, we have $Tor_1^{\mathcal{O}_S}(\mathcal{B}', \mathcal{A}) = 0$, hence (since Tor commutes with filtered direct limits) that $Tor_1^{\mathcal{O}_S}(\mathcal{B}, \mathcal{A}) = 0$, as claimed.

In fact, in our situation, instead of being given that the entire morphism

$$f'_*\{(\mathcal{F} \otimes_{\mathcal{O}_{C'_d}} \mathcal{P})|_{E'_d}\} \otimes_{\mathcal{O}_S} \mathcal{A} \to f'_*\{(\mathcal{F} \otimes_{\mathcal{O}_{C'_d}} \mathcal{P})|_{E'_d} \otimes_{\mathcal{O}_S} \mathcal{A}\}$$

is surjective when $\mathcal{A} = \mathcal{O}/(\alpha)$, we will instead just be given that its $\boldsymbol{\mu}_N$*-invariant portion* is surjective when $\mathcal{A} = \mathcal{O}/(\alpha)$. But since $\boldsymbol{\mu}_N$ is of multiplicative type, it is clear that the argument of the preceding paragraph then allows one to conclude the surjectivity of the $\boldsymbol{\mu}_N$-invariant portion of this morphism for any $\mathcal{A}$ on which β acts injectively. In other words, we have proven the following:

Lemma 2.3. *Suppose that S is the spectrum of a* regular local ring of dimension 2. *Suppose that α, β are a system of regular parameters for this regular local ring, and that $V(\beta) \subseteq S$ is set-theoretically equal to the divisor at infinity $D \subseteq S$. Suppose, moreover, that we know that the morphism*

$$f'_*\{(\mathcal{F} \otimes_{\mathcal{O}_{C'_d}} \mathcal{P})|_{E'_d}\} \otimes_{\mathcal{O}_S} \mathcal{A} \to f'_*\{(\mathcal{F} \otimes_{\mathcal{O}_{C'_d}} \mathcal{P})|_{E'_d} \otimes_{\mathcal{O}_S} \mathcal{A}\}$$

is surjective *on $\boldsymbol{\mu}_N$-invariant parts when $\mathcal{A}$ is taken to be $\mathcal{O}_S/(\alpha)$. Then it follows that this morphism is surjective on $\boldsymbol{\mu}_N$-invariant parts for all $\mathcal{A}$ on which β acts injectively.*

§3. The Comparison Isomorphism in Characteristic Zero

The purpose of the present § is to show that for appropriate choices of the metrized line bundle $\overline{\mathcal{L}}$ (cf. Chapter V, §1), *certain slightly modified versions of the evaluation maps of Chapter V, Propositions 2.2, 2.3, are isomorphisms in characteristic zero.*

In this §, we use the notation of §1. Let us suppose that we are also given an *m-torsion point*

$$\eta \in E_{\infty,S}(S_\infty)$$

(where $m \geq 1$ is an integer) as in Chapter V, §1. Then, by pulling back the metrized line bundles constructed in the universal case in Chapter V, §1, we obtain metrized line bundles

$$\overline{\mathcal{L}}_{\mathrm{st},\eta}; \quad \overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta}$$

on $E_{\infty,S}$. We are now ready to state and prove the following result:

Theorem 3.1. (Comparison Isomorphism in Characteristic Zero) *Let $d, m \geq 1$ be integers such that m does* not *divide d. Suppose that $S^{\log}$ is a fine noetherian log scheme, and let*

$$C^{\log} \to S^{\log}$$

be a log elliptic curve *over $S^{\log}$ such that the "divisor at infinity" $D \subseteq S$ (i.e., the pull-back of the divisor at infinity of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ via the classifying morphism $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$) is a* Cartier divisor *on S. Also, let us assume that étale locally on the completion of S along D,* the pull-back of the Tate parameter q to this completion admits a d-th root*, and that the classifying morphism $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ associated to $C^{\log} \to S^{\log}$ is* flat *in a neighborhood of D. (Note that this flatness* automatically holds *whenever S is of characteristic zero.) Suppose, moreover, we are given a* torsion point

$$\eta \in E_{\infty,S}(S_\infty)$$

of order precisely m *which (according to the discussion preceding Chapter V, Proposition 1.1) defines line bundles $\overline{\mathcal{L}}_{\mathrm{st},\eta}$, $\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta}$. If d is* odd *(respectively,* even*), then let $\overline{\mathcal{L}} \stackrel{\mathrm{def}}{=} \overline{\mathcal{L}}_{\mathrm{st},\eta}$ (respectively, $\overline{\mathcal{L}} \stackrel{\mathrm{def}}{=} \overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta}$). Then:*

(1) *For* any *collection $\mathbf{v} = \{\mathbf{v}_\iota\}_\iota$ of elements of $\mathbf{Q}_{\geq 0} \cdot \log(q) \bigcup \infty$ (one for each connected component ι of the divisor at infinity D), the* evaluation map *of Chapter V, Proposition 2.2 (for "α" taken to be* 0*)* factors

through the metrized vector bundle $(f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\mathbf{v}, \mathrm{et}\}$ *of* §*1* to define a natural morphism

$$\Xi\{\mathbf{v}, \mathrm{et}\} : (f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\mathbf{v}, \mathrm{et}\} \to (f_S)_*(\overline{\mathcal{L}}|_{({}_dE^\dagger_\infty)})$$

(2) *If* $\mathbf{v}_\iota = \infty$ *for all* ι, *then*

$$\Xi\{\mathbf{v}, \mathrm{et}\} \otimes \mathbf{Q}$$

is an isomorphism. *Moreover, at each* ι, *there exists a unique sequence* $\mathbf{a}_\iota = \{(\mathbf{a}_\iota)_0, \ldots, (\mathbf{a}_\iota)_{d-1}\}$ *of elements of* $\mathbf{Q}_{\geq 0} \cdot \log(q)$ *such that the natural inclusions*

$$F^j((f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{(\mathbf{a}_\iota)_{j-1}, \mathrm{et}\}) \;\subseteq\; F^j((f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\infty, \mathrm{et}\})$$

are all equalities *and the natural projections*

$$(F^j/F^{j-1})((f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{(\mathbf{a}_\iota)_{j-1}, \mathrm{et}\}) \longrightarrow \frac{1}{(j-1)!} \cdot \exp(-(\mathbf{a}_\iota)_{j-1}) \cdot (f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}}) \otimes_{\mathcal{O}_S} \tau_E^{\otimes j-1}$$

(where $j = 1, \ldots, d$*) are* surjective *(hence bijective).*

(3) *Finally, these sequences* $\mathbf{a}_\iota$ *satisfy the following condition: for each connected component* ι, *there is an* $X_\iota \in \{I, II, III\}$ *such that*

$$(\mathbf{a}_\iota)_{j-1} = \frac{1}{d} \cdot c_j$$

where c_j *is as in Chapter V,* §*4, for Case* X_ι.

Proof. First, let us observe that since the classifying morphism $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ is *flat* in a neighborhood of infinity, *the various push-forwards* $(f_S)_*$ *(along with their filtrations)* *are compatible with base-change* (cf. Lemma 2.2). (Note that in characteristic zero, the classifying morphism factors through $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Q}}$, which is regular of dimension one. Thus, flatness follows from the assumption that the q-parameter is a non-zero divisor, i.e., that the closed subscheme D is a *Cartier divisor* on S.) Thus, it suffices to prove Theorem 3.1 in the *"universal case"* – for instance, in the case of $S \stackrel{\mathrm{def}}{=} T$, where $T \to B$ is any "nice covering," as in Chapter V, Proposition 1.2, such that at each ι, the corresponding q-parameter has a d-th root in $\mathcal{O}_T$.

Next, note that to see that the evaluation map of Chapter V, Proposition 2.2, factors as claimed in (1), it suffices to prove that it factors in *characteristic zero*. Indeed, once it factors in characteristic zero, the fact that it also factors *away from the divisor at infinity* (cf. Proposition 1.2; the integrality statement at the end of Chapter V, Theorem 3.1) implies – since $S = T$ is finite, flat over $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$, which is regular – that it factors over all of $S = T$. Similarly, the natural inclusions of (2) are equalities if and only if they are equalities in *characteristic zero*. Indeed, if they are equalities in characteristic zero, then both sides are vector bundles (cf. Lemma 2.1) which are tautologically equal away from the divisor at infinity.

Thus, in summary, *except for the surjectivity of the natural projections in (2), it suffices to prove Theorem 3.1 in characteristic zero*. In particular, for the remainder of the proof (except where specified otherwise),

We assume that S is $T \otimes \mathbf{Q}$, where T is as above.

Also, to simplify the discussion, we assume that S is *connected*. Thus, the base S is a smooth, proper curve over a field of characteristic zero.

Now we divide the proof into parts:

Local Assertions:

Let us first show that *the evaluation map factors as stated in (1)* for any choice of $\mathbf{v}$, and that *the natural projections of (2)*

$$(F^j/F^{j-1})((f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<d}\{(\mathbf{a}_\iota)_{j-1}, \mathrm{et}\}) \longrightarrow$$
$$\frac{1}{(j-1)!} \cdot \exp(-(\mathbf{a}_\iota)_{j-1}) \cdot (f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}}) \otimes_{\mathcal{O}_S} \tau_E^{\otimes j-1}$$

are surjective for the special choice of $\mathbf{v}$ referred in (2) and (3).

Since these are *local issues at infinity*, we assume (just for this "Local Assertions" part of the proof) that

$$S = \mathrm{Spec}(A)$$

where $A = \mathcal{O}[[q^{\frac{1}{N\cdot d}}]]$ (where $N \stackrel{\mathrm{def}}{=} 2m \cdot d$), and $\mathcal{O}$ is a Zariski localization of the ring of integers of a number field. Let $K_H = \mathbf{Z}/d\mathbf{Z} \subseteq {}_dE$ be the subgroup corresponding to the d-th root $q^{\frac{1}{d}}$ of q. Thus, we have a finite étale isogeny

$$E_d \to E_d/K_H = E_H$$

One checks easily (using Chapter IV, Lemma 5.4 – cf. the argument in the "Alternate Proof of Theorem 6.2" in Chapter V, §6) that $\overline{\mathcal{L}}$ descends to a metrized line bundle $\overline{\mathcal{L}}_H$ of relative degree 1 on $E_{\infty_H,S}$. Thus, there exists a *Lagrangian subgroup* (cf. the discussion of such subgroups in Chapter IV, §1)

$$H \subseteq \mathcal{G}_{\overline{\mathcal{L}}}$$

Note that the $\{\mathbf{v}, \mathrm{et}\}$-push forwards, as well as the evaluation maps $\Xi\{\mathbf{v}, \mathrm{et}\}$ all have "subscript H versions" (obtained by taking "H-invariants" of the original versions discussed above). We leave the routine details to the reader.

Next, let us note that ${}_dE^{\dagger}_{\infty_H} \subseteq E^{\dagger}_{\infty_H,[d]}$ *lies inside the canonical section derived from* κ. Indeed, this follows from the facts that:

> (i) the canonical section is a *group homomorphism* (Chapter III, Theorem 2.1);
>
> (ii) W_E has no torsion since our base S is $\mathbf{Z}$-flat; and
>
> (iii) the image of ${}_dE^{\dagger}_{\infty_H}$ in $E_{\infty_H,S}$ lies in the identity component of the special fiber of $E_{\infty_H,S}$.

Since the modifications in integral structure that one performs in order to construct the $\{\mathbf{v}, \mathrm{et}\}$-push-forward (for *any* $\mathbf{v}$) are all done using the splittings defined by the canonical section (derived from κ), it is a *tautology* that all sections of the $\{\mathbf{v}, \mathrm{et}\}$-push-forward which have "denominators" relative to the usual push-forward *vanish* (i.e., modulo the sections which do not have denominators relative to the usual push-forward) when restricted to the canonical section, hence when restricted to ${}_dE^{\dagger}_{\infty_H}$. Thus, it follows that the evaluation map of Chapter V, Proposition 2.2, factors as claimed in (1) (for *any* $\mathbf{v}$).

Next, let us observe that "metrically speaking" $\overline{\mathcal{L}}_H$ is the same as the metrized line bundle $\mathcal{O}_{E_{\infty_H,S}}([e_{E_{\infty_H,S}}])$. More precisely, we have:

$$\overline{\mathcal{L}}_H^{\otimes m\cdot d} \cong \mathcal{O}_{E_{\infty_H,S}}(m \cdot d \cdot [e_{E_{\infty_H,S}}])$$

Indeed, if we forget about metrics, this follows from the linear equivalence $m(d \cdot [\eta]) \sim m \cdot d \cdot [e]$ on $E_{\infty,S}|_{U_S}$ pushed forward from $E_{\infty,S}$ to $E_{\infty_H,S}$. Moreover, it follows from the computation of the curvature of $\overline{\mathcal{L}}$ (in the discussion preceding Chapter V, Proposition 1.1) that both sides have the same curvature. Thus, by Chapter IV, Proposition 4.3, it follows that these two metrized line bundles are isomorphic. (Indeed, it is easy to check that the constant "C" in Chapter IV, Proposition 4.3, may be taken to be zero in this case).

In particular, it follows that $\overline{\mathcal{L}}_H$ differs from $\mathcal{O}_{E_{\infty_H,S}}([e_{E_{\infty_H,S}}])$ by a metrized line bundle on $E_{\infty_H,S}$ whose $(m \cdot d)$*-th power is trivial.* But, in the language of Chapter V, §4, if we think of E_H here as the curve "$\widetilde{E}$" of Chapter V, §4, then this amounts to saying that $\overline{\mathcal{L}}_H$ is the same as the "standard line bundle $\mathcal{L}_{\widetilde{E}}$" of Chapter V, §4, twisted by some appropriate character $\chi_{\mathcal{L}}$ of order "n" where we take the "n" of Chapter V, §4, to be $N = 2m \cdot d$, relative to the notation used here. This character $\chi_{\mathcal{L}}$ falls into one of the three "Cases" treated in Chapter V, §4. We take $X_\iota \in \{I, II, III\}$ to be the "Case" that $\chi_{\mathcal{L}}$ falls under (cf. Chapter V, Theorem 4.6).

Now let

$$(\mathbf{a}_\iota)_{j-1} \stackrel{\mathrm{def}}{=} \frac{1}{d} \cdot c_j \cdot \log(q)$$

(where c_j is as in Chapter V, §4), and apply Chapter V, Theorem 4.6. Note that the "q" of Chapter V, §4, which we write $q_{\text{Ch. V,§4}}$, is such that $q_{\text{Ch. V,§4}}^N$ is the q-invariant of $\widetilde{E}$, which corresponds to E_H in the present discussion. Since the q-invariant q_{E_H} of E_H is equal to $q^{\frac{1}{d}}$, we thus see that

$$q^{\frac{1}{d}} = q_{E_H} = q_{\text{Ch. V,§4}}^N$$

Thus, the various adjustments of integral structure used to form the $\{\mathbf{v}, \mathrm{et}\}$-push-forward are of the form

$$\exp((\mathbf{a}_\iota)_{j-1}) = q^{\frac{1}{d} \cdot c_j} = q_{E_H}^{c_j} = q_{\text{Ch. V,§4}}^{N \cdot c_j}$$

Note, moreover, that

$$c_j = \mathrm{Max}_{j_0 \leq j}(c_{j_0})$$

(where j_0 ranges over all integers $\leq j$ which are admissible as "j_0's" for Chapter V, Theorem 4.6, and we define the "Max" of the empty set to be 0). Note that Chapter V, Theorem 4.6, addresses the case of "torsorial degree" (cf. Chapter III, Definition 2.2) $< j$ and states that in this case, there exist *generators* of the portion of the "usual" push-forward (i.e., without the $\{\mathbf{v}_\iota\}$) corresponding to torsorial degrees $j_0 - 1, \ldots, j - 1$ which *vanish* modulo $\exp((\mathbf{a}_\iota)_{j_0-1}) = q^{\frac{1}{d} \cdot c_j}$ when restricted to the canonical section derived from κ over $(E_H)_{\widehat{S}}$. Moreover, if j_0 is the largest "admissible j_0" (for Chapter V, Theorem 4.6) which is $\leq j$, then $\exp((\mathbf{a}_\iota)_{j'}) = \exp((\mathbf{a}_\iota)_{j_0-1})$, for all $j' \in \{j_0 - 1, \ldots, j-1\}$. Thus, if we divide *these* generators by $\exp((\mathbf{a}_\iota)_{j_0-1})$, we see that we get *integral* sections of the $(\mathbf{a}_\iota)_{j-1}$-push-forward. In particular, we see that for this choice of $(\mathbf{a}_\iota)_{j-1}$, the natural projections

$$(F^j/F^{j-1})((f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{(\mathbf{a}_\iota)_{j-1}, \mathrm{et}\}) \longrightarrow \frac{1}{(j-1)!} \cdot \exp(-(\mathbf{a}_\iota)_{j-1}) \cdot (f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}}) \otimes_{\mathcal{O}_S} \tau_E^{\otimes j-1}$$

are *surjective*, as desired. This completes the proof of the *local assertions*.

Reduction to the Computation of Degrees:

Let us now return to our *global, universal, characteristic zero* S.

Next, let us observe that, by Chapter V, Theorems 5.6 and 6.2, it follows from the fact that m does not divide d (so $d \cdot \eta \neq 0$) that in the present *universal* situation, *the evaluation map of Chapter V, Proposition 2.2, is an isomorphism over the generic point of* S. Thus, now that we have shown that the evaluation map factors as stated, *to show that* $\Xi\{\mathbf{v}, \mathrm{et}\}$ *is an isomorphism over all of* S*, it will suffice to compute the degrees of the metrized vector bundles (of rank* d^2*) on both sides, and show that:*

$$\deg(\text{domain}) \geq \deg(\text{range})$$

Note, moreover, that:

> *The surjectivity assertion of (2) (proven above) is necessary in order to bound from below the degree of the* $\{\mathbf{v}, \mathrm{et}\}$*-push-forward* (for $\mathbf{v} = \infty$) *by the sum of the degrees of the various* $\exp(-(\mathbf{a}_\iota)_{j-1}) \cdot (f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}}) \otimes_{\mathcal{O}_S} \tau_E^{\otimes j-1}$.

It turns out that the computation of the degree differs quite substantially, depending on whether d is *odd* or *even*. Thus, in the following we carry out the computations in these two cases separately.

Computation of the Degree in the Odd Case:

To complete the proof of Theorem 3.1 for d odd, it suffices (as noted above) to show that in the situation discussed above, the degree of the domain of $\Xi\{\mathbf{v}, \mathrm{et}\}$ is $\geq$ the degree of the range. Let us first observe that Chapter V, Proposition 1.2, implies that:

The degree of the range of $\Xi\{\mathbf{v}, \mathrm{et}\}$ *is equal to* 0.

Thus, it remains to show that the degree of the $\{\mathbf{v}, \text{et}\}$-push-forward (for $\mathbf{v} = \infty$) is ≥ 0. First, let us note that the surjectivity assertion already proven implies that:

$$\begin{aligned}
\deg(\mathbf{v}\text{−push − forward}) - \deg(\text{usual push − forward}) &\geq d \cdot \sum_{\iota} \sum_{j=0}^{d-1} (\mathbf{a}_\iota)_j \\
&= \sum_{\iota} \sum_{j=1}^{d} c_j(\text{Case } X_\iota) \\
&= \frac{1}{24} d(d^2 - 1)
\end{aligned}$$

(in $\log(q)$ units), where:

(1) In the first inequality, the extra factor of d out in front arises from the fact that the push-forward of the metrized line bundle $\overline{\mathcal{L}}$ on $E_{\infty,S}$ has *rank* d.

(2) In the last equality, we apply Chapter V, Lemma 4.2.

On the other hand, the usual push-forward admits a filtration (by torsorial degree) whose subquotients are $\tau_E^{\otimes j} \otimes_{\mathcal{O}_S} (f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}})$, for $j = 0, \ldots, d-1$. Since the line bundle τ_E on S has degree (in $\log(q)$ units) given by $-\frac{1}{12}$ (cf. the proof of Chapter IV, Theorem 5.8), we thus obtain that the degree of the usual push-forward is given by:

$$-\frac{d}{12} \cdot (\sum_{j=0}^{d-1} j) + d \cdot \deg((f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}})) = -\frac{1}{24} d(d^2 - 1)$$

(cf. the Remark following Chapter V, Proposition 1.1). Thus, putting everything together, we obtain that the $\{\mathbf{v}, \text{et}\}$-push-forward has degree ≥ 0, as desired.

This completes the proof of the fact that $\Xi\{\mathbf{v}, \text{et}\}$ is an isomorphism for $\mathbf{v} = \infty$. It also shows that the *natural inclusions* of (2)

$$F^j((f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{(\mathbf{a}_\iota)_{j-1}\}) \subseteq F^j((f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\infty\})$$

are all equalities (for if any one of them were not an inequality, it would follow that the degree of the domain of $\Xi\{\infty\}$ is $>$ the degree of the range, which is absurd). (Note that here we use the fact that the function $j \mapsto (\mathbf{a}_\iota)_j$ is "monotone increasing" (i.e., preserves the relation "$\leq$").) This completes the proof of Theorem 3.1, for d *odd*.

Computation of the Degree in the Even Case:

In the *even* case, the computations become somewhat more complicated. First, let us note that by Chapter V, Proposition 1.2, we have:

$$\begin{aligned}\deg(\text{range}) &= -d \sum_{\iota} \phi_1(-d \cdot \eta_\iota + \frac{1}{2}) \\ &= -d \sum_{\iota} \{\phi_1(-d \cdot \eta_\iota + \frac{1}{2}) - \phi_1(-d \cdot \eta_\iota)\}\end{aligned}$$

where we note that:

(1) The intersection number of Chapter V, Proposition 1.2, must be multiplied by d^2, since we have d^2 torsion points annihilated by d.

(2) In the second line, the sum over all ι of the second term is *zero*, since m does not divide d – cf. the (proof of the) odd part of Chapter V, Proposition 1.2.

Next, observe that for $|\theta| \leq \frac{1}{2}$, the function $\theta \mapsto \phi_1(\theta + \frac{1}{2}) - \phi_1(\theta)$ is linear, except at 0 and $\frac{1}{2}$ (cf. Chapter IV, Corollary 4.5), and takes the value $-\frac{1}{8}$ (respectively, $\frac{1}{8}$) at $\theta = 0$ (respectively, $\theta = \frac{1}{2}$) – cf. Chapter IV, Proposition 4.4. It is easy to see that these properties determine this function uniquely, namely:

$$\phi_1(\theta + \frac{1}{2}) - \phi_1(\theta) = \frac{1}{2}|\theta| - \frac{1}{8}$$

for $|\theta| \leq \frac{1}{2}$.

On the other hand, we have

$$\begin{aligned}\deg(\text{domain}) &\geq -\frac{1}{24}d(d^2-1) + \sum_{\iota}\sum_{j=1}^{d} c_j(\text{Case } X_\iota) \\ &= -\frac{1}{24}d(d^2-1) + \sum_{\iota}\sum_{j=1}^{d-1} c_j(\text{Case } X_\iota) + \sum_{\iota} c_d(\text{Case } X_\iota) \\ &= -\frac{1}{24}d(d^2-1) + \frac{1}{24}(d-1)((d-1)^2-1) + \sum_{\iota} c_d(\text{Case } X_\iota) \\ &= \frac{1}{24}d(d-1)\{-(d+1)+d-2\} + \sum_{\iota} c_d(\text{Case } X_\iota) \\ &= -\frac{1}{8}d(d-1) + \sum_{\iota} c_d(\text{Case } X_\iota)\end{aligned}$$

where:

(1) The first line follows by the same general reasoning as in the odd case.

(2) The third line follows from the computation of the sum of the c_j's in the odd case, applied to the *odd* number $d-1$.

Thus, to summarize, we would like to prove that

$$-\frac{1}{2}k(k-\frac{1}{2})+\sum_{\iota} c_{2k}(\text{Case } X_\iota) \geq -d \sum_{\iota} \{\phi_1(-d\cdot\eta_\iota+\frac{1}{2})-\phi_1(-d\cdot\eta_\iota)\}$$

where we set $k \stackrel{\text{def}}{=} \frac{d}{2}$.

Now, for each ι, we would like to define a number i_ι, as follows. If $X_\iota = I$ (i.e., we are in Case I at ι), then let

$$i_\iota \stackrel{\text{def}}{=} m\cdot d = m_{\text{Ch. V},\S4}$$

(i.e., "$m\cdot d$" in the notation of the present discussion corresponds to "m" in the notation of Chapter V, §4 ($= m_{\text{Ch. V},\S4}$) – cf. the discussion of the "Local Assertions" part of the proof). If $X_\iota = II$, then let

$$i_\iota \stackrel{\text{def}}{=} 0$$

If $X_\iota = III$, then let

$$i_\iota \stackrel{\text{def}}{=} |i_\chi|$$

(where $|i_\chi| < m_{\text{Ch. V},\S4}$ is as in the discussion of Case III in Chapter V, §4). Then let us observe that it follows immediately from the explicit formulas for $c_d(\text{Case } X_\iota)$ in Chapter V, §4, that we have

$$c_d(\text{Case } X_\iota) = \frac{1}{2}k^2 - \frac{1}{2}k\cdot\frac{i_\iota}{m_{\text{Ch. V},\S4}} = \frac{1}{2}k(k-\frac{i_\iota}{m_{\text{Ch. V},\S4}})$$

(where we use here the fact that $d = 2k$ is *even*). Thus, we see that the left-hand side of the inequality that we would like to show may be written as follows:

$$-\frac{1}{2}k(k-\frac{1}{2})+\sum_{\iota}\frac{1}{2}k(k-\frac{i_\iota}{m_{\text{Ch. V},\S4}}) = \sum_{\iota}\frac{1}{2}k(\frac{1}{2}-\frac{i_\iota}{m_{\text{Ch. V},\S4}}) = -\sum_{\iota}\frac{1}{8}d(2\frac{i_\iota}{m_{\text{Ch. V},\S4}}-1)$$

(where we note that in our conventions, the "sums" over ι are "to be in $\log(q)$ units," hence amount, in fact, to *averages*; thus, a sum over ι of a constant C is equal to the constant C itself). Thus, by dividing by $-d$, we see that it suffices to show that

$$\sum_{\iota} \frac{1}{8}(\frac{2i_\iota}{m_{\text{Ch. V,§4}}} - 1) = \sum_{\iota} (\frac{1}{2}|\theta_\iota| - \frac{1}{8})$$

where θ_ι is the unique rational number of absolute value $\leq \frac{1}{2}$ that defines the same element as $-d \cdot \eta_\iota$ in $\mathbf{S}^1_\iota$. In particular, it suffices to verify that

$$\frac{1}{2}\frac{i_\iota}{m_{\text{Ch. V,§4}}} = |\theta_\iota|$$

for each ι. But if one unravels the definitions, it is easy to see that this is essentially a *tautology*: Indeed, it follows from the even part of Chapter IV, Lemma 5.4, that the character $\chi_{\mathcal{L}}$ of Chapter V, §4, is precisely that determined by the torsion point $-d \cdot \eta_\iota + \frac{1}{2}$ of $\mathbf{S}^1_\iota$, i.e.,

$$\chi_{\mathcal{L}} \longleftrightarrow -d \cdot \eta_\iota + \frac{1}{2}$$

On the other hand, it follows from the definitions of Chapter V, §4, that $\pm\frac{i_\iota}{2m_{\text{Ch. V,§4}}}$ is precisely the "coordinate" on $\mathbf{S}^1_\iota$ corresponding to the character $\chi_{\mathcal{M}}$, i.e.,

$$\chi_{\mathcal{M}} \longleftrightarrow \pm\frac{i_\iota}{2m_{\text{Ch. V,§4}}}$$

Moreover, the character $\chi_{\mathcal{M}}$ is the result of shifting the character $\chi_{\mathcal{L}}$ by the character defined by $\frac{1}{2}$:

$$\chi_{\mathcal{M}} \otimes {\chi_{\mathcal{L}}}^{-1} \longleftrightarrow \frac{1}{2}$$

Thus, putting everything together, the two "$\frac{1}{2}$'s" cancel, and we see that the rational number $\frac{i_\iota}{2m_{\text{Ch. V,§4}}}$ ($\leq \frac{1}{2}$) defines the same element (up to sign) as $-d \cdot \eta_\iota$ in $\mathbf{S}^1_\iota$, hence is equal to $|\theta_\iota|$, as desired.

This completes the proof of the fact that $\Xi[\infty]$ *is an isomorphism in the case of* d *even.* The fact that the natural inclusions of Theorem 3.1, (2), are, in fact, equalities follows by the same argument as in the odd case. Thus, the proof of Theorem 3.1 is complete. $\bigcirc$

Remark. One may regard the computation of the sequences $\mathbf{a}_\iota$ in Theorem 3.1, (2), (3), such that the natural inclusions and natural projections of (2) are surjective as a computation of the *analytic torsion* of the $\{\mathbf{v}, \mathrm{et}\}$-push forwards (i.e., the push-forwards equipped with the new integral structure/metric defined at the beginning of this §). This point of view is very much in line with what we will do in Chapter VII, Chapter VIII, where we compute analogous analytic torsion-type quantities at the archimedean primes of a number field.

§4. The Comparison Isomorphism in Mixed Characteristic

In this §, we prove the *Comparison Isomorphism in Mixed Characteristic*. Put another way, we show in this § that the characteristic zero isomorphism of §3, Theorem 3.1, (2), *preserves certain natural integral structures (at all finite primes) on both sides.*

In this §, we shall continue to use the notations of §3. The following result is the *main result of this Chapter*:

Theorem 4.1. (Comparison Isomorphism in Mixed Characteristic) *Let $d, m \geq 1$ be integers such that m does* not *divide d. Suppose that $S^{\log}$ is a fine noetherian log scheme, and let*

$$C^{\log} \rightarrow S^{\log}$$

be a log elliptic curve *over $S^{\log}$ such that the "divisor at infinity" $D \subseteq S$ (i.e., the pull-back of the divisor at infinity of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ via the classifying morphism $S \rightarrow (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$) is a* Cartier divisor *on S. Also, let us assume that étale locally on the completion of S along D,* the pull-back of the Tate parameter q to this completion admits a d-th root*, and that we are given a* torsion point

$$\eta \in E_{\infty,S}(S_\infty)$$

of order precisely m *which defines line bundles $\overline{\mathcal{L}}_{\mathrm{st},\eta}$, $\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta}$ (cf. Chapter V, §1). If d is* odd *(respectively,* even*), then let $\overline{\mathcal{L}} \stackrel{\mathrm{def}}{=} \overline{\mathcal{L}}_{\mathrm{st},\eta}$ (respectively, $\overline{\mathcal{L}} \stackrel{\mathrm{def}}{=} \overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta}$). Then:*

(1) **(Compatibility with Base-Change)** *For any collection $\mathbf{v} = \{\mathbf{v}_\iota\}_\iota$ of elements of $\mathbf{Q}_{\geq 0} \cdot \log(q) \bigcup \infty$ (one for each connected component ι of the divisor at infinity D) such that for each ι, $\mathbf{v}_\iota \geq (\mathbf{a}_\iota)_{d-1}$ (notation of Theorem 3.1, (3)),* the formation of the push-forward

$$(f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\mathbf{v}, \mathrm{et}\}$$

(along with its filtration) commutes with base-change *(among bases $S^{\log}$ satisfying the hypotheses given above).*

(2) **(Zero Locus of the Determinant)** *Assume that S is $\mathbf{Z}$-flat. If $\mathbf{v}_\iota = \infty$ for all ι, then the scheme-theoretic* zero locus of $\det(\Xi\{\mathbf{v}, \mathrm{et}\})$, *i.e., the determinant of the morphism*

$$\Xi\{\mathbf{v}, \mathrm{et}\} : (f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\mathbf{v}, \mathrm{et}\} \to (f_S)_*(\overline{\mathcal{L}}|_{({}_dE^\dagger_\infty)})$$

of Theorem 3.1, (1), is given by the divisor

$$d \cdot [\eta \bigcap ({}_dE)]$$

(where ${}_dE$ is the kernel of multiplication by d on E_d). In fact, the divisor of poles of the inverse morphism to $\Xi\{\mathbf{v}, \mathrm{et}\}$ is contained in the divisor $[\eta \bigcap ({}_dE)]$.

(3) **(Analytic Torsion Properties)** *For each ι, let us write*

$$\mathbf{a}_\iota = \{(\mathbf{a}_\iota)_0, \ldots, (\mathbf{a}_\iota)_{d-1}\}$$

for the sequence of elements of $\mathbf{Q}_{\geq 0} \cdot \log(q)$ discussed in Theorem 3.1, (2), (3). Then the natural inclusions

$$F^{j+1}((f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{(\mathbf{a}_\iota)_j, \mathrm{et}\}) \ \subseteq \ F^{j+1}((f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\infty, \mathrm{et}\})$$

are all equalities *and the natural projections*

$$(F^{j+1}/F^j)((f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{(\mathbf{a}_\iota)_j, \mathrm{et}\}) \ \longrightarrow \ \frac{1}{j!} \cdot \exp(-(\mathbf{a}_\iota)_j) \cdot (f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}}) \otimes_{\mathcal{O}_S} \tau_E^{\otimes j}$$

(where $j = 0, \ldots, d-1$) are bijective. *Moreover, the sections of $\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}}$ that realize these bijections have q-expansions in a neighborhood of infinity that are given* explicitly *in Chapter V, Theorem 4.8.*

Remark. Relative to the bound on the poles of the inverse morphism to $\Xi\{\mathbf{v}, \text{et}\}$ (i.e., Theorem 4.1, (2)), we remark that *these denominators tend to be very small.* Indeed, typically an intersection between torsion points will look like the zero locus of some $1-\omega$, where ω is a root of unity. In fact, by taking m to have *at least two* prime factors that do not divide d, it is easy to see that one can arrange that $\eta \bigcap ({}_dE) = \emptyset$.

Remark. Note that *compatibility with base-change is extremely important for applications.* For instance, base-change from $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ to a point of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ valued in the ring of integers of a number field is a typical case of a situation where one must use the "full power" of compatibility with base-change.

Remark. The poles "$\exp(-(\mathbf{a}_\iota)_j)$" appearing in Theorem 4.1, (3), are clearly of fundamental importance in Theorems 3.1 and 4.1, and indeed, of fundamental importance to this entire paper. We shall refer to these poles as *Gaussian poles*, since they grow roughly like a Gaussian (i.e., $(\mathbf{a}_\iota)_j$ is roughly *quadratic* in j).

Proof. First, let us observe that once one proves the base-change assertion (1), assertions (2) and (3) will follow as soon as they have been proven in the "universal case" – i.e., say, for S regular of dimension 2 such that the classifying morphism $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ is flat. But in this "universal case," assertion (3) already follows from Theorem 3.1, (2). Also, in the "universal case," assertion (2) follows from Theorem 3.1, (2), and Chapter V, Theorem 6.2. Indeed, both the domain and range of $\Xi\{\mathbf{v}, \text{et}\}$ are vector bundles of the same rank over a two-dimensional regular base (cf. Lemma 2.1), so the scheme-theoretic zero locus of the determinant of $\Xi\{\mathbf{v}, \text{et}\}$ is known as soon as it is known at the height 1 primes. But at the height 1 primes, the zero locus of $\det(\Xi\{\mathbf{v}, \text{et}\})$ is known by Theorem 3.1, (2), and Chapter V, Theorem 6.2. Thus, in summary, to complete the proof of Theorem 4.1, *it suffices to prove assertion (1) under the hypothesis that assertions (2) and (3) are known in the "universal case."*

On the other hand, by what we did in §2, we know that *assertion (1) holds away from the divisor at infinity.* Thus, by Lemma 2.3, it suffices to prove compatibility with base-change in the special case where S is *regular of dimension* 2, and *the classifying morphism* $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ *is finite and flat* (cf. the theory of [KM], Chapter 5). Let $\mathcal{O}$ be the normalization of $\mathbf{Z}$ in S. Then we may even assume that the fibers of $S \to \text{Spec}(\mathcal{O})$ are *geometrically reduced*, and even *smooth near infinity.* Indeed, by flatness over $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$, it suffices to check this near infinity, where S may be taken to be such that its completion at infinity is the spectrum of

$$\mathcal{O}[[q^{\frac{1}{M}}]]$$

for some positive integer M. In fact, Lemma 2.3 tells us that it suffices to prove compatibility with base-change for the particular base-change

$$S \times_{\mathcal{O}} \mathrm{Spec}(\mathcal{O}/\mathfrak{p}) \to S$$

where $\mathfrak{p}$ is an arbitrary maximal prime of $\mathcal{O}$. In the following, we will denote the result of base-changing objects via $\mathcal{O} \to k \stackrel{\mathrm{def}}{=} \mathcal{O}/\mathfrak{p}$ by means of a subscript k.

Next, let us observe that the morphism "$f'_*(\mathcal{F} \otimes_{\mathcal{O}_{C'_d}} \mathcal{P}) \otimes_{\mathcal{O}_S} \mathcal{A} \to f'_*(\mathcal{F} \otimes_{\mathcal{O}_{C'_d}} \mathcal{P} \otimes_{\mathcal{O}_S} \mathcal{A})$" studied in §2 is always *injective*. Indeed, this follows from the fact that both sides inject into their restrictions to $U_S = S \backslash D$, where they are equal. Thus, compatibility with base-change issues amount to the *surjectivity* of the "$\boldsymbol{\mu}_N$-invariant part of" (terminology of §2) this morphism.

Thus, in summary, it suffices to show that the morphism

$$(f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<j}\{\mathbf{v}, \mathrm{et}\} \otimes_{\mathcal{O}} k \to (f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}} \otimes_{\mathcal{O}} k)^{<j}\{\mathbf{v}, \mathrm{et}\}$$

is *surjective* for all $j = 1, \ldots, d$. Let us write $\mathcal{R}_j$ (respectively, $\mathcal{D}_j$; $\mathcal{I}_j$) for the *range* (respectively, *domain*; *image*) of this morphism. Note that $\mathcal{R}_j$, $\mathcal{D}_j$, and $\mathcal{I}_j$ are all *vector bundles* on S_k. Indeed, this is clear away from infinity; near infinity, it follows from the fact that S_k is smooth over k near infinity. Thus, we have an inclusion

$$\mathcal{I}_j \subseteq \mathcal{R}_j$$

of vector bundles on S_k which is an equality away from infinity. We wish to show that this inclusion is an equality over all of S_k.

Now let us suppose that:

$$\mathcal{I}_d = \mathcal{R}_d$$

Then I *claim* that it follows that $\mathcal{I}_j = \mathcal{R}_j$ for all $j = 1, \ldots, d$. Indeed, this follows by considering *degrees*, as in the proof of Theorem 3.1. That is to say, the fact that the natural projections of (3) are *surjective* (over S), combined with the fact that the push-forward $(f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}})$ commutes with base-change (cf. the last part of Lemma 2.2), implies that if there were a single j such that $\mathcal{I}_j \neq \mathcal{R}_j = F^j(\mathcal{R}_d)$, then summing up over the various $(F^j/F^{j-1})(\mathcal{R}_d)$'s and $\mathcal{I}_j/\mathcal{I}_{j-1}$'s, we obtain that

$$
\begin{aligned}
\deg(\mathcal{R}_d) &> \sum_{\iota} \sum_{j=1}^{d} \{d \cdot (\mathbf{a}_\iota)_{j-1} + \deg((f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}} \otimes_{\mathcal{O}_S} \tau_E^{\otimes j-1}))\} \\
&= \deg((f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<j}\{\mathbf{v}, \mathrm{et}\} \otimes_{\mathcal{O}} \mathbf{Q}) \\
&= \deg((f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<j}\{\mathbf{v}, \mathrm{et}\} \otimes_{\mathcal{O}} k) \\
&= \deg(\mathcal{D}_d) \\
&= \deg(\mathcal{R}_d)
\end{aligned}
$$

(where the *first equality* follows from Theorem 3.1, (2); the *second equality* follows from the fact that the degree of a vector bundle on S is locally constant over $\mathrm{Spec}(\mathcal{O})$; the *third equality* follows from the definition of $\mathcal{D}_d$; and the *fourth equality* follows from the assumption $\mathcal{I}_d = \mathcal{R}_d$). Thus, we get a contradiction; this completes the proof of the claim.

Thus, it suffices to prove that

$$\mathcal{I}_d = \mathcal{R}_d$$

Since this assertion is local at infinity, *we will assume in the rest of the proof that*

$$S = \mathrm{Spec}(\mathcal{O}[[q^{\frac{1}{M}}]])$$

for some positive integer M. To conclude that $\mathcal{I}_d = \mathcal{R}_d$, we would like to use the fact that $\Xi\{\mathbf{v}, \mathrm{et}\}$ is an *isomorphism* over S (by (2) in the "universal case"), so long as η does not intersect ${}_dE$. Now of course, in the case under consideration η might intersect ${}_dE$. Note, however, that if we modify η by some torsion point $\eta_\delta \in E_d(S)$ whose restriction to the special fiber of E_d lies in the same connected component as that of the identity, then this problem can be avoided. Note that (after possibly enlarging $\mathcal{O}$) such an η_δ always exists (since we have localized near infinity). Moreover, we may even assume that η_δ is such that $\eta' \stackrel{\mathrm{def}}{=} \eta + \eta_\delta$ does not intersect ${}_dE$. Next, note that, by (2) in the "universal case," the "$\Xi\{\mathbf{v}, \mathrm{et}\}$" corresponding to such an η' is an *isomorphism*. Since in fact, we are interested in the situation with the original η, we observe that (by translating by $-\eta_\delta$) we can interpret the fact that we get an isomorphism when η is replaced by η' as saying that the evaluation map

$$(f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\mathbf{v}, \mathrm{et}\} \to (f_S)_*(\overline{\mathcal{L}}|_{(-\eta^\dagger_\delta + {}_dE^\dagger_\infty)})$$

(where $\eta^\dagger_\delta$ is the natural lifting of η_δ to $E^\dagger_{\infty,[d]}$) is an isomorphism. Tensoring with k, we thus obtain that the evaluation map

$$(f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<d}\{\mathbf{v},\mathrm{et}\} \otimes_{\mathcal{O}} k = \mathcal{D}_d \to (f_S)_*(\overline{\mathcal{L}}|_{(-\eta^{\dagger}_{\delta} + {}_d E^{\dagger}_{\infty})} \otimes_{\mathcal{O}} k)$$

is an *isomorphism* (since push-forwards for finite morphisms always commute with base-change). But *this morphism clearly factors through* $\mathcal{R}_d$ (since sections of

$$(\overline{\mathcal{L}} \otimes_{\mathcal{O}_{E_{\infty,S}}} \mathcal{R}^{\mathrm{et}}_{E^{\dagger}_{\infty,[d]}}\{\mathbf{v}\}) \otimes_{\mathcal{O}} k$$

over $E_{\infty,S} \otimes_{\mathcal{O}} k$ can always be restricted to $(-\eta^{\dagger}_{\delta} + {}_d E^{\dagger}_{\infty}) \otimes_{\mathcal{O}} k$). In other words, the morphism $\mathcal{D}_d \to \mathcal{R}_d$ is *bijective*, so $\mathcal{I}_d = \mathcal{R}_d$, as desired. This completes the proof of Theorem 4.1. ○

Remark. In fact, compatibility of base-change, i.e., Theorem 4.1, (1), may itself be regarded as a sort of "analytic torsion property." Thus, it is not surprising that the proof of Theorem 4.1, (1), makes essential use of Theorem 4.1, (3), in the "universal case."

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Chapter VII: The Geometry of Function Spaces: Systems of Orthogonal Functions

§0. Introduction

The purpose of the present Chapter is to prepare for the discussion of the Arakelov-theoretic extension of the theory of Chapter VI in Chapter VIII by developing the theory of *systems of orthogonal functions* in the form in which we will use it in this paper. Such systems of orthogonal functions are central to the archimedean portion of the theory of this paper. The reason for this is that the main result of this paper, i.e., the *comparison isomorphism* between "*de Rham functions*" (i.e., functions on the universal extension of an elliptic curve, which is a sort of de Rham cohomology group of the elliptic curve) and "*étale functions*" (i.e., functions on the torsion points of the elliptic curve) is, after all, an isomorphism between *function spaces*. To consider such function spaces in an Arakelov-theoretic context thus amounts to considering such function spaces equipped with some *metric*. Thus, in order to develop the natural Arakelov-theoretic extension of the theory of Chapter VI, we must study the relationship between the natural metrics induced from the de Rham and étale sides of the comparison isomorphism. Because function spaces tend to be rather large and unwieldy, in order to study their *geometry as metrized spaces*, it is of crucial importance to have natural and explicit *orthonormal coordinate systems* at our disposal which allow us to express the difference between the geometries induced by different metrics in a precise and quantitative fashion. Such orthonormal coordinate systems are provided by systems of orthogonal functions.

It turns out that the natural metrics arising from the de Rham and étale sides of the comparison isomorphism are rather alien in nature to one another. It is thus difficult to compare them to one another directly. Thus, in order to *interpolate* the difference between them, we must introduce various *intermediate systems of orthogonal functions*. This is the main goal of the present Chapter. In §1, we begin by discussing the "general nonsense" aspects of the theory of orthogonal functions. In §2, we review the theory of *Legendre and Hermite polynomials*, not only because they are perhaps the most fundamental examples of orthogonal functions, but also because they play a crucial role in the theory of the comparison isomorphism at the infinite prime. In §3, we discuss the theory of the "*discrete Tchebycheff polynomials*," which are a sort of discrete analogue of Legendre polynomials. These polynomials also play an important role in the theory of this paper. In §4, we discuss the *metric arising from the de Rham side of the comparison isomorphism*. Then in §5, we discuss the *relationship between this geometry and the theory of canonical Schottky-Weierstrass zeta functions*. Finally, in §6, we begin our study of the geometry of the space of canonical Schottky-Weierstrass zeta functions. The geometry of this function space will be studied further in Chapter VIII.

In fact, the introduction of various types of canonical Schottky-Weierstrass zeta functions is the first and most crucial step in the *interpolation between de Rham and étale metrics* referred to above. One way to think of this interpolation is the following: The de Rham side of the comparison isomorphism involves functions on *continuous spaces*, i.e., more precisely, on the underlying real analytic manifold – which we denote $E_{\mathbf{R}}$ – of a complex elliptic curve E. On the other hand, the étale side of the comparison isomorphism involves functions on some *finite, discrete set* of torsion points. Thus, in order to develop an Arakelov-theoretic extension of the theory of Chapter VI, we must somehow bridge this gap between continuous and finite, discrete spaces. Now note that $E_{\mathbf{R}}$ has precisely *two real dimensions*. The passage (§5) from the de Rham-theoretic function space geometry of §4 to the canonical Schottky-Weierstrass zeta function-based geometry of §6 – which is a sort of function theory on $\mathbf{S^1}$ (which has precisely *one* real dimension) – then amounts to the discretization of one of these two real dimensions. This discretization, which we refer to as the *first discretization*, turns out to be the most essential. The *second discretization*, i.e., the discretization of the remaining $\mathbf{S^1}$, will be carried out in Chapter VIII. Thus, schematically:

$$\textit{de Rham-based function theory on } E_{\mathbf{R}} \ (\dim/\mathbf{R} = 2)$$

$$\Downarrow$$

$$\textit{canonical Schottky-Weierstrass zeta function-based function theory on } \mathbf{S^1} \ (\dim/\mathbf{R} = 1)$$

$$\Downarrow$$

$$\textit{function theory of functions on a discrete set of torsion points } (\dim/\mathbf{R} = 0)$$

In some sense, this diagram represents the main theme behind the theory of the present and following Chapters.

§1. The Orthogonalization Problem

In this §, we introduce the basic ideas and terminology surrounding the theory of systems of orthogonal functions. We begin by discussing what we call the *orthogonalization problem*. That is to say, given a *Hilbert space equipped with a filtration and an operator* satisfying certain properties, one wishes to construct a corresponding *system of orthogonal elements* of the Hilbert space. We then discuss various basic invariants that one can associate to such data – namely, the *means* and *submeans* of the system (cf. Definition 1.1). When the given operator is *self-adjoint up to a constant multiple*, it turns out that the system of orthogonal elements is completely determined by a relatively small amount of data, namely, the means and "principal submeans" (cf. Proposition 1.2). Finally, we discuss various examples of this theory. We remark that the material that we treat here is in essence well-known (cf., e.g., [Sze], [Bate]), but I do not know of a reference that treats this material from the point of view that we take here.

We begin with a *Hilbert space* $\mathcal{H}$ over a base field K, which we assume to be either $\mathbf{R}$ or $\mathbf{C}$. We will denote the inner product on $\mathcal{H}$ by $(-,-)$ and its corresponding norm by $|-|$. Let us assume that $\mathcal{H}$ is equipped with a *filtration* of (not necessarily closed) K-subspaces:

$$F^{-1}(\mathcal{H}) = 0 \subseteq F^0(\mathcal{H}) \subseteq F^1(\mathcal{H}) \subseteq \ldots \subseteq F^n(\mathcal{H}) \subseteq \ldots \subseteq \mathcal{H}$$

where n ranges over all integers $<$ some fixed N (where $N \in \mathbf{Z} \bigcup \infty$). Moreover, we assume that the subquotients $Q^n(\mathcal{H}) \stackrel{\text{def}}{=} (F^n/F^{n-1})(\mathcal{H})$ satisfy:

$$\dim_K(Q^n(\mathcal{H})) = 1$$

for all $0 \leq n < N$. Thus, in particular, if $0 \leq n < N$, $\dim_K(F^n(\mathcal{H})) = n+1$. In the following, we will write

$$F^\infty(\mathcal{H}) \stackrel{\text{def}}{=} \bigcup_{i=0}^{N-1} F^i(\mathcal{H}); \quad F^{\infty-1}(\mathcal{H}) \stackrel{\text{def}}{=} \bigcup_{i=0}^{N-2} F^i(\mathcal{H})$$

for the union of all the (respectively, all but the last) subspace(s) in the filtration.

Note that this data of Hilbert space plus filtration already determines *splittings of the filtration*

$$F^n(\mathcal{H}) = \bigoplus_{i=0}^{n} Q^i(\mathcal{H})$$

(for $0 \leq n < N$) given by taking the orthogonal complement of $F^{n-1}(\mathcal{H})$ in $F^n(\mathcal{H})$. Thus, one can think of the $Q^n(\mathcal{H})$ as *subspaces* of $\mathcal{H}$. In its most basic form,

> The *orthogonalization problem* is the problem of *explicitly determining* these splittings of the filtration $F^*(\mathcal{H})$.

What does to mean to "explicitly determine" the splittings? To give meaning to this expression, it is typical to assume that one is also given a (K-linear, but not necessarily continuous) *operator*

$$X : F^{\infty-1}(\mathcal{H}) \to F^\infty(\mathcal{H})$$

Moreover, we assume that, for $0 \leq n < N$, $X(F^{n-1}(\mathcal{H})) \subseteq F^n(\mathcal{H})$, and, moreover, that the induced morphism

$$Q^{n-1}(\mathcal{H}) \to Q^n(\mathcal{H})$$

is an *isomorphism*. Put another way, if $F^0(\mathcal{H}) = K \cdot \varpi$, then

$$F^n(\mathcal{H}) = \{\phi(X) \cdot \varpi\}$$

where $\phi(X)$ ranges over all polynomials in X with coefficients in K of degree $\leq n$. Thus, if one is given such an operator X, then it makes sense to ask:

> Can one explicitly determine some *set of polynomials* $\{\phi_n(X)\}$ (where $\deg(\phi_n(X)) = n$; n ranges over all nonnegative integers) such that $\phi_n(X) \cdot \varpi$ generates the one-dimensional K-subspace $Q_n(\mathcal{H}) \subseteq \mathcal{H}$?

This is the form of the orthogonalization problem that will appear most frequently in the present Chapter. A mild variant of this problem involves finding $\phi_n(X)$ as above such that $\phi_n(X) \cdot \varpi$ not only generates $Q_n(\mathcal{H})$, but has *norm equal to* 1.

Let us assume that the set $\{\phi_n(X) \cdot \varpi\}$ is *orthonormal*. Then we shall denote the coefficient of X^{n-i} in $\phi_n(X)$ by $\kappa_{n,i} \in K$ (cf. [Sze], Theorem 12.7.1). Thus,

$$\phi_n(X) = \kappa_{n,0} \cdot X^n + \kappa_{n,1} \cdot X^{n-1} + \ldots + \kappa_{n,n}$$

Note (cf. [Sze], §12.3) that the number $\mu_n \stackrel{\text{def}}{=} |\kappa_{n,0}|^{-2}$ may also be characterized as the *infimum*

$$\inf_{\phi(X)} \ |\phi(X) \cdot \varpi|^2$$

where $\phi(X) = X^n + \ldots$ is a polynomial in X (with coefficients in K) of degree n with leading coefficient equal to 1. Note that this infimum is attained for $\phi(X) \stackrel{\text{def}}{=} \kappa_{n,0}^{-1} \cdot \phi_n(X)$.

Definition 1.1. We shall refer to the number $\mu_n = |\kappa_{n,0}|^{-2}$ as the *n-th mean* of the data $(\mathcal{H}, F^*(\mathcal{H}), X)$. We shall refer to the number $r_{n,i} \stackrel{\text{def}}{=} \kappa_{n,i}/\kappa_{n,0}$ (for $0 \leq i \leq n < N$) as the *((n,i)-)submean* of the data $(\mathcal{H}, F^*(\mathcal{H}), X)$. The $(n,1)$-submean will be referred to as the *n-th principal submean.*

Note that the means and submeans are uniquely determined by the data $(\mathcal{H}, F^*(\mathcal{H}), X)$. Moreover, if one knows the means and submeans of this data, then one can immediately reconstruct the system $\{\phi_n(X)\}$, up to multiplication (of each $\phi_n(X)$) by an element of K of absolute value 1.

The reason for the terminology "*means*" is the following: In a certain very basic and representative case (cf. Example (3) below; [Sze], §12.3) involving polynomials which are orthogonal with respect to a given weight function on the circle $\mathbf{S^1}$, the zeroth mean μ_0 is, in fact, the *arithmetic mean* of the weight function (i.e., in the notation of Example (3) below, the quantity $\frac{1}{2\pi} \cdot \int_0^{2\pi} \ w(\theta) \ d\theta$). Moreover, in this case, the higher means satisfy $\mu_n \leq \mu_{n-1}$ (for all positive integers n) and their limit $\mu_\infty \stackrel{\text{def}}{=} \lim_{n\to\infty} \ \mu_n$ is the *geometric mean* of the weight function (i.e., in the notation of Example (3) below, the quantity $\exp(\frac{1}{2\pi} \cdot \int_0^{2\pi} \ \log(w(\theta)) \ d\theta)$).

One of the most important cases of the orthogonalization problem is the case where the operator X is *self-adjoint up to a constant multiple*, i.e., there exists some fixed constant $\alpha_X \in K$ such that $(X \cdot v, w) = \alpha_X \cdot (v, X \cdot w)$, for all $v, w \in F^\infty(\mathcal{H})$. In this case, the polynomials $\{\phi_n(X)\}$ satisfy a *recurrence relation*:

Proposition 1.2. *Assume that the operator X is* self-adjoint up to a constant multiple *$\alpha_X \in K$. Then for $1 \leq n < N-1$, we have*

$$\phi_{n+1}(X) = (A_n X + B_n) \cdot \phi_n(X) - C_n \cdot \phi_{n-1}(X)$$

where $A_n = \kappa_{n+1,0}/\kappa_{n,0}$; $B_n = A_n \cdot (r_{n+1,1} - r_{n,1})$; $C_n = \alpha_X \cdot A_n/\overline{A}_{n-1}$ (where the bar denotes complex conjugation). That is to say, the polynomials $\phi_n(X)$ are entirely determined by the means and principal submeans.

Proof. The proof is essentially given in [Bate], pp. 158-9. In fact, [Bate] only treats the case where $\mathcal{H}$ is some space of real-valued functions on an interval $(a, b) \subseteq \mathbf{R}$ equipped with the norm $(f, g) \stackrel{\text{def}}{=} \int_a^b \ f \cdot g \cdot w$ arising from some weight function w on (a, b), and X is the operator given by multiplying by the standard coordinate on $\mathbf{R}$. In fact, however, it is not difficult to see that the only properties of this data that are used in this proof are the properties that we have assumed here. For instance, to see that $(X \cdot \phi_n(X) \cdot \varpi, \phi(X) \cdot \varpi) = 0$ for all $\phi(X)$ of degree $\leq n-2$, it suffices to apply the *self-adjointness* (up to a constant multiple) of X, which implies that

$$(\phi_n(X) \cdot \varpi, X \cdot \phi(X) \cdot \varpi) = 0$$

since $X \cdot \phi(X)$ is of degree $\leq n-1$. The coefficients A_n, B_n, and C_n may then be determined in the obvious fashion – cf. [Bate], pp. 158-9. ○

We conclude this § by listing some *examples* of the theory discussed so far. Some examples are well-known; others are to be treated in the present and following Chapters.

Examples of Orthogonal Systems:

(1) *Polynomials on intervals of the real line:* Let $(a, b) \subseteq \mathbf{R}$ be an open interval in $\mathbf{R}$ (where $a, b \in \mathbf{R} \bigcup \{\pm\infty\}$). Write x for the standard coordinate on $\mathbf{R}$. Suppose that $w(x)$ is positive integrable function on (a, b). Then we may form the *Hilbert space* $L^2_w(a, b)$ of K-valued functions f on (a, b) which satisfy $\int_a^b |f|^2 \cdot w \cdot dx < \infty$. The inner product of this Hilbert space is given by

$$(f, g) \stackrel{\text{def}}{=} \int_a^b f \cdot \overline{g} \cdot w \cdot dx$$

(where $\overline{g}$ denotes the complex conjugate of g). Let us assume that for all $n \geq 0$, $x^n \in L^2_w(a, b)$. Then if we take

$$\mathcal{H} \stackrel{\text{def}}{=} L^2_w(a, b); \quad F^n(\mathcal{H}) \stackrel{\text{def}}{=} \{c_n x^n + \ldots + c_1 x + c_0 \mid c_0, \ldots, c_n \in K\}; \quad X \stackrel{\text{def}}{=} x\cdot$$

(i.e., X is the *self-adjoint* operator given by multiplication by x), we obtain data as in the above discussion. The resulting $\phi_n(x) \stackrel{\text{def}}{=} \phi_n(X) \cdot 1$'s are thus orthonormal with respect to the *weight function* $w(x)$. This special case of the above discussion is the most fundamental. Two of the most basic examples of this sort of orthogonal system are the *Legendre polynomials* (where $a = -1, b = 1, w(x) = 1$) and the *Hermite polynomials* (where $a = -\infty, b = \infty, w(x) = e^{-\frac{1}{2}x^2}$). These two cases will be discussed in §2 below.

(2) *Polynomials on a finite set of points:* This case is similar to (1). Instead of working with an interval in $\mathbf{R}$, however, we work with the finite set of points $\{0, \ldots, d-1\} \subseteq \mathbf{R}$. We then consider a positive *weight function* $w(x)$ on $\{0, \ldots, d-1\}$. The *Hilbert space* $\mathcal{H}$ is taken to be the set of K-valued functions f on $\{0, \ldots, d-1\}$ with the inner product given by

$$(f, g) \stackrel{\text{def}}{=} \sum_{x=0}^{d-1} f(x) \cdot \overline{g}(x) \cdot w(x)$$

Here, we take $N = d$, and for $0 \leq n < N$, we let $F^n(\mathcal{H}) \subseteq \mathcal{H}$ be the subspace of polynomials in x of degree $\leq n$; and X the *self-adjoint* operator given by multiplication by x. Just as was the case with (1), the present set-up (2) also has a long history, dating back to the nineteenth century. The most basic example of this sort of orthogonal system is the system of *discrete Tchebycheff polynomials* (where $w(x) = 1$), to be reviewed in §3 below (cf. also [Sze], §2.8; [Bate], p. 223). As one might expect from the weight functions, the discrete Tchebycheff polynomials are a sort of discrete analogue of the Legendre polynomials. Another basic example of (2) is the case of *Krawtchouk polynomials*, which are a sort of discrete analogue of the Hermite polynomials (cf. [Sze], §2.82).

(3) *Polynomials on the circle* $\mathbf{S^1}$*:* Another important case is the case of a positive weight function $w(\theta)$ on the unit circle $\mathbf{S^1}$. We let $\mathcal{H} \stackrel{\text{def}}{=} L^2_w(\mathbf{S^1})$ be the Hilbert space of complex square integrable functions with respect to w on $\mathbf{S^1}$; $F^n(\mathcal{H}) \stackrel{\text{def}}{=} \{c_n z^n + \ldots + c_1 z + c_0 \mid c_0, \ldots, c_n \in \mathbf{C}\}$, where $n \geq 0$, and $z \stackrel{\text{def}}{=} e^{i\theta}$; and X the operator given by multiplication by z. There is a rich and well-developed concerning the resulting orthogonal system $\phi_n(z)$ – see, e.g., [Sze], Chapters XI, XII – which is in some senses more transparent then the theory of Example (1). For instance, this theory gives rise to *asymptotic formulas* for the $\phi_n(z)$ in terms of $w(\theta)$ (cf. [Sze], Chapter XII). Historically, one of the first cases in which this situation was investigated in detail was the case in which $w(\theta)$ is given by a *theta function on* $\mathbf{S^1}$ (cf. [Sze2]). This case is particularly interesting in that it foreshadowed the subsequent development of *q-analogues of the classical orthogonal polynomials* (such as those of Legendre and Hermite).

(4) *Derivatives of a weight function on the circle* $\mathbf{S^1}$*:* In this case, we let $\mathcal{H} \stackrel{\text{def}}{=} L^2(\mathbf{S^1})$ be the usual L^2-space of square-integrable complex functions on $\mathbf{S^1}$. Moreover, we assume that we are given a function $w(\theta)$ on $\mathbf{S^1}$ which is *infinitely differentiable* (i.e., of class C^∞). Let $X \stackrel{\text{def}}{=} \frac{\partial}{\partial\theta}$ be the usual differentiation operator, and let $F^n(\mathcal{H}) \stackrel{\text{def}}{=} \{c_n X^n + \ldots + c_1 X + c_0 \mid c_0, \ldots, c_n \in \mathbf{C}\} \cdot w(\theta)$, for $n \geq 0$. Note that the adjoint of X is $-X$, so Proposition 1.2 applies. Thus, the $\phi_n(X)$ are *entirely determined by the means and principal submeans.* In the present paper, the case where $w(\theta)$ is a *theta function* will play an important role (cf. §6, Definition 6.3; Chapter VIII).

(5) *Sections of a line bundle on an elliptic curve:* Let E be an *elliptic curve* over $\mathbf{C}$, and $\mathcal{L}$ be the line bundle defined by the origin. Then $\mathcal{L}$ admits a Hermitian metric (unique up to constant multiple) $|| \sim ||_{\mathcal{L}}$ whose curvature is translation invariant (cf. §4 for more details). This Hermitian metric allows us to define the L^2-space of square-integrable sections s of $\mathcal{H}$ satisfying $\int_E \ ||s||^2_{\mathcal{L}} < \infty$. We take $\mathcal{H}$ to be this L^2-space, and (for $n \geq 0$) $F^n(\mathcal{H})$ to be the space of sections annihilated by $\overline{\partial}^{n+1}$. It may be shown that $\dim_{\mathbf{C}}(F^n(\mathcal{H})) = n + 1$. Now *one wishes to compute the splittings of this filtration defined by the inner product on* $\mathcal{H}$. This is done in §4 by using the adjoint $\overline{\partial}^*$ of $\overline{\partial}$ (with respect to the inner product of $\mathcal{H}$). In particular, it is shown that if s is a generator of $F^0(\mathcal{H})$, then $Q^n(\mathcal{H}) \subseteq \mathcal{H}$ is generated by $(\overline{\partial}^*)^n(s)$ (cf. Theorem 4.5 of §4 below). Thus, if we take X to be $\overline{\partial}^*$, then we see that in this case, $\phi_n(X)$ *is a constant multiple of* X^n. This example will also play a key role in the present paper and will be treated in detail in §4.

(6) *The discrete version of (5):* Let E and $\mathcal{L}$ be as in (5), and write $\mathcal{H}_{\mathbf{R}}$ for the Hilbert space "$\mathcal{H}$" of (5). Fix a positive integer N, and let $G \subseteq E$ be a cyclic subgroup of order N. Let $\eta \in E$ be a torsion point which is $\notin G$. Then it follows from Chapter VI, Theorem 3.1 (2), together with the theory of §4 of the present Chapter, that the restriction map

$$\Xi_{\mathbf{R}} : F^{N-1}(\mathcal{H}_{\mathbf{R}}) \to \mathcal{L}|_{\eta+G}$$

is *bijective*. Note that $|| \sim ||_{\mathcal{L}}$ defines a natural inner product on the N-dimensional $\mathbf{C}$-vector space $\mathcal{L}|_{\eta+G}$. Thus, $\mathcal{L}|_{\eta+G}$ is a finite-dimensional Hilbert space $\mathcal{H}$. Let $F^n(\mathcal{H}) \stackrel{\text{def}}{=} \Xi_{\mathbf{R}}(\mathrm{F}^n(\mathcal{H}_{\mathbf{R}})) \cong F^n(\mathcal{H}_{\mathbf{R}})$ for $0 \leq n < N$, and write $X : F^{\infty-1}(\mathcal{H}) \to F^{\infty}(\mathcal{H})$ for the operator induced by the "X" of (5) (via the isomorphisms $F^n(\mathcal{H}) \cong F^n(\mathcal{H}_{\mathbf{R}})$). Then one wishes to compute the resulting $\phi_n(X)$. Indeed,

> *The computation of the $\phi_n(X)$ in this case is the ultimate goal of the theory of the present and following Chapters. In fact, it is essentially a tautology that concretely speaking, the comparison of the metrized function spaces of "de Rham functions" and "étale functions" (cf. §0) amounts to estimating the coefficients of the $\phi_n(X)$.*

Since this ultimate goal of comparing Examples (5) and (6) is too difficult to achieve directly, we relate Examples (5) and (6) by using a certain intermediate orthogonal system which is a special case of Example (4) (i.e., the case where "$w(\theta)$" is a theta function). Finally, we analyze this special case of Example (4) (cf. §6; Chapter VIII) by using the well-known theories of Examples (1) and (2).

§2. Review of Legendre and Hermite Polynomials

In this §, we review the classical orthogonal polynomials of *Legendre* and *Hermite*. These examples are important not only because they serve to illustrate the general theory of orthogonal systems of functions, but also because they appear naturally in the theory of the *Hodge-Arakelov comparison isomorphism* (Chapter VIII) as *limiting cases of the orthogonal functions arising from differential calculus on the "theta-weighted circle"*.

We begin with the polynomials of *Legendre*:

Proposition 2.1. (Legendre Polynomials) *For every integer $n \geq 0$, we define:*

$$P_n(T) \stackrel{\text{def}}{=} \frac{1}{2^n} \cdot \sum_{j=0}^{[\frac{n}{2}]} (-1)^j \binom{n}{j} \binom{2n-2j}{n} \cdot T^{n-2j}$$

(where $[\sim]$ denotes the greatest integer less than the real number in brackets). Thus, $P_n(T)$ is a polynomial of degree n whose leading term is given by $2^{-n} \cdot \binom{2n}{n} \cdot T^n$ and which satisfies $P_n(1) = 1$. In particular,

$$P_0(T) = 1; \quad P_1(T) = T$$

Moreover, the system of polynomials $\{P_n(T)\}$ satisfies the following properties:

(1) **(Orthogonality and Norms)** $\int_{[-1,1]} P_m(T) \cdot P_n(T) = \frac{2}{2n+1} \cdot \delta_{mn}$.

(2) **(Recurrence Formula)**

$$(n+1) \cdot P_{n+1}(T) - (2n+1)T \cdot P_N(T) + n \cdot P_{n-1}(T) = 0$$

for $n \geq 1$.

(3) **(Differential Equation)**

$$(1 - T^2)P_n''(T) - 2T \cdot P_n'(T) + n(n+1) \cdot P_n(T) = 0.$$

(4) **(Rodrigues' Formula)** $P_n(T) = \frac{1}{2^n \cdot n!} \cdot (\frac{d}{dT})^n \{(T^2 - 1)^n\}$.

Proof. See, e.g., [Rice], pp. 47-48. Note that up to constant multiples, the $P_n(T)$ are determined uniquely by the following two properties: (i) $P_n(T)$ is of degree precisely n; (ii) if $m \neq n$, then $t_m(T)$ and $t_n(T)$ are orthogonal with respect to $L^2([-1,1])$. ○

Next, we consider the polynomials of *Hermite*:

Proposition 2.2. **(Hermite Polynomials)** *Let* X *be the standard coordinate on* $\mathbf{R}$. *Write* $D \stackrel{\text{def}}{=} \frac{d}{dX}$. *For* $n \geq 0$, *define*

$$H_n(X) \stackrel{\text{def}}{=} e^{\frac{1}{2}X^2} \cdot D^n(e^{-\frac{1}{2}X^2})$$

Thus, $H_0(X) = 1$; $H_1(X) = -X$. *Let* $d\alpha \stackrel{\text{def}}{=} e^{-\frac{1}{2}X^2} \cdot dX$, *which we think of as a metric/measure on* $\mathbf{R}$. *Then for* $n \geq 0$,

(1) **(Orthogonality and Norms)**

$$\int_{-\infty}^{+\infty} H_n(X) \cdot H_m(X) \cdot d\alpha = (2\pi)^{\frac{1}{2}} \cdot n! \cdot \delta_{mn}.$$

(2) **(Explicit Formula)**

$$H_n(X) = (-1)^n \cdot n! \cdot \sum_{m=0}^{[n/2]} \frac{(-2)^{-m} \cdot X^{n-2m}}{m! \ (n-2m)!}$$

where $[n/2]$ is $n/2$ (respectively, $(n-1)/2$) if n is even (respectively, odd).

(3) **(Alternate Definition)** $\quad H_n(X) = (D - X)^n(1)$.

(4) **(Recurrence Formula)**

$$H_{n+1}(X) + X \cdot H_n(X) = D(H_n(X)) = -n \cdot H_{n-1}(X).$$

(5) **(Differential Relations)** *If we set* $\Phi_n(X) \stackrel{\text{def}}{=} H_n(X) \cdot e^{-\frac{1}{4}X^2}$, *then*

$$(D - \frac{1}{2}X) \cdot \Phi_n(X) = \Phi_{n+1}(X); \quad (D + \frac{1}{2}X) \cdot \Phi_n(X) = -n \cdot \Phi_{n-1}(X);$$

$$(D^2 - \frac{1}{4} \cdot X^2 - \frac{1}{2}) \cdot \Phi_n(X) = -(n+1) \cdot \Phi_n(X).$$

Proof. The properties here all follow essentially from [Bate], p. 193. Since, however, we use different normalizations here from those of [Bate], we sketch the proofs here. First, define the operator:

$$D^{\exp} \stackrel{\text{def}}{=} e^{\frac{1}{2}X^2} \cdot D \cdot e^{-\frac{1}{2}X^2}$$

Then by definition, $H_n(X) = (D^{\exp})^n(1)$. But one computes easily that

$$\begin{aligned} D^{\exp} &= e^{\frac{1}{2}X^2} \cdot (e^{-\frac{1}{2}X^2} \cdot D + [D, e^{-\frac{1}{2}X^2}]) \\ &= D + e^{\frac{1}{2}X^2} \cdot D(e^{-\frac{1}{2}X^2}) \\ &= D - X \end{aligned}$$

This proves the "Alternate Definition." The "Recurrence Formula" follows by applying the operator-theoretic relation

$$\begin{aligned} D \cdot (D - X)^n &= (D - X)^n \cdot D + [D, (D - X)^n] \\ &= (D - X)^n \cdot D + n \cdot (D - X)^{n-1} \cdot [D, D - X] \\ &= (D - X)^n \cdot D - n \cdot (D - X)^{n-1} \end{aligned}$$

to the constant function 1. The "Differential Relations" follow by using $(D - X)(H_n) = H_{n+1}$; $D(H_n) = -n \cdot H_{n-1}$; and the operator-theoretic relation $(D^2 - \frac{1}{4}X^2) - \frac{1}{2} = D^2 - \frac{1}{4}X^2 - \frac{1}{2}[D, X] = (D + \frac{1}{2}X)(D - \frac{1}{2}X)$. This operator

$$(D + \frac{1}{2}X)(D - \frac{1}{2}X)$$

is a sort of *Laplacian operator*, and is self-adjoint for the L^2-norm on $\mathbf{R}$ equipped with the usual measure (i.e., dX). Thus, the fact that the $\Phi_n(X)$'s are *eigenfunctions for this operator* implies that they are *mutually orthogonal.* The integrals

$$\int_{-\infty}^{+\infty} \Phi_n^2 \; dX$$

may then be computed by using the fact that $D + \frac{1}{2}X$ (respectively, $D - \frac{1}{2}X$) is adjoint to $-(D - \frac{1}{2}X)$ (respectively, $-(D + \frac{1}{2}X)$). Finally, the "Explicit Formula" may be proven by applying induction on n and the relation $H_{n+1}(X) = (D - X) \cdot H_n(X)$. ○

Remark 1. In fact, the explicit formula given above (i.e., Proposition 2.2, (2)) may be interpreted as a formula for any two operators A and B satisfying

$$[A, B] = c \in \mathbf{C}$$

and $A \cdot v = 0$, for some vector v. That is to say, the explicit formula implies that

$$(A + B)^n \cdot v = n! \cdot \sum_{m=0}^{[n/2]} \frac{(c/2)^m \cdot B^{n-2m}}{m! \; (n - 2m)!} \cdot v$$

Indeed, this follows by thinking of A as $\lambda^{-1} \cdot D$ and of B as $-\lambda^{-1} \cdot X$, where $\lambda^{-2} = -c$.

Remark 2. Thus, the theory of Hermite polynomials essentially amounts to *harmonic analysis with respect to the metric*

$$d\alpha \stackrel{\mathrm{def}}{=} e^{-\frac{1}{2}X^2} \cdot dX$$

on $\mathbf{R}$. This is interesting relative to the main theme of this work, i.e., of a *"comparison isomorphism via theta functions,"* since this sort of factor $e^{-\frac{1}{2}X^2}$ (i.e., an exponential with a quadratic exponent) appears frequently in the theory of theta functions (cf., e.g., [Mumf3], §12, proof of Lemma 2, (II)). Indeed, from this point of view, one may think of this factor as being essentially:

$$\exp\text{ (Chern class of } \mathcal{L}) \;\; = \;\; \exp\text{ (Riemann form of } \mathcal{L})$$

Note that this sort of factor also appears in the *adjustment of integral structure at infinity* in Chapter VI, Theorem 3.1, (3), i.e., this adjustment of integral structure varies roughly as the *exponential* of the "c_j's" of *loc. cit.*, which (cf. Chapter V, Schola 4.1) are *quadratic* functions of j. This appearance of the exponential of a quadratic function in all of these places is by no means a coincidence, as will become increasingly apparent to the reader as he/she studies the material of this and the following Chapters.

§3. Discrete Tchebycheff Polynomials and the Fundamental Combinatorial Model

In this §, we discuss the theory of *discrete Tchebycheff polynomials*. This theory may be regarded as the *archimedean portion* of the *fundamental combinatorial model* underlying the theory of this paper. This combinatorial model is the *evaluation map*

$$\Xi^{\mathrm{cb}} : \mathbf{Z}[T]^{<d} \longrightarrow \bigoplus_{i=0}^{d-1} \mathbf{Z}$$

$$T \mapsto (0, 1, 2, \ldots, d-1)$$

given by evaluating polynomials in the indeterminate T of degree $< d$ (for d a positive integer) with integral coefficients at the points $T = 0, 1, 2, \ldots, d-1$. In Chapter V (cf., especially, Chapter V, §6), we discussed the evaluation map Ξ^{cb} at *finite primes*. In particular, we saw that if we adjust the integral structure in an appropriate fashion (cf. Chapter III, §6; Chapter V, §3) – i.e., if we consider the submodule of $\mathbf{Z}[T] \otimes_{\mathbf{Z}} \mathbf{Q}$ generated by the polynomials

$$T^{[n]} \stackrel{\mathrm{def}}{=} \frac{1}{n!} T(T-1)(T-2) \cdot \ldots \cdot (T-n+1)$$

– then the matrix corresponding to the evaluation map Ξ^{cb} takes on the simple upper triangular form

$$\{T^{[i]}|_{T=j}\}_{\{i,j=0,1,2,\ldots,d-1\}} = \begin{pmatrix} 1 & * & \ldots & * & * \\ 0 & 1 & \ldots & * & * \\ \ldots & & \ldots & & \ldots \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

which is manifestly both *integral and invertible* at all finite primes. Unfortunately, however, although the polynomials $T^{[n]}$ are very well-suited to analyzing the effect of Ξ^{cb} on the integral structures at finite primes, they are not well-suited to analyzing Ξ^{cb} at the infinite (archimedean) prime. At the infinite prime, the range of Ξ^{cb} has a natural "L^2-metric" given by:

$$||(a_0, \ldots, a_{d-1})|| = (a_0^2 + \ldots + a_{d-1}^2)^{\frac{1}{2}}$$

(where $a_0, \ldots, a_{d-1} \in \mathbf{R}$). In the present §, we would like to analyze Ξ^{cb} relative to this L^2-metric on the range, i.e., *we would like to discuss the metric induced on* $\mathbf{Z}[T]^{<d}$ *via* Ξ^{cb} *by this* L^2*-metric.*

In the theory of Ξ^{cb} at archimedean primes, the polynomials that play the role of the $T^{[n]}$ are the *discrete Tchebycheff polynomials* discussed in [Bate], [Sze]. We summarize their definition and basic properties in the following Proposition:

Proposition 3.1. (Discrete Tchebycheff Polynomials) *Fix a* positive integer d. *Then we define, for* $n = 0, 1, 2, \ldots, d-1$,

(1) **("Rodrigues' Formula")** $\quad t_n(T) \stackrel{\mathrm{def}}{=} n! \cdot \delta^n \left[\binom{T}{n} \binom{T-d}{n} \right]$

where, as usual, if $f(T)$ *is a polynomial in* T*, then* $\delta(f)(T) \stackrel{\mathrm{def}}{=} f(T+1) - f(T)$*. (When it is necessary to specify the dependence on* d*, we write* $t_{n,d}(T)$ *for* $t_n(T)$*.) Thus, for instance,*

$$t_0(T) = 1; \quad t_1(T) = 2T - (d-1)$$

and (for all $n = 0, 1, 2, \ldots, d-1$*) the leading term of* $t_n(T)$ *is* $\binom{2n}{n} \cdot T^n$*. The polynomials* $t_n(T)$ *form an* orthogonal *system of polynomials for the* L^2*-norm (as defined above) on the space of functions on the set* $\{0, 1, 2, \ldots, d-1\}$*:*

(2) **(Orthogonality and Norms)**

$$\langle\ t_n(T), t_m(T)\ \rangle \stackrel{\mathrm{def}}{=} \sum_{T=0}^{d-1} t_n(T) \cdot t_m(T) = (2n+1)^{-1} \cdot d(d^2 - 1^2)(d^2 - 2^2) \ldots (d^2 - n^2) \cdot \delta_{mn}$$

for $m, n = 0, 1, 2, \ldots, d-1$ *(where* δ_{mn} *is* $= 0$ *if* $m \neq n$ *and* $= 1$ *if* $m = n$*).*

Moreover, these polynomials also satisfy the following properties:

(3) **(Recurrence Formula)**

$$(n+1) \cdot t_{n+1}(T) - (2n+1)(2T - d + 1) \cdot t_n(T) + n(d^2 - n^2) t_{n-1}(T) = 0$$

for $n = 1, 2, \ldots, d-1$.

(4) **(Difference Equation)**

$$(T+2)(T-d+2)\ \delta^2(t_n(T)) + [2T - d + 3 - n(n+1)]\ \delta(t_n(T)) - n(n+1)\ t_n(T) = 0$$

for $n = 0, 1, 2, \ldots, d-1$.

(5) **(Connection with Legendre Polynomials)**

$$\lim_{d \to \infty} d^{-n}\ t_n(d \cdot T) = P_n(2T - 1)$$

for $n = 0, 1, 2, \ldots, d-1$. *Here* $P_n(T)$ *is the* n*-th* Legendre polynomial *(cf. Proposition 2.1).*

(6) **(Connection with usual Monomials)** *For any real number* $\epsilon > 0$*,*

$$\lim_{d \to \infty} d^{-n(1+\epsilon)}\ t_n(d^{1+\epsilon} \cdot T) = \binom{2n}{n} \cdot T^n$$

for $n = 0, 1, 2, \ldots, d-1$.

Proof. See, e.g., [Bate], p. 223. The symbol T (respectively, $\delta(-)$; d) in our notation corresponds to x (respectively, Δ; N) in the notation of *loc. cit.* Note that up to constant multiples, the $t_n(T)$ are determined uniquely by the following two properties: (i) $t_n(T)$ is of degree precisely n; (ii) if $m \neq n$, then $t_m(T)$ and $t_n(T)$ are orthogonal with respect to $L^2(\{0, 1, \ldots, d-1\})$. The only result listed in Proposition 3.1 that cannot be found in [Bate] is the connection with "the usual monomials T^n." But this follows immediately from Rodrigues' formula, which, in the limit, takes on the form

$$\frac{1}{n!} \cdot (\frac{d}{dT})^n (T^{2n}) = \binom{2n}{n} \cdot T^n$$

as desired. ○

Remark 1. Note that the various properties of the Legendre polynomials listed in Proposition 2.1 may all be obtained by passing to the limit $d \to \infty$ from the corresponding properties listed in Proposition 3.1. Thus, the discrete Tchebycheff polynomials are discrete versions of the Legendre polynomials. The Legendre polynomials appear (by substituting "$\cos(\theta)$" for T) as eigenfunctions of the Laplacian operator on the sphere (equipped with its unique rotation invariant metric). Indeed, the differential equation which states that $P_n(\cos(\theta))$ is an eigenfunction of this Laplacian with eigenvalue $n(n+1)$ may be written entirely in terms of T and is, in fact, the differential equation that appears in Proposition 2.1. Thus, *the discrete Tchebycheff polynomials may be regarded as discrete versions of the eigenfunctions of the Laplacian on the sphere.*

Remark 2. Note that while the limit of the discrete Tchebycheff polynomials as the length of the interval under consideration remains fixed and equal to 1 is given by the Legendre polynomials (Proposition 3.1, (5)), *the limit of the discrete Tchebycheff polynomials as the length of the interval under consideration* $\rightarrow 0$ *is simply the system of "usual monomials"* T^n (up to constant multiples – cf. Proposition 3.1, (6)). That is to say, just as the Legendre polynomials are "metrically suited" to reflect the geometry of the unit interval, *the usual monomials are functions which are "metrically suited" to reflect the geometry of an infinitesimal neighborhood of a point.* It is for this reason that such functions appear naturally in the theory of Taylor series, i.e., the theory of expansions of functions in an infinitesimal neighborhood of a point.

The main result that we wish to prove in this § is the following (Proposition 3.2), concerning a *bound on the coefficients of the discrete Tchebycheff polynomials.* First, let us introduce some notation: Write $c^n_k[d]$ for the *coefficient of* T^k *in the polynomial* $d^{-n} \cdot t_{n,d}(d \cdot T)$ (cf. Proposition 2.1). Thus,

$$c^n_k[d] = \mathrm{Coeff}_{T^k}\left\{\frac{1}{d^n} \cdot t_n(d \cdot T)\right\} = \frac{1}{d^{n-k}} \cdot \mathrm{Coeff}_{T^k}\Big\{t_n(T)\Big\}$$

(where "$\mathrm{Coeff}_A(B)$" denotes "the coefficient of the monomial A in the polynomial B").

Proposition 3.2. *Let us write*

$$\widetilde{t}_n(T) \stackrel{\mathrm{def}}{=} \frac{d^{\frac{1}{2}}}{||\ t_n\ ||} \cdot t_n(d \cdot T)$$

(where "$||\sim||$" denotes the L^2-norm on $\{0, 1, \ldots d-1\}$). Then:

(i.) $c^n_k[d] \leq e^{2n}$;

(ii.) $1 \leq \frac{d^{n+\frac{1}{2}}}{||\ t_n\ ||} \leq (2n+1)^{\frac{1}{2}} \cdot e^{n+1} \leq 3 \cdot e^{2n}$;

$$\mathrm{Coeff}_{T^k}\Big\{\widetilde{t}_n(T)\Big\} \leq (2n+1)^{\frac{1}{2}} \cdot e^{3n+1} \leq 3 \cdot e^{4n};$$

(iii.) $\mathrm{Coeff}_{T^n}\Big\{\widetilde{t}_n(T)\Big\} \geq c^n_n[d] = \binom{2n}{n} \geq 1$;

for all integers d, n, k *such that* $0 \leq k \leq n \leq d-1$.

Proof. Let us first prove (i.). First, note that $d^0 \cdot t_0(d \cdot T) = 1$; $d^{-1} \cdot t_1(d \cdot T) = d^{-1}(2d \cdot T + 1 - d) = T - 1 + d^{-1}$, so the inequality of (i.) is satisfied for $n = 0, 1$. Now we apply the "Recurrence Formula" (cf. Proposition 3.1, (3)), and induction on n. This gives us:

$$\begin{aligned}
\left| \ \mathrm{Coeff}_{T^k}\left\{d^{-n-1} \cdot t_{n+1}(d \cdot T)\right\} \ \right| &= \left| \ \mathrm{Coeff}_{T^k}\left\{(\frac{2n+1}{n+1})(2d \cdot T - d + 1) \cdot d^{-n-1} \cdot t_n(d \cdot T) \right.\right. \\
&\qquad \left.\left. - (\frac{n}{n+1})(d^2 - n^2)d^{-n-1} \cdot t_{n-1}(d \cdot T)\right\} \ \right| \\
&\leq \left| \ \mathrm{Coeff}_{T^k}\left\{2(2T - 1 + d^{-1}) \cdot d^{-n} \cdot t_n(d \cdot T)\right\} \ \right| \\
&\qquad + \left| \ \mathrm{Coeff}_{T^k}\left\{d^{-(n-1)} \cdot t_{n-1}(d \cdot T)\right\} \ \right| \\
&\leq 4 \cdot \left| \ \mathrm{Coeff}_{T^{k-1}}\left\{d^{-n} \cdot t_n(d \cdot T)\right\} \ \right| \\
&\qquad + 2 \cdot \left| \ \mathrm{Coeff}_{T^k}\left\{d^{-n} \cdot t_n(d \cdot T)\right\} \ \right| \\
&\qquad + \left| \ \mathrm{Coeff}_{T^k}\left\{d^{-(n-1)} \cdot t_{n-1}(d \cdot T)\right\} \ \right| \\
&\leq 4 \cdot 7^n + 2 \cdot 7^n + 7^{n-1} \leq 7^n
\end{aligned}$$

Since $e^2 \geq 7$, this completes the proof of (i.).

To derive (ii.) from (i.), it suffices (by Proposition 3.1, (2)) to bound

$$1 \leq \frac{(2n+1)^{\frac{1}{2}} \cdot d^n}{(d^2 - 1^2)^{\frac{1}{2}} \cdot (d^2 - 2^2)^{\frac{1}{2}} \cdot \ldots \cdot (d^2 - n^2)^{\frac{1}{2}}} = (2n+1)^{\frac{1}{2}} \cdot d^n \cdot \left\{\frac{(d-n-1)! \cdot d}{(d+n)!}\right\}^{\frac{1}{2}}$$

By *Stirling's Formula* (reviewed below – cf. Lemma 3.5), it follows that this last expression may be bounded by

$$(2n+1)^{\frac{1}{2}} \cdot \left\{\frac{d^{2n+1} \cdot (d-n-1)^{d-n-1} \cdot e^{-(d-n-1)} \cdot (d-n-1)^{\frac{1}{2}} \cdot e}{(d+n)^{d+n} \cdot e^{-(d+n)} \cdot (d+n)^{\frac{1}{2}}}\right\}^{\frac{1}{2}} \leq (2n+1)^{\frac{1}{2}} \cdot e^{n+1} \leq 3 \cdot e^{2n}$$

as desired. This completes the proof of (ii.) (as well as the proof of the first inequality of (iii.)). That $c^n_n[d] = \binom{2n}{n} \geq 1$ follows immediately from Proposition 3.1, (1) (and the fact that the leading term of $\binom{T}{n}$, $\binom{T-d}{n}$ is $\frac{1}{n!}T^n$). ○

Before proceeding, we would also like to *bound the coefficients of the polynomials* $T^{[n]}$.

Proposition 3.3. *Let n be a positive integer. Then the coefficients of the polynomial*

$$T^{[n]} \stackrel{\text{def}}{=} \frac{1}{n!} \cdot T(T-1) \cdot \ldots \cdot (T-(n-1))$$

satisfy (for $k = 0, \ldots, n$)

$$\text{Coeff}_{T^k}(T^{[n]}) \leq \frac{e^{2n}}{n^k}$$

In particular, if $n \leq d$, then

$$\text{Coeff}_{(T/d)^k}(T^{[n]}) \leq e^{2n} \cdot \left(\frac{d}{n}\right)^k \leq e^{2n} \cdot \left(\frac{d}{n}\right)^n \leq e^{2n+d}$$

Proof. If one expands the polynomial $T(T-1) \cdot \ldots \cdot (T-(n-1))$, one sees that there are $\leq \binom{n}{k}$ terms of degree k, each of which has a coefficient whose absolute value is a product of $n-k$ positive integers $\leq n$. It thus follows that

$$\begin{aligned} \text{Coeff}_{T^k}(T^{[n]}) &\leq \binom{n}{k} \cdot \frac{n^{n-k}}{n!} \\ &\leq 2^n \cdot \frac{n^{n-k}}{n^n \cdot e^{-n}} \\ &\leq \frac{e^{2n}}{n^k} \end{aligned}$$

(where in the second inequality, we use Lemma 3.5 below). If $n \leq d$, then $(d/n)^k \leq (d/n)^n$. Finally, by Lemma 3.6, $(d/n)^n \leq e^d$. ◯

Next, we return to the *combinatorial evaluation map*:

$$\Xi^{\text{cb}} : \mathbf{Z}[T]^{<d} \longrightarrow \bigoplus_{i=0}^{d-1} \mathbf{Z}$$

As discussed earlier, the range admits a natural "L^2-norm," hence defines an *arithmetic vector bundle* (cf. Chapter I, Definition 1.1) over $\mathbf{Q}$, which we denote by Φ_{et} (for "étale function space"). Next, let us denote by Φ_{DR} (for "de Rham function space") the arithmetic vector bundle defined by $\mathbf{Q}[T]^{<d}$ equipped with: (i.) the metric induced by the L^2-metric on Φ_{et} via Ξ^{cb} at the infinite prime of $\mathbf{Q}$; (ii.) the integral structure defined

by $T^{[0]}, T^{[1]}, \ldots, T^{[d-1]}$ at the finite primes of $\mathbf{Q}$. Thus, Ξ^{cb} induces an *isomorphism of arithmetic vector bundles*:

$$\Phi_{\mathrm{DR}} \cong \Phi_{\mathrm{et}}$$

Let us denote the filtration on Φ_{DR} given by considering polynomials of degree $< n$ by $F^n(\Phi_{\mathrm{DR}})$. Note that since Φ_{et} is a direct sum of trivial arithmetic line bundles, it follows that

$$\sum_{n=0}^{d-1} \deg\{(F^{n+1}/F^n)(\Phi_{\mathrm{DR}})\} = \deg(\Phi_{\mathrm{DR}}) = \deg(\Phi_{\mathrm{et}}) = 0$$

In fact, if we compute the "zeroes and poles" of the rational section of the arithmetic line bundle

$$L_n \stackrel{\mathrm{def}}{=} (F^{n+1}/F^n)(\Phi_{\mathrm{DR}})$$

defined by T^n, we see (cf. Proposition 3.1, (2); Proposition 3.2, (iii.)) that

$$\begin{aligned}\deg(L_n) &= \log(n!) + \log(||t_n||^{-1} \cdot \mathrm{Coeff}_{T^n}(t_n(T))) \\ &= \log(n!) + \log\Big(\binom{2n}{n} \cdot \frac{(2n+1)^{\frac{1}{2}} \cdot ((d-n-1)!)^{\frac{1}{2}}}{((d+n)!)^{\frac{1}{2}}}\Big)\end{aligned}$$

(where the first (respectively, second) "log" represents the contribution arising from the finite (respectively, infinite) places). In fact, the fact that the sum of the $\deg(L_n)$'s is 0 may be checked directly as follows:

Proposition 3.4. *The degree of* L_n

$$\deg(L_n) = \log(n!) + \log(||t_n||^{-1} \cdot \mathrm{Coeff}_{T^n}(t_n(T)))$$

satisfies

$$|\ n \cdot \log(n/d) + \frac{1}{2} \cdot \log(d) - \deg(L_n)\ | \ \leq\ 4n + 2$$

Moreover, the sum of the $\deg(L_n)$*'s satisfies:*

$$\prod_{n=0}^{d-1} \exp(\deg(L_n)) = \prod_{n=0}^{d-1} n! \cdot \binom{2n}{n} \cdot \Big(\frac{(2n+1) \cdot (d-1-n)!}{(d+n)!}\Big)^{\frac{1}{2}} = 1$$

(for any positive integer n).

Proof. The inequality concerning $\deg(L_n)$ follows immediately from Proposition 3.2, (ii.), and Lemma 3.5 below. Thus, it remains to prove the assertion concerning the sum of the $\deg(L_n)$'s. Although it is not logically necessary for the proof, we first observe that the fact that the sum of the $\deg(L_n)$'s is zero is *compatible* with the inequalities just proven. Indeed, we have

$$\Big(\prod_{n=1}^{d-1} \left(\frac{d}{n}\right)^n \cdot d^{\frac{1}{2}}\Big) \geq 1$$

while (by Lemma 3.6 below)

$$\Big(\prod_{n=1}^{d-1} \left(\frac{d}{n}\right)^n \cdot d^{\frac{1}{2}}\Big) \cdot \Big(\prod_{n=0}^{d-1} e^{-4n-2}\Big) \leq \Big(\prod_{n=1}^{d-1} e^d \cdot e^{\frac{1}{2}\cdot\log(d)}\Big) \cdot e^{-2d^2} \leq e^{d(d+\frac{1}{2}\cdot d)-2d^2} \leq 1$$

as desired.

Now we prove *exact equality*. For simplicity, we treat the case of d *odd*; the case of d even is similar (only easier). We compute:

$$\begin{aligned}
\prod_{n=0}^{d-1} \frac{(2n+1)\cdot(d-1-n)!}{(d+n)!} &= 1\cdot 3\cdot 5\cdot \ldots \cdot(2d-1)\cdot\Big(\frac{1!\cdot 2!\cdot\ldots\cdot(d-1)!}{d!\cdot(d+1)!\cdot\ldots\cdot(2d-1)!}\Big) \\
&= \{1\cdot 3\cdot 5\cdot\ldots\cdot(d-2)\}^2\cdot \\
&\quad \Big(\frac{(2!)^2\cdot(4!)^2\cdot\ldots\cdot((d-3)!)^2\cdot(d-1)!}{(d-1)!\cdot((d+1)!)^2\cdot((d+3)!)^2\cdot\ldots\cdot((2d-2)!)^2}\Big)
\end{aligned}$$

If we multiply the square root of this expression by $\prod_{n=0}^{d-1} \binom{2n}{n}$, we obtain:

$$\begin{aligned}
&\Big(\frac{2!\cdot 4!\cdot\ldots\cdot(2d-2)!}{(1!\cdot 2!\cdot\ldots\cdot(d-1)!)^2}\Big)\cdot\Big(1\cdot 3\cdot\ldots\cdot(d-2)\cdot\frac{(2!)\cdot(4!)\cdot\ldots\cdot(d-3)!\cdot(d-1)!}{(d-1)!\cdot(d+1)!\cdot(d+3)!\cdot\ldots\cdot(2d-2)!}\Big) \\
&\qquad = \Big(\frac{2!\cdot 4!\cdot\ldots\cdot(d-1)!}{1!\cdot 2!\cdot 3!\cdot\ldots\cdot(d-1)!}\Big)^2\cdot\Big(\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(d-2)}{(d-1)!}\Big) \\
&\qquad = \Big(\frac{(2!\cdot 4!\cdot\ldots\cdot(d-3)!)^2}{1!\cdot 2!\cdot 3!\cdot\ldots\cdot(d-1)!}\Big)\cdot\Big(\frac{1}{(2!\cdot 4!\cdot\ldots\cdot(d-3)!)^2}\Big) \\
&\qquad = \frac{1}{1!\cdot 2!\cdot 3!\cdot\ldots\cdot(d-1)!}
\end{aligned}$$

as desired. ◯

Before continuing, we review the following well-known elementary Lemmas, which we will use often throughout this Chapter:

Lemma 3.5. **(Stirling's Formula)** *If n is a positive integer, then*

$$n! = n^n \cdot e^{-n} \cdot \sqrt{n} \cdot \sqrt{2\pi} \cdot \Theta_n$$

where Θ_n is a real number satisfying $1 \leq \Theta_n \leq e$.

Proof. For a precise statement of Stirling's formula, we refer to [Ahlf], Chapter 5, §2.5, Ex. 2. The inequalities stated here are formal consequences of Stirling's formula stated in this form. ◯

Lemma 3.6. *If x, y are positive real numbers, then $(\frac{y}{x})^x \leq e^{\frac{y}{e}}$.*

Proof. By letting $u \stackrel{\text{def}}{=} \frac{y}{x}$, we see that it suffices to prove that $u \leq e^{\frac{u}{e}}$, or, equivalently, that $f(u) \stackrel{\text{def}}{=} u - e \cdot \log(u)$ is ≥ 0 for all positive real u. Now note that $f'(u) = 1 - \frac{e}{u}$ is 0 if and only if $u = e$. Moreover, since $f(e) = 0$, $f(0) = +\infty$, $f(+\infty) = +\infty$, we thus conclude that $f(u) \geq 0$ for all positive real u, as desired. ◯

The above discussion of the fundamental combinatorial model at finite and infinite primes, together with Remark 2 following Proposition 3.1, motivate the following *point of view*: Fix an integer $n \geq 2$. Then let us consider the *limit*

$$\lim_{d \to \infty} \frac{t_{n,d}(d^\lambda \cdot T)}{d^{\lambda \cdot n} \cdot \text{Coeff}_{T^n}(t_{n,d}(T))}$$

where $\lambda \in \mathbf{R}$, and "$\text{Coeff}_{T^n}(-)$" denotes the coefficient of T^n in the polynomial in parentheses. Proposition 3.1, (6), states that when $\lambda > 1$, *this limit exists and is equal to* T^n. Proposition 3.1, (5), states that when $\lambda = 1$, *this limit exists and is equal to a polynomial of degree* n whose leading term is T^n, but which has (as one sees from the explicit formula for the Legendre polynomials given in Proposition 2.1) *nonzero terms of lower degree.* The fact that there exist nonzero terms of lower degree implies that if $\lambda < 1$, then the above limit *diverges*. Thus, we see that "$\lambda = 1$" is a distinguished value associated to the family of systems of discrete Tchebycheff polynomials that one obtains as d varies.

Terminology 3.7. We shall refer to this special exponent λ of the scaling factor d as the *slope* of this family of orthogonal systems on finite discrete sets.

Later in this Chapter (cf. §6, especially Theorem 6.7), we shall encounter another family of orthogonal systems on finite discrete sets whose *slope is* $\frac{1}{2}$ and whose limit is the Hermite polynomials. On the other hand, the natural system of polynomials at the finite primes, i.e., the *binomial coefficient polynomials* $T^{[n]}$ (cf. the discussion at the beginning of this §), have *slope* 0, i.e., the exact *same* polynomials $T^{[n]}$ capture the integral structure at finite primes of $\mathbf{Z}$-valued functions on the set $\{0, \ldots, d-1\}$, regardless of the size of d.

> It turns out that the combinatorics of these three types of systems, i.e., *Legendre, Hermite, and binomial*, are *fundamental* to the theory of the comparison isomorphism at archimedean places (cf. Chapter VIII). Moreover, in Chapter VIII, we shall see that these three types of systems form the fundamental *models* of approximation that allow us to estimate the difference between the natural metrics on the spaces of "de Rham functions" and "étale functions" (cf. §0).

Also, it is interesting to note that the *slopes* of these three models, i.e., 1, $\frac{1}{2}$, and 0, are exactly the same as the *slopes of Frobenius* that appear in the Frobenius action on the first crystalline cohomology group associated to an elliptic curve in characteristic p. Indeed, it is precisely because we feel that there is a *deep analogy between these two notions of "slope"* that we chose the name "slope" for the invariant λ of the above discussion. For more on this analogy, we refer to the discussion of §6.

§4. The Kähler Geometry of a Polarized Elliptic Curve

In this §, we study the *Kähler geometry* of a polarized elliptic curve (i.e., an elliptic curve equipped with a line bundle of degree 1) over $\mathbf{C}$. Since the underlying real analytic manifold of such an elliptic curve is a two (real) dimensional *torus*, it is natural to expect that the *topology of the torus* will play a fundamental role in our analysis. In fact, the key technology that we will use in this § is a sort of *Lie algebra version of the theta groups discussed in Chapter IV, §1.* That is to say, just as the theta group of a line bundle may essentially be thought of as the extension of (a certain portion of) the étale fundamental group of the elliptic curve determined by the étale cohomological first Chern class ($\in H^2_{\mathrm{et}}$ – i.e., the "Weil pairing") of the line bundle in question, the "differential theta group" that we consider here may be thought of as the extension of the topological fundamental group tensored with $\mathbf{C}$ determined by the *differential-geometric first Chern class* of the line bundle in question. This technology will allow us to analyze (in terms of various differential operators) the structure of certain natural metrics on the space of sections of the line bundle in question over the universal extension of the elliptic curve. This analysis will be important in the following §'s since it will allow us to relate these natural metrics to the *canonical Schottky-Weierstrass zeta functions* of Chapter III, §7; Chapter IV, §3. We remark that the material of this § is "in principle well-known," but we do not know an adequate reference for it.

Let E be an *elliptic curve* over $\mathbf{C}$. We shall often write E for the complex manifold defined by the given (algebraic) elliptic curve. Let us write $E_{\mathbf{R}}$ (respectively, $E^{\dagger}_{\mathbf{R}}$) for the *underlying real analytic manifold* associated to E (respectively, the universal extension $E^{\dagger}$ of E). Then recall (cf. Chapter III, Definition 3.2) that we have a canonical *real analytic section*

$$\kappa_{\mathbf{R}} : E_{\mathbf{R}} \to E^{\dagger}_{\mathbf{R}}$$

of the universal extension $E^{\dagger} \to E$ of E. This section $\kappa_{\mathbf{R}}$ is the unique real analytic section that respects the group structures of $E_{\mathbf{R}}$, $E^{\dagger}_{\mathbf{R}}$.

Let us write $\mathcal{R}_{E^{\dagger}}$ for the subsheaf of the push-forward of $\mathcal{O}_{E^{\dagger}}$ to $\mathcal{O}_E$ consisting of sections of *finite torsorial degree* (cf. Chapter III, Definition 2.2). Thus, $\mathcal{R}_{E^{\dagger}}$ admits an *exhaustive* filtration $\{F^n(\mathcal{R}_{E^{\dagger}})\}$, where $F^n(\mathcal{R}_{E^{\dagger}})$ is the subsheaf of sections of torsorial degree $< n$. In particular, $F^n(\mathcal{R}_{E^{\dagger}})$ is a holomorphic vector bundle of rank n on E. Moreover,

$$(F^{n+1}/F^n)(\mathcal{R}_{E^{\dagger}}) = \tau_E^{\otimes n} \otimes_{\mathbf{C}} \mathcal{O}_E$$

(where $\tau_E = \omega_E^{\vee}$, and ω_E is the cotangent space to E at the origin). Next, let us observe that the real analytic splitting $\kappa_{\mathbf{R}}$ of the torsor $E^{\dagger} \to E$ induces a real analytic splitting of the filtration on $\mathcal{R}_{E^{\dagger}}$:

$$\mathcal{R}_{E^{\dagger}} \otimes_{\mathcal{O}_E} \mathcal{O}_{E_{\mathbf{R}}} = \bigoplus_{n \geq 0} \tau_E^{\otimes n} \otimes_{\mathbf{C}} \mathcal{O}_{E_{\mathbf{R}}}$$

Here, we write $\mathcal{O}_{E_{\mathbf{R}}}$ for the sheaf of *complex-valued real analytic functions on* $E_{\mathbf{R}}$.

Now let $d\mu$ be the unique (real analytic) $(1,1)$-form on E which is invariant with respect to translation by elements of E and whose integral satisfies:

$$\int_E d\mu = 1$$

Write

$$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(e_E)$$

(where e_E is the origin of E). Thus, $\mathcal{L}$ is a holomorphic line bundle on E. Write $\mathcal{L}_{\mathbf{R}} \stackrel{\text{def}}{=} \mathcal{L} \otimes_{\mathcal{O}_E} \mathcal{O}_{E_{\mathbf{R}}}$. Let $|| \sim ||_{\mathcal{L}}$ be a *Hermitian metric* on $\mathcal{L}$ whose *curvature* $\Theta_{\mathcal{L}}$ (i.e., the $(1,1)$-form given locally by $\partial\overline{\partial} \log(||s||_{\mathcal{L}})$, where s is a nonvanishing holomorphic section of $\mathcal{L}$) satisfies:

$$\frac{1}{2\pi i} \cdot \Theta_{\mathcal{L}} = d\mu$$

(Such a Hermitian metric always exists, and, moreover, is unique up to a positive real constant multiple – cf., e.g., [Mumf1,2,3], §12, Corollary to Lemma 1.) Note that $d\mu$ induces a *metric* $|| \sim ||_\tau$ *on* τ_E; $|| \sim ||_{\mathcal{L}}$ and $d\mu$ induce a *metric* $|| \sim ||_{V_{\mathbf{R}}}$ *on* $V_{\mathbf{R}} \stackrel{\mathrm{def}}{=} \Gamma(E_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$:

$$||\phi||^2_{V_{\mathbf{R}}} \stackrel{\mathrm{def}}{=} \int_E ||\phi||^2_{\mathcal{L}} \cdot d\mu$$

(for $\phi \in V_{\mathbf{R}}$).

In the following discussion, it will be useful to *trivialize* ω_E once and for all by some differential form θ such that $||\theta||_\omega = 1$ (where $|| \sim ||_\omega$ is the dual metric to $|| \sim ||_\tau$). Thus, θ also defines a trivialization $\theta^\vee$ of τ_E, as well as trivializations $\overline{\theta}$, $\overline{\theta}^\vee$ of the complex conjugates of ω_E and τ_E. Moreover, one checks easily that $\theta \wedge \overline{\theta}$ is *imaginary* (i.e., complex conjugation acts on it by multiplication by -1). Thus,

$$\theta \wedge \overline{\theta} = -i \cdot d\mu$$

(where the sign preceding "i" may be checked by computing using local coordinates). Moreover, the trivialization $\theta^\vee$ allows us to think of

$$\mathcal{R}_{E^\dagger} \otimes_{\mathcal{O}_E} \mathcal{O}_{E_{\mathbf{R}}} = \bigoplus_{n \geq 0} \tau_E^{\otimes n} \otimes_{\mathbf{C}} \mathcal{O}_{E_{\mathbf{R}}}$$

as the $\mathcal{O}_{E_{\mathbf{R}}}$-algebra $\mathcal{O}_{E_{\mathbf{R}}}[T_{\mathrm{DR}}]$ of *polynomials in the indeterminate* T_{DR}.

Now write

$$V_n \stackrel{\mathrm{def}}{=} \Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^n(\mathcal{R}_{E^\dagger}))$$

Then we can define *two natural metrics on* V_n, as follows:

(1) Let $s \in V_n$. Then the above real analytic splitting of the filtration on $\mathcal{R}_{E^\dagger}$ induces a decomposition of s into components: $s = s[0] + s[1] \cdot T_{\mathrm{DR}} + \ldots + s[n-1] \cdot T_{\mathrm{DR}}^{n-1}$, where $s[i] \in V_{\mathbf{R}}$. Now we define

$$||s||_{L^2_{\mathrm{DR}}} \stackrel{\mathrm{def}}{=} \sum_i ||s[i]||_{V_{\mathbf{R}}}$$

We shall refer to this metric as the L^2_{DR}-*metric* on V_n.

(2) Let $s \in V_n$. Then one may think of s as a (holomorphic) section of $\mathcal{L}$ over $E^\dagger$. If we pull-back this section via $\kappa_{\mathbf{R}}$, we obtain a *real analytic* section $s_{\mathbf{R}} \in V_{\mathbf{R}}$. (Note that in the notation of (1) above, $s_{\mathbf{R}} = s[0]$.) Then we define

$$||s||_{L^2_{\mathbf{R}}} \stackrel{\text{def}}{=} ||s_{\mathbf{R}}||_{V_{\mathbf{R}}}$$

We shall refer to this metric as the $L^2_{\mathbf{R}}$-*metric* on V_n.

One of the goals of this § is *to describe the precise relationship between the L^2_{DR}- and $L^2_{\mathbf{R}}$-metrics.*

To do this, we must introduce various *differential operators*, as follows. First, let us denote by

$$\nabla_{\mathbf{R}} : \mathcal{L}_{\mathbf{R}} \to \mathcal{L}_{\mathbf{R}} \otimes_{\mathcal{O}_{E_{\mathbf{R}}}} \Omega_{E_{\mathbf{R}}}$$

(where $\Omega_{E_{\mathbf{R}}}$ is the rank *two* locally free $\mathcal{O}_{E_{\mathbf{R}}}$-module of (complex-valued) differentials on $E_{\mathbf{R}}$) naturally associated to the metric $|| \sim ||_{\mathcal{L}}$ (cf., e.g., [Wells], Chapter III, Theorem 2.1). Then the decomposition

$$\Omega_{E_{\mathbf{R}}} = \Omega^{1,0}_{E_{\mathbf{R}}} \oplus \Omega^{0,1}_{E_{\mathbf{R}}}$$

(where $\Omega^{1,0}_{E_{\mathbf{R}}} = \Omega_E \otimes_{\mathcal{O}_E} \mathcal{O}_{E_{\mathbf{R}}}$, and $\Omega^{0,1}_{E_{\mathbf{R}}}$ is the complex conjugate of $\Omega^{1,0}_{E_{\mathbf{R}}}$) induces a decomposition

$$\nabla_{\mathbf{R}} = \nabla^{1,0} \oplus \nabla^{0,1}$$

where $\nabla^{0,1}$ is simply the $\overline{\partial}$-operator on $\mathcal{L}$. If we think of $E^\dagger \to E$ as the torsor of holomorphic connections on $\mathcal{L}^{-1}$ (cf. Chapter III, Theorem 4.2), then we see that the $(1,0)$-component $\nabla^{1,0}$ induces a real analytic section

$$\kappa^{\nabla}_{\mathbf{R}} : E_{\mathbf{R}} \to E^\dagger_{\mathbf{R}}$$

Moreover:

Proposition 4.1. *We have: $\kappa_{\mathbf{R}} = \kappa^{\nabla}_{\mathbf{R}}$, i.e., the canonical real analytic section $\kappa_{\mathbf{R}}$ of Chapter III, Definition 3.2, is the same as the real analytic section defined by the canonical connection $\nabla_{\mathbf{R}}$ associated to $|| \sim ||_{\mathcal{L}}$.*

Proof. Since $E^\dagger \to E$ is a *holomorphic torsor*, we may consider $\overline{\partial}$ of real analytic sections s of this torsor. Such a $\overline{\partial}(s)$ will be a real analytic $(1,1)$-form on E whose integral

$$\int_E \overline{\partial}(s) = -1$$

is precisely the class in $H^1(E, \Omega_E) \cong \mathbf{C}$ corresponding to the torsor $E^\dagger \to E$. Next, let us observe that (it follows from the definitions that) $\overline{\partial}(\kappa_{\mathbf{R}}^\nabla)$ is *a constant multiple of the curvature* $\Theta_{\mathcal{L}}$, hence is translation invariant. On the other hand, one sees immediately from the definition of $\kappa_{\mathbf{R}}$ (cf. Chapter III, Definition 3.2) – which is essentially a matter of *linear* algebra – that $\overline{\partial}(\kappa_{\mathbf{R}})$ is also translation invariant. Thus, if we write $\eta \stackrel{\mathrm{def}}{=} \kappa_{\mathbf{R}} - \kappa_{\mathbf{R}}^\nabla$ for the $(1,0)$-form on E given by the difference between the two sections in question, we see that $\overline{\partial}(\eta)$ is a translation invariant $(1,1)$-form whose integral is *zero*. Thus, $\overline{\partial}(\eta) = 0$, i.e., η is *holomorphic*. Moreover, $\eta \in \Gamma(E, \Omega_E) = \omega_E$ will be zero if it is invariant under pull-back by the automorphism $\alpha : E \to E$ given by multiplication by -1. On the other hand, it is clear from the definitions that both $\kappa_{\mathbf{R}}$ and $\kappa_{\mathbf{R}}^\nabla$ are invariant with respect to α. Thus, $\alpha(\eta) = \eta$, which implies that $\eta = 0$, as desired (cf. the proof of Chapter III, Theorem 5.6). $\bigcirc$

Remark. Note that if $\mathbf{G}_{\mathrm{m}} \to E = \mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$ is a *Schottky uniformization* (as in the discussion preceding Chapter III, Definition 3.3), then the splitting

$$\kappa_{\Lambda_1} : \mathbf{G}_{\mathrm{m}} \to E^\dagger|_{\mathbf{G}_{\mathrm{m}}}$$

of Chapter III, Definition 3.3, *coincides* with $\kappa_{\mathbf{R}}$ on $\mathbf{S^1} \subseteq \mathbf{C}^\times = \mathbf{G}_{\mathrm{m}}$. (Here, note that since $\mathbf{S^1} \bigcap q^{\mathbf{Z}} = \{1\}$, we also have an inclusion $\mathbf{S^1} \hookrightarrow \mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}} = E$.) Indeed, this follows from the fact that both $\kappa_{\mathbf{R}}|_{\mathbf{S^1}}$ and $\kappa_{\Lambda_1}|_{\mathbf{S^1}}$ are continuous group homomorphisms. Thus, their difference is a continuous group homomorphism $\mathbf{S^1} \to \omega_E$ ($\cong \mathbf{C}$), hence is zero. *This coincidence of* $\kappa_{\mathbf{R}}$ *and* κ_{Λ_1} *on* $\mathbf{S^1} \subseteq \mathbf{G}_{\mathrm{m}}, E$ will be important in the computations of this Chapter since it *will allow us to relate the real analytic theory of the present § to the arithmetic Schottky uniformizations over* $\mathbf{Z}$ *that appeared throughout Chapters III-VI.*

Now we would like to study the various differential operators introduced above. Recall that in the above discussion, we chose trivializations $\theta, \overline{\theta}$ of $\omega_E, \overline{\omega}_E$ (the complex conjugate of ω_E). These trivializations allow us to regard the operators $\nabla^{1,0}$ and $\nabla^{0,1}$ as *operators on* $V_{\mathbf{R}} = \Gamma(E_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$. Note that the L^2-norm $|| \sim ||_{V_{\mathbf{R}}}$ defined above on $V_{\mathbf{R}}$ corresponds naturally to an inner product $\langle -, - \rangle_{V_{\mathbf{R}}}$ on $V_{\mathbf{R}}$.

Proposition 4.2. *We have:* $(\nabla^{1,0})^* = \nabla^{0,1}$; $(\nabla^{0,1})^* = \nabla^{1,0}$ *(where "$*$" denotes the adjoint of an operator on* $V_{\mathbf{R}}$ *equipped with the inner product* $\langle -, - \rangle_{V_{\mathbf{R}}}$*).*

Proof. Indeed, if $\phi, \psi \in V_{\mathbf{R}}$, then

$$
\begin{aligned}
\langle\ \nabla^{1,0}(\phi), \psi\ \rangle_{V_{\mathbf{R}}} &= \int_E\ i \cdot\ \langle\ \nabla^{1,0}(\phi) \cdot \theta, \psi \cdot \theta\ \rangle_{\mathcal{L}} \\
&= \int_E\ i \cdot\ \langle\ \nabla_{\mathbf{R}}(\phi), \psi \cdot \theta\ \rangle_{\mathcal{L}} \\
&= -\int_E\ i \cdot\ \langle\ \phi, \nabla_{\mathbf{R}}(\psi \cdot \theta)\ \rangle_{\mathcal{L}} + \int_E\ i \cdot d\ \langle\ \phi, \psi \cdot \theta\ \rangle_{\mathcal{L}} \\
&= -\int_E\ i \cdot\ \langle\ \phi, \nabla_{\mathbf{R}}(\psi \cdot \theta)\ \rangle_{\mathcal{L}} \\
&= \int_E\ i \cdot\ \langle\ \phi, \nabla^{0,1}(\psi) \cdot \theta \wedge \overline{\theta}\ \rangle_{\mathcal{L}} \\
&= \int_E\ \langle\ \phi, \nabla^{0,1}(\psi)\ \rangle_{\mathcal{L}} \cdot d\mu =\ \langle\ \phi, \nabla^{0,1}(\psi)\ \rangle_{V_{\mathbf{R}}}
\end{aligned}
$$

Here, the second (respectively, third; fourth; fifth) equality follows from the fact that all $(0,2)$-forms are identically zero (respectively, the fact that $\nabla_{\mathbf{R}}$ preserves the metric $||\sim||_{\mathcal{L}}$; Stokes' theorem; the fact that all $(2,0)$-forms are identically zero), and we recall that

$$
\theta \wedge \overline{\theta} = -i \cdot d\mu
$$

Thus, $(\nabla^{1,0})^* = \nabla^{0,1}$, as desired. The relation $(\nabla^{0,1})^* = \nabla^{1,0}$ follows similarly. ◯

Thus, in the following, we shall simply write:

$$
\overline{\partial}^* \stackrel{\text{def}}{=} \nabla^{1,0}; \quad \overline{\partial} \stackrel{\text{def}}{=} \nabla^{0,1}
$$

Put another way, $\overline{\partial}^*$ (respectively, $\overline{\partial}$) is simply $\nabla_{\mathbf{R}}$ evaluated in the direction $\theta^\vee$ (respectively, $\overline{\theta}^\vee$).

Proposition 4.3. *We have:* $[\overline{\partial}^*, \overline{\partial}] = -2\pi$.

Proof. Note that $[\theta^\vee, \overline{\theta}^\vee] = 0$. (Indeed, this follows from the fact that the Lie algebra of the complex Lie group $E^\dagger$ is abelian!) Thus, it follows from the definitions (cf. [Wells], Chapter III, Proposition 1.9) that $[\overline{\partial}^*, \overline{\partial}]$ is simply the curvature $\Theta_{\mathcal{L}}$ (which is a $(1,1)$-form) divided by $\theta \wedge \overline{\theta} = -i \cdot d\mu$, i.e., $[\overline{\partial}^*, \overline{\partial}] = -2\pi$, as desired. ◯

Remark. One can now define the *Lie algebra* $\mathcal{G}^{\text{diff}}_{\mathcal{L}}$ to be the Lie algebra (over $\mathbf{C}$) of operators on $V_{\mathbf{R}}$ generated by $1, \overline{\partial}, \overline{\partial}^*$. Thus, we have an exact sequence

$$
0 \to \mathbf{C} \cdot 1 \to \mathcal{G}^{\text{diff}}_{\mathcal{L}} \to \text{Lie}(E^\dagger) \to 0
$$

where $\mathbf{C} \cdot 1$ is central in $\mathcal{G}^{\mathrm{diff}}_{\mathcal{L}}$, and $\mathrm{Lie}(E^{\dagger}) = \mathbf{C} \cdot \overline{\partial}^* + \mathbf{C} \cdot \overline{\partial}$ is the Lie algebra of $E^{\dagger}$. Moreover, the commutator product of $\mathcal{G}^{\mathrm{diff}}_{\mathcal{L}}$ defines a bilinear morphism

$$\mathrm{Lie}(E^{\dagger}) \times \mathrm{Lie}(E^{\dagger}) \to \mathbf{C}$$

that maps $(\overline{\partial}^*, \overline{\partial}) \mapsto -2\pi$ (by Proposition 4.3). This bilinear form is essentially the "differential geometric" first Chern class of the line bundle $\mathcal{L}$. Thus, this "differential theta group" $\mathcal{G}^{\mathrm{diff}}_{\mathcal{L}}$ is very much analogous to the "étale theta groups" reviewed in Chapter IV, §1.

Next, we would like to consider the following *Laplacian operator* on $V_{\mathbf{R}}$:

$$\Delta \stackrel{\mathrm{def}}{=} \frac{1}{2\pi} \cdot \overline{\partial}^* \cdot \overline{\partial}$$

Then we have the following:

Proposition 4.4. *We have:*

(i.) $\Delta^* = \Delta$

(ii.) $\Delta \cdot \overline{\partial} = \overline{\partial} \cdot (\Delta - 1)$

(iii.) $\Delta \cdot \overline{\partial}^* = \overline{\partial}^* \cdot (\Delta + 1)$

In particular, the eigenvalues of Δ *are all* real and nonnegative, *and eigenfunctions of* Δ *with distinct eigenvalues are orthogonal to one another (with respect to* $\langle -,- \rangle_{V_{\mathbf{R}}}$*). Moreover, if, for* $s \in V_{\mathbf{R}}$, $\lambda \in \mathbf{R}$, *we have* $\Delta(s) = \lambda \cdot s$, *then* $\Delta(\overline{\partial}(s)) = (\lambda - 1) \cdot (\overline{\partial}(s))$; $\Delta(\overline{\partial}^*(s)) = (\lambda + 1) \cdot (\overline{\partial}^*(s))$.

Proof. (i.) Since $(\overline{\partial})^* = \overline{\partial}^*$, $(\overline{\partial}^*)^* = \overline{\partial}$, we obtain immediately that $\Delta^* = \Delta$.

(ii.) Compute (using Proposition 4.3): $2\pi \cdot \Delta \cdot \overline{\partial} = \overline{\partial}^* \cdot \overline{\partial} \cdot \overline{\partial} = \overline{\partial} \cdot \overline{\partial}^* \cdot \overline{\partial} - 2\pi \cdot \overline{\partial} = 2\pi \cdot (\overline{\partial} \cdot \Delta - \overline{\partial})$.

(iii.) Compute (using Proposition 4.3): $2\pi \cdot \Delta \cdot \overline{\partial}^* = \overline{\partial}^* \cdot \overline{\partial} \cdot \overline{\partial}^* = \overline{\partial}^* \cdot \overline{\partial}^* \cdot \overline{\partial} + 2\pi \cdot \overline{\partial}^* = 2\pi \cdot (\overline{\partial}^* \cdot \Delta + \overline{\partial}^*)$.

The final remarks — except for the fact that the eigenvalues are ≥ 0, but this follows from the well-known argument:

$$\Delta(s) = \lambda \cdot s \implies \lambda \langle s, s \rangle_{V_{\mathbf{R}}} = \langle s, \Delta(s) \rangle_{V_{\mathbf{R}}} = \langle \overline{\partial}(s), \overline{\partial}(s) \rangle_{V_{\mathbf{R}}} \geq 0$$

— in Proposition 4.4 are formal consequences of (i.), (ii.), and (iii.). ○

We are now ready to prove the following result:

Theorem 4.5. *The pull-back morphism*

$$\kappa_{\mathbf{R}}^* : V_n \to V_{\mathbf{R}}$$

is injective, and its image is equal to the kernel of $\overline{\partial}^n$ *(as an operator on* $V_{\mathbf{R}}$*). In fact, to apply* $\overline{\partial}$ *to an element of the image of* $\kappa_{\mathbf{R}}^* : V_n \to V_{\mathbf{R}}$ *corresponds to applying the relative exterior differential operator (in the direction* $\overline{\theta}^{\vee}$*) of the morphism* $E^{\dagger} \to E$ *to the corresponding element of* V_n*. In particular,* $\mathrm{Ker}(\overline{\partial}) \cong V_1$ *is a one-dimensional complex vector space. If* $\zeta_0^{\mathrm{DR}} \in \mathrm{Ker}(\overline{\partial})$ *is nonzero, write*

$$\zeta_n^{\mathrm{DR}} \stackrel{\mathrm{def}}{=} \frac{1}{n!} \cdot (\overline{\partial}^*)^n(\zeta_0^{\mathrm{DR}})$$

if $n > 0$, $\zeta_n^{\mathrm{DR}} \stackrel{\mathrm{def}}{=} 0$ *if* $n < 0$*. Then for* $n \geq 0$*, we have:* $\mathbf{C} \cdot \zeta_n^{\mathrm{DR}} = \mathrm{Ker}(\Delta - n)$*, and* $\zeta_0^{\mathrm{DR}}, \ldots, \zeta_n^{\mathrm{DR}}$ *form an orthogonal basis of* $\mathrm{Ker}(\overline{\partial}^{n+1})$ *such that* $\overline{\partial}^*(\zeta_n^{\mathrm{DR}}) = (n+1) \cdot \zeta_{n+1}^{\mathrm{DR}}$*;* $\overline{\partial}(\zeta_n^{\mathrm{DR}}) = 2\pi \cdot \zeta_{n-1}^{\mathrm{DR}}$*;* $||\zeta_n^{\mathrm{DR}}||_{V_{\mathbf{R}}}^2 = \frac{(2\pi)^n}{n!} \cdot ||\zeta_0^{\mathrm{DR}}||_{V_{\mathbf{R}}}^2$*.*

Proof. That $\kappa_{\mathbf{R}}^* : V_n \to V_{\mathbf{R}}$ is injective follows from the fact that if it were not, then the image of $\kappa_{\mathbf{R}}$ would be contained inside a one-dimensional complex subvariety of $E^{\dagger}$, which is absurd (cf. the definition of $\kappa_{\mathbf{R}}$ in Chapter III, §3). The statement concerning the correspondence between $\overline{\partial}$ and the relative exterior differential operator for $E^{\dagger} \to E$ follows immediately from the definitions (cf. also the discussion of Chapter III, §3).

To see that the image $\kappa_{\mathbf{R}}^*(V_n)$ is equal to $\mathrm{Ker}(\overline{\partial}^n)$, we reason as follows: First, the statement concerning the relationship between $\overline{\partial}$ and the relative exterior differential operator of $E^{\dagger} \to E$ implies that $\kappa_{\mathbf{R}}^*(V_n) \subseteq \mathrm{Ker}(\overline{\partial}^n)$. Thus, in particular, we see that $\dim_{\mathbf{C}}(\mathrm{Ker}(\overline{\partial}^n)) \geq n$. On the other hand, one proves easily by induction on n that $\dim_{\mathbf{C}}(\mathrm{Ker}(\overline{\partial}^n)) \leq n$: Indeed, this is clear if $n = 1$, since (by definition) $\mathrm{Ker}(\overline{\partial}) = \Gamma(E, \mathcal{L})$. If $n \geq 2$, then note that $\overline{\partial}$ defines a morphism $\mathrm{Ker}(\overline{\partial}^n) \to \mathrm{Ker}(\overline{\partial}^{n-1})$ whose kernel has dimension ≤ 1. Thus, by the induction hypothesis, we conclude that $\dim_{\mathbf{C}}(\mathrm{Ker}(\overline{\partial}^n)) \leq 1 + (n-1) = n$, as desired. This completes the proof that $\kappa_{\mathbf{R}}^*(V_n) = \mathrm{Ker}(\overline{\partial}^n)$.

That $\Delta(\zeta_n^{\mathrm{DR}}) = n \cdot \zeta_n^{\mathrm{DR}}$ (for $n \geq 0$) follows from Proposition 4.4, (iii.), induction on n, and the fact that $\Delta(\zeta_0^{\mathrm{DR}}) = 0$. If, for $s \in V_{\mathbf{R}}$, we have $\Delta(s) = n \cdot s$ (for $n \geq 1$), then by Proposition 4.4, (ii.), $\Delta(\overline{\partial}(s)) = (n-1) \cdot \overline{\partial}(s)$. Thus, by induction on n, we obtain that $\Delta(s) = n \cdot s$ (for $n \geq 1$) implies $\overline{\partial}^n(s) \in \mathrm{Ker}(\Delta)$. But if $\Delta(t) = 0$, then

$$\langle\ \overline{\partial}(t), \overline{\partial}(t)\ \rangle_{V_{\mathbf{R}}} =\ \langle\ t, \overline{\partial}^* \cdot \overline{\partial}(t)\ \rangle_{V_{\mathbf{R}}} =\ \langle\ t, \Delta(t)\ \rangle_{V_{\mathbf{R}}} = 0$$

so $\overline{\partial}(t) = 0$. Thus, in summary, $\mathrm{Ker}(\Delta - n) \subseteq \mathrm{Ker}(\overline{\partial}^{n+1})$. On the other hand, this implies that $\zeta_0^{\mathrm{DR}}, \ldots, \zeta_n^{\mathrm{DR}}$ (which are mutually orthogonal by Proposition 4.4, hence linearly independent over $\mathbf{C}$) are $\in \mathrm{Ker}(\overline{\partial}^{n+1})$, which (as we saw above) has dimension $n+1$. Thus, we conclude that $\zeta_0^{\mathrm{DR}}, \ldots, \zeta_n^{\mathrm{DR}}$ form an orthogonal basis of $\mathrm{Ker}(\overline{\partial}^{n+1})$, and that $\mathrm{Ker}(\Delta - n) = \mathbf{C} \cdot \zeta_n^{\mathrm{DR}}$, as desired.

That $\overline{\partial}^*(\zeta_n^{\mathrm{DR}}) = (n+1) \cdot \zeta_{n+1}^{\mathrm{DR}}$ follows from the definitions. That $\overline{\partial}(\zeta_n^{\mathrm{DR}}) = 2\pi \cdot \zeta_{n-1}^{\mathrm{DR}}$ follows by computing (using Proposition 4.3):

$$\overline{\partial}(\zeta_n^{\mathrm{DR}}) = n^{-1} \cdot \overline{\partial} \cdot \overline{\partial}^*(\zeta_{n-1}^{\mathrm{DR}}) = 2\pi \cdot n^{-1} \cdot (\Delta + 1)(\zeta_{n-1}^{\mathrm{DR}}) = 2\pi \cdot \zeta_{n-1}^{\mathrm{DR}}$$

Finally, $||\zeta_n^{\mathrm{DR}}||^2_{V_{\mathbf{R}}} = \frac{(2\pi)^n}{n!} \cdot ||\zeta_0^{\mathrm{DR}}||^2_{V_{\mathbf{R}}}$ follows by computing:

$$2\pi \cdot n \cdot \langle\ \zeta_n^{\mathrm{DR}}, \zeta_n^{\mathrm{DR}}\ \rangle\ = 2\pi \cdot \langle\ \Delta(\zeta_n^{\mathrm{DR}}), \zeta_n^{\mathrm{DR}}\ \rangle\ =\ \langle\ \overline{\partial}(\zeta_n^{\mathrm{DR}}), \overline{\partial}(\zeta_n^{\mathrm{DR}})\ \rangle\ = (2\pi)^2 \cdot \langle\ \zeta_{n-1}^{\mathrm{DR}}, \zeta_{n-1}^{\mathrm{DR}}\ \rangle$$

This completes the proof of Theorem 4.5. ○

We are now ready to analyze the *relationship between the L^2_{DR}- and $L^2_{\mathbf{R}}$-metrics*:

Corollary 4.6. *Let $\zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}] \in V_n$ be the unique element such that $\kappa^*_{\mathbf{R}}(\zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}]) = \zeta_n^{\mathrm{DR}}$. Then, relative to the above decomposition of $\mathcal{R}_{E^\dagger} \otimes_{\mathcal{O}_E} \mathcal{O}_{E_{\mathbf{R}}}$, we have:*

$$\begin{aligned}\zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}] &= \sum_{i=0}^{n} \zeta_i^{\mathrm{DR}} \cdot \Big(\frac{(2\pi \cdot T_{\mathrm{DR}})^{n-i}}{(n-i)!}\Big) \\ &= \zeta_0^{\mathrm{DR}} \cdot \Big(\frac{(2\pi \cdot T_{\mathrm{DR}})^n}{n!}\Big) + \zeta_1^{\mathrm{DR}} \cdot \Big(\frac{(2\pi \cdot T_{\mathrm{DR}})^{n-1}}{(n-1)!}\Big) + \ldots + \zeta_{n-1}^{\mathrm{DR}} \cdot (2\pi \cdot T_{\mathrm{DR}}) + \zeta_n^{\mathrm{DR}}\end{aligned}$$

In particular, $\zeta_0^{\mathrm{DR}}[T_{\mathrm{DR}}], \zeta_1^{\mathrm{DR}}[T_{\mathrm{DR}}], \ldots, \zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}], \ldots$ are mutually orthogonal with respect to both the L^2_{DR}- and $L^2_{\mathbf{R}}$-metrics, and their norms are given as follows: If we rescale *the metric on τ_E so that $|T_{\mathrm{DR}}| = \rho \in \mathbf{R}_{>0}$ (note that in the above discussion $\rho = 1$), then*

$$||\ \zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}]\ ||_{L^2_{\mathbf{R}}} = ||\zeta_n^{\mathrm{DR}}||_{V_{\mathbf{R}}} = \Big(\frac{(2\pi)^n}{n!}\Big)^{\frac{1}{2}} \cdot ||\zeta_0^{\mathrm{DR}}||_{V_{\mathbf{R}}}$$

$$\begin{aligned}
\|\ \zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}]\ \|_{L^2_{\mathrm{DR}}} &= \Big\{ \sum_{i=0}^{n} \ \Big(\frac{(2\pi\cdot\rho)^{n-i}}{(n-i)!}\Big)^2 \cdot \Big(\frac{(2\pi)^i}{i!}\Big)\Big\}^{\frac{1}{2}} \cdot \|\zeta_0^{\mathrm{DR}}\|_{V_{\mathbf{R}}} \\
&= \|\ \zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}]\ \|_{L^2_{\mathbf{R}}} \cdot \Big\{ \sum_{i=0}^{n} \ \binom{n}{i}\Big(\frac{(2\pi\cdot\rho^2)^i}{i!}\Big)\Big\}^{\frac{1}{2}} \\
&= \|\ \zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}]\ \|_{L^2_{\mathbf{R}}} \cdot C_n(\rho)
\end{aligned}$$

where $1 \leq C_n(\rho) \leq e^{\pi} \cdot 2^{\frac{n}{2}} \cdot \mathrm{Max}(1, \rho^n)$.

Proof. Let us first show that

$$\zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}] = \zeta_0^{\mathrm{DR}} \cdot \Big(\frac{(2\pi\cdot T_{\mathrm{DR}})^n}{n!}\Big) + \zeta_1^{\mathrm{DR}} \cdot \Big(\frac{(2\pi\cdot T_{\mathrm{DR}})^{n-1}}{(n-1)!}\Big) + \ldots + \zeta_{n-1}^{\mathrm{DR}} \cdot (2\pi\cdot T_{\mathrm{DR}}) + \zeta_n^{\mathrm{DR}}$$

We use induction on n. The result is clear for $n = 0$, so assume the result known for "$n-1$." The fact that the constant term of $\zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}]$ is ζ_n^{DR} follows from the definition of $\zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}]$. Thus, to prove the above equality, it suffices to prove it *after applying* $\frac{\partial}{\partial T_{\mathrm{DR}}}$. Since applying $\frac{\partial}{\partial T_{\mathrm{DR}}}$ corresponds (relative to pull-back via $\kappa_{\mathbf{R}}^*$) to applying $\overline{\partial}$ to the corresponding element of $V_{\mathbf{R}}$ (cf. Theorem 4.5), it follows (since $\overline{\partial}(\zeta_n^{\mathrm{DR}}) = 2\pi\cdot\zeta_{n-1}^{\mathrm{DR}}$ by Theorem 4.5) that it suffices to show that

$$2\pi\cdot\zeta_{n-1}^{\mathrm{DR}}[T_{\mathrm{DR}}] = \zeta_0^{\mathrm{DR}} \cdot \Big(\frac{2\pi\cdot(2\pi\cdot T_{\mathrm{DR}})^{n-1}}{(n-1)!}\Big) + \zeta_1^{\mathrm{DR}} \cdot \Big(\frac{2\pi\cdot(2\pi\cdot T_{\mathrm{DR}})^{n-2}}{(n-2)!}\Big) + \ldots + \zeta_{n-1}^{\mathrm{DR}} \cdot (2\pi)$$

But this follows from the induction hypothesis. This completes the computation of the components of $\zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}]$.

The statement concerning orthogonality then follows from the statement concerning the orthogonality of the ζ_n^{DR}'s in Theorem 4.5. The computation of the norms also follows from the norm computation of Theorem 4.5 (together with the elementary facts that $\binom{n}{i} \leq 2^n$, $\sum_{i=0}^{n} \frac{x^i}{i!} \leq e^x$ for $x \geq 0$). ○

Remark 1. One can regard the results of Corollary 4.6 as the computation of the *analytic torsion* of $V_n \stackrel{\mathrm{def}}{=} \Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^n(\mathcal{R}_{E^\dagger}))$. If there were "no analytic torsion," then the norm of $\zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}]$ would go roughly as the norm of its "leading term," i.e., roughly as

$$\frac{(2\pi)^n}{n!}$$

not as $(\frac{(2\pi)^n}{n!})^{\frac{1}{2}}$ (as in Corollary 4.6).

Remark 2. The reader may already have noticed many formal similarities between Theorem 4.5, Corollary 4.6 above and Chapter III, Theorem 7.4. These similarities are by no means an accident. In §5 below, we will discuss *the relationship between the theory of the present § and the theory of the canonical Schottky-Weierstrass functions of Chapter III, §7.*

Remark 3. Just as in Chapter III, Remark 3 (following Chapter III, Theorem 7.4), one can give a more "*generating function-theoretic*" formulation of the Corollary 4.6 as follows: Let $\mathbf{s}$ be an indeterminate. Let us write $\overline{\partial}_{E^\dagger}$ for the "$\overline{\partial}$-operator" acting on real analytic sections of the holomorphic line bundle $\mathcal{L}$ *over* $E^\dagger_{\mathbf{R}}$. Thus, we may regard $\overline{\partial}_{E^\dagger}$ as an operator on

$$\mathcal{L}_{\mathbf{R}}[T_{\mathrm{DR}}]$$

If we write $\overline{\partial}$, $\overline{\partial}^*$ for the operators obtained on $\mathcal{L}_{\mathbf{R}}[T_{\mathrm{DR}}]$ by *acting on the coefficients*, then one computes easily that

$$\overline{\partial}_{E^\dagger} = \overline{\partial} - \frac{\partial}{\partial T_{\mathrm{DR}}}$$

hence that

$$[\ \overline{\partial}_{E^\dagger},\ \overline{\partial}^* + 2\pi \cdot T_{\mathrm{DR}}\ ]\ =\ [\ \overline{\partial},\ \overline{\partial}^*\ ]\ -\ [\ \frac{\partial}{\partial T_{\mathrm{DR}}},\ 2\pi \cdot T_{\mathrm{DR}}\ ]\ =\ 2\pi - 2\pi =\ 0$$

(by Proposition 4.3). Thus, since $(\overline{\partial}_{E^\dagger})(\zeta_0^{\mathrm{DR}}) = 0$, $\overline{\partial}_{E^\dagger}$ also annihilates

$$\begin{aligned} e^{(\overline{\partial}^* + 2\pi \cdot T_{\mathrm{DR}})\cdot \mathbf{s}} \cdot \zeta_0^{\mathrm{DR}} &= \sum_{n \geq 0} \ \left(\sum_{i=0}^{n} \zeta_i^{\mathrm{DR}} \cdot \frac{(2\pi \cdot T_{\mathrm{DR}})^{n-i}}{(n-i)!} \right) \cdot \mathbf{s}^n \\ &= \zeta_0^{\mathrm{DR}}[T_{\mathrm{DR}}] + \zeta_1^{\mathrm{DR}}[T_{\mathrm{DR}}] \cdot \mathbf{s} + \zeta_2^{\mathrm{DR}}[T_{\mathrm{DR}}] \cdot \mathbf{s}^2 + \ldots + \zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}] \cdot \mathbf{s}^n \\ &\quad + \ldots \end{aligned}$$

The coefficients of this series are precisely the functions discussed in Corollary 4.6. The norms computed in Corollary 4.6 may also be computed directly from these exponential expressions. We leave this as an exercise for the reader.

§5. The Relationship Between de Rham and Canonical Schottky-Weierstrass Zeta Functions

In this §, we study *the relationship between the canonical Schottky-Weierstrass* ("SW " for short) *zeta functions of Chapter III, §7, and the de Rham* ("DR" for short) *zeta functions of §4* (i.e., the ζ_n^{DR}'s). In a word, the combinatorics of this relationship are described by the *Hermite polynomials* (cf. Proposition 2.2). Using the well-known explicit form of the Hermite polynomials, we give explicit equations relating the two types of zeta functions, and use these equations *to bound the norm of the canonical SW zeta functions in terms of the* L^2_{DR}*-metric* (of §4). One way to think about the material of the present § is to regard it as corresponding to the usual *Hodge comparison theorem* (i.e., the "de Rham isomorphism" between de Rham cohomology and singular cohomology – cf. Chapter VIII, §1) over $\mathbf{C}$, expressed at the level of *differential operators*, i.e.,

> as the relationship between the *"de Rham-theoretic" operators* $\overline{\partial}, \overline{\partial}^*$ of §4, which correspond to differentiation in the direction of the Hodge filtration and its complex conjugate, and the operator δ^* of Chapter III, §7, which corresponds to differentiation along a *topological cycle* of the elliptic curve.

This relationship between differential operators is expressed at the function-theoretic level as the relationship between the DR zeta functions of §4 and the canonical SW zeta functions of Chapter III, §7.

We continue with the notation of §4. Thus, E is an *elliptic curve* over $\mathbf{C}$. Recall from the discussion of Chapter III, §3, that one has a natural isomorphism – called the *de Rham isomorphism* –

$$H^1_{\mathrm{DR}}(E, \mathcal{O}_E) \cong H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} = \Lambda \otimes_{\mathbf{Z}} \mathbf{C}$$

(where $\Lambda \stackrel{\mathrm{def}}{=} H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbf{Z})$) of complex vector spaces. The left-hand side of this isomorphism, i.e., $H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$, may be regarded as the *tangent space* to the universal extension $E^\dagger$, and thus admits a natural surjection $H^1_{\mathrm{DR}}(E, \mathcal{O}_E) \to \tau_E$ arising from the projection $E^\dagger \to E$. Note that since the right-hand side of this isomorphism $H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbf{C})$ admits a natural *real structure* (given by $\Lambda_{\mathbf{R}} \stackrel{\mathrm{def}}{=} \Lambda \otimes_{\mathbf{Z}} \mathbf{R}$), it follows that the isomorphism induces on $H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$ a natural *complex conjugation automorphism.* If we use this automorphism to conjugate the projection $H^1_{\mathrm{DR}}(E, \mathcal{O}_E) \to \tau_E$, we thus obtain a natural direct sum decomposition:

$$H^1_{\mathrm{DR}}(E, \mathcal{O}_E) = \tau_E \oplus \tau_E^{\mathrm{c}}$$

(where τ_E^{c} is the complex conjugate $\mathbf{C}$-vector space to τ_E). Note that here we use the notation "τ_E^{c}" rather than $\overline{\tau}_E$ because we wish to think of τ_E^{c} here as a "*holomorphic* dimension" (which just happens to be isomorphic to the complex conjugate of τ_E), not as an "anti-holomorphic dimension" (which is the interpretation most commonly given to a "bar," as in $\overline{\tau}_E$).

Next, suppose that we are given a rank one $\mathbf{Z}$-submodule

$$\Lambda_1 \subseteq \Lambda$$

(cf. Chapter III, §3, the discussion of the "$\mathbf{G}_{\mathrm{m}}$-splitting"). Choose a generator $\lambda_1 \in \Lambda_1$. Then there exists a unique isomorphism of τ_E with $\mathbf{C}$ relative to which E may be written in the form

$$E = \mathbf{C}/ \langle 1, \tau \rangle = \mathbf{C}^\times / q^{\mathbf{Z}}$$

where the "1" corresponds to λ_1, $q \stackrel{\mathrm{def}}{=} e^{2\pi i \tau}$, and $\mathrm{Im}(\tau) > 0$. Relative to the de Rham isomorphism and the direct sum decomposition of $H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$ discussed above, λ_1 corresponds to some element

$$(v, u \cdot v^{\mathrm{c}}) \in \tau_E \oplus \tau_E^{\mathrm{c}} = H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$$

where v^{c} is the element of τ_E^{c} defined by v; $u \in \mathbf{C}$ satisfies $|u| = 1$. Moreover, the length of v (relative to the metric $|| \sim ||_\tau$) of §4) may be computed as follows:

Lemma 5.1. *If λ_1 corresponds under the de Rham isomorphism to $(v, u \cdot v^{\mathrm{c}})$ (where $u \in \mathbf{C}$ satisfies $|u| = 1$), then the length of v is given by:*

$$||v||_\tau = \frac{1}{\{8\pi^2 \cdot \mathrm{Im}(\tau)\}^{\frac{1}{2}}}$$

Proof. Let us think of E as $\mathbf{C}/ \langle 1, \tau \rangle = \mathbf{C}^\times / q^{\mathbf{Z}}$, and write z (respectively, U) for the standard coordinate on $\mathbf{C}$ (respectively, $\mathbf{C}^\times$). Thus, $U = \exp(2\pi i z)$. Moreover, the differential corresponding to $\lambda_1 \in \Lambda_1$ is given by $d\ \log(U) \stackrel{\mathrm{def}}{=} \frac{dU}{U} = 2\pi i \cdot dz$. That is to say, $||v||_\tau^{-1} = ||d\ \log(U)||_\omega = 2\pi \cdot ||dz||_\omega$. Moreover, the definition of $|| \sim ||_\omega$ is such that

$$||dz||_\omega^2 \stackrel{\mathrm{def}}{=} \left| \int_E dz \wedge d\overline{z} \right| = 2 \cdot \mathrm{Im}(\tau)$$

as desired. ○

Now we would like to consider *the relationship between the canonical SW zeta functions of Chapter III, §7, and the DR zeta functions introduced in §4.* To avoid confusion, let us denote the indeterminate "T" of Chapter III, §7, by T_{SW}. Thus, we would like to investigate the relationship between $\zeta_n^{\mathrm{PD}}[T_{\mathrm{SW}}]$ (cf. Chapter III, Theorem 7.4) and $\zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}]$ (Corollary 4.6). It follows immediately from the definitions that this essentially amounts to investigating the relationship between the *differential operators*:

$$\overline{\partial}^* + T_{\mathrm{DR}} \qquad \text{and} \qquad \delta^* + T_{\mathrm{SW}}$$

(where δ^* is as in Chapter III, §7). This relationship may be described as follows:

Lemma 5.2. *As operators on the* holomorphic sections *of $\mathcal{L}$ over $E^\dagger$, the above two operators are related as follows:*

$$L \cdot (\overline{\partial}^* + u \cdot \overline{\partial} + T_{\mathrm{DR}}) = \delta^* + T_{\mathrm{SW}}$$

for some $u, L \in \mathbf{C}$, where $|u| = 1$, $|L| = \{8\pi^2 \cdot \mathrm{Im}(\tau)\}^{-\frac{1}{2}}$.

Proof. Note that $\delta^* + T_{\mathrm{SW}}$ is (by definition) the tautological connection on $\mathcal{L}|_{E^\dagger}$ applied in the direction $(d\ \log(U))^\vee$. Put another way, this operator is the operator given by differentiation in the direction defined by $\lambda_1 \in \Lambda_1 \subseteq \Lambda_{\mathbf{C}} \cong H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$ (where we regard $H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$ as the tangent space to $E^\dagger$). By definition, the tangent space to $E_{\mathbf{R}} \subseteq E^\dagger$ is given by $\Lambda \otimes \mathbf{R} \hookrightarrow H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$. Thus, it follows that this tangent direction is contained inside the real analytic submanifold $E_{\mathbf{R}} \subseteq E^\dagger$ defined by the section $\kappa_{\mathbf{R}}$. Moreover, since λ_1 corresponds to $(v, u \cdot v^{\mathrm{c}})$ under the de Rham isomorphism, differentiation in this tangent direction on $E_{\mathbf{R}}$ is given by the operator $L \cdot (\overline{\partial}^* + u \cdot \overline{\partial})$, for some L, u as in the statement of Lemma 5.2. Note that here, L is defined by the relation $L \cdot \theta = (d\ \log(U))^\vee$. Similarly, relative to the decomposition of $E^\dagger$ induced by $\kappa_{\mathbf{R}}$, the tautological connection (applied in the tangent direction in question) at points of $E^\dagger$ with relative coordinate (i.e., the coordinate of $E^\dagger$ over E) T_{DR} is given by adding to the connection at the section $\kappa_{\mathbf{R}}$ the quantity "T_{DR} times $\frac{(d\ \log(U))^\vee}{\theta}$." But this quantity is simply $L \cdot T_{\mathrm{DR}}$, as desired. ○

Remark. Note that the equality of operators in Lemma 5.2 does not necessarily hold when applied to arbitrary *real analytic* sections of $\mathcal{L}$ over $E^\dagger$. Indeed, roughly speaking, the operator on the right-hand side of the formula in Lemma 5.2 is the "holomorphic portion" of the operator on the left-hand side of this formula.

We are now ready to *compute the precise relationship between $\zeta_n^{\mathrm{PD}}[T_{\mathrm{SW}}]$ and $\zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}]$*. It turns out that *the combinatorics of this relationship are described by the Hermite polynomials* (cf. Proposition 2.2).

Let us begin by observing first of all that both $\zeta_n^{\mathrm{PD}}[T_{\mathrm{SW}}]$ and $\zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}]$ are *holomorphic* (hence, in particular, real analytic) *sections of the line bundle $\mathcal{L}$ over $E^\dagger$*. Thus, it makes sense to compare them. Here, by $\zeta_n^{\mathrm{PD}}[T_{\mathrm{SW}}]$ we mean the *complex analytic version* of the canonical Schottky-Weierstrass zeta functions (cf. Chapter III, Remark 1 following Theorem 7.4). That is to say, these functions are the functions obtained by regarding the (clearly convergent – since differentiation does not effect convergence) formal series in q of Chapter III, §7, as holomorphic functions of $q = e^{2\pi i\tau}$ (where the family of elliptic curves in question is given by $\mathbf{C}^\times/q^{\mathbf{Z}}$, and $\tau \in \mathfrak{H} \stackrel{\mathrm{def}}{=} \{\tau \in \mathbf{C} \mid \mathrm{Im}(\tau) > 0\}$.

Theorem 5.3. *Let $n \geq 0$ be an integer. Choose $\zeta_0^{\mathrm{DR}}, \zeta_0^{\mathrm{PD}}$ to be the* same *section $\zeta_0 \in \Gamma(E, \mathcal{L})$. Then we have the following equality of holomorphic (hence, in particular, real analytic) sections of $\mathcal{L}$ over $E^\dagger$:*

$$\begin{aligned}\zeta_n^{\mathrm{PD}}[T_{\mathrm{SW}}] \stackrel{\mathrm{def}}{=} \frac{1}{n!} \cdot (\delta^* + T_{\mathrm{SW}})^n(\zeta_0^{\mathrm{PD}}) &= L^n \cdot \sum_{m=0}^{[n/2]} \frac{(\pi \cdot u)^m \cdot (\overline{\partial}^* + T_{\mathrm{DR}})^{n-2m}(\zeta_0^{\mathrm{DR}})}{m!\ (n-2m)!} \\ &= L^n \cdot \sum_{m=0}^{[n/2]} \frac{(\pi \cdot u)^m}{m!} \cdot \zeta_{n-2m}^{\mathrm{DR}}[T_{\mathrm{DR}}]\end{aligned}$$

where $u, L \in \mathbf{C}$ satisfy $|u| = 1$, $|L| = \{8\pi^2 \cdot \mathrm{Im}(\tau)\}^{-\frac{1}{2}}$.

Similarly,

$$\begin{aligned}\zeta_n^{\mathrm{DR}}[T_{\mathrm{DR}}] \stackrel{\mathrm{def}}{=} \frac{1}{n!} \cdot (\overline{\partial}^* + T_{\mathrm{DR}})^n(\zeta_0^{\mathrm{DR}}) &= \sum_{m=0}^{[n/2]} \frac{(-\pi \cdot u)^m \cdot (\delta^* + T_{\mathrm{SW}})^{n-2m}(\zeta_0^{\mathrm{PD}})}{L^{n-2m} \cdot m!\ (n-2m)!} \\ &= \sum_{m=0}^{[n/2]} \frac{(-\pi \cdot u)^m}{L^{n-2m} \cdot m!} \cdot \zeta_{n-2m}^{\mathrm{PD}}[T_{\mathrm{SW}}]\end{aligned}$$

Proof. Indeed, the first formula follows immediately by applying the identity

$$(A + B)^n \cdot v = n! \cdot \sum_{m=0}^{[n/2]} \frac{(c/2)^m \cdot B^{n-2m}}{m!\ (n-2m)!} \cdot v$$

(cf. Remark 1 following Proposition 2.2) with $A \stackrel{\mathrm{def}}{=} u \cdot \overline{\partial}$ and $B \stackrel{\mathrm{def}}{=} \overline{\partial}^* + T_{\mathrm{DR}}$ (which satisfy $A(\zeta_0) = u \cdot \overline{\partial}(\zeta_0) = 0$ and $[A, B] = u \cdot [\overline{\partial}, \overline{\partial}^*] = 2\pi \cdot u$ (by Proposition 4.3)) to compute

$$\frac{1}{n!} \cdot (\overline{\partial}^* + u \cdot \overline{\partial} + T_{\mathrm{DR}})^n(\zeta_0^{\mathrm{DR}}) = L^{-n} \cdot \frac{1}{n!} \cdot (\delta^* + T_{\mathrm{SW}})^n(\zeta_0^{\mathrm{PD}}) = L^{-n} \cdot \zeta_n^{\mathrm{PD}}[T_{\mathrm{SW}}]$$

(where the first equality follows from Lemma 5.2).

The second formula follows similarly, by letting $A \stackrel{\text{def}}{=} -u \cdot \overline{\partial}$, $B \stackrel{\text{def}}{=} L^{-1} \cdot (\delta^* + T_{\text{SW}})$. Thus, $A + B = \overline{\partial}^* + T_{\text{DR}}$ (cf. Lemma 5.2), $[A, B] = -2\pi \cdot u$ (cf. Proposition 4.3). Note that Lemma 5.2 applies here since the operator $\overline{\partial}$ (as we have defined it here) maps all of the holomorphic sections of $\mathcal{L}$ (over $E^{\dagger}$) in question to holomorphic sections of $\mathcal{L}$ (over $E^{\dagger}$). Indeed, this may be verified immediately by applying $\overline{\partial}$ to the expression on the right-hand side of the first formula of Corollary 4.6 (cf. also the proof of Corollary 4.6). $\bigcirc$

Corollary 5.4. *Let $n \geq 0$ be an integer. Choose $\zeta_0^{\text{DR}}, \zeta_0^{\text{PD}}$ to be the* same *section $\zeta_0 \in \Gamma(E, \mathcal{L})$. Then if we* rescale *the metric on τ_E so that $|T_{\text{DR}}| = \rho \in \mathbf{R}_{>0}$ (note that in the above discussion $\rho = 1$; cf. Corollary 4.6), then*

$$|| \ n! \cdot \zeta_n^{\text{PD}}[T_{\text{SW}}] \ ||_{L^2_{\text{DR}}} \leq ||\zeta_0||_{L^2_{\text{DR}}} \cdot e^{\pi} \cdot \text{Max}(1, \rho^n) \cdot \left(\frac{8\pi \cdot n}{\text{Im}(\tau)} \right)^{\frac{n}{2}}$$

Proof. We compute, using Theorem 5.3 and Corollary 4.6:

$$\begin{aligned}
|| \ n! \cdot \zeta_n^{\text{PD}}[T_{\text{SW}}] \ ||_{L^2_{\text{DR}}} &\leq |n^{\frac{1}{2}} \cdot L|^n \cdot \pi^n \cdot \sum_{m=0}^{[n/2]} \frac{n!}{n^{\frac{n}{2}} \cdot (m!)} \cdot || \ \zeta_{n-2m}^{\text{DR}}[T_{\text{DR}}] \ ||_{L^2_{\text{DR}}} \\
&\leq \left(\frac{n}{8\pi^2 \cdot \text{Im}(\tau)} \right)^{\frac{n}{2}} \cdot (2 \cdot \pi^2 \cdot 2\pi)^{\frac{n}{2}} \cdot e^{\pi} \cdot \text{Max}(1, \rho^n) \cdot || \ \zeta_0 \ ||_{L^2_{\text{DR}}} \\
&\qquad \cdot \sum_{m=0}^{[n/2]} \left(\frac{n! \cdot n!}{n^n \cdot (m!)^2 \cdot (n-2m)!} \right)^{\frac{1}{2}} \\
&\leq \left(\frac{n}{\text{Im}(\tau)} \right)^{\frac{n}{2}} \cdot (\pi/2)^{\frac{n}{2}} \cdot e^{\pi} \cdot \text{Max}(1, \rho^n) \cdot || \ \zeta_0 \ ||_{L^2_{\text{DR}}} \\
&\qquad \cdot \sum_{m=0}^{[n/2]} \left(\frac{n!}{(m!)^2 \cdot (n-2m)!} \right)^{\frac{1}{2}} \\
&\leq \left(\frac{n}{\text{Im}(\tau)} \right)^{\frac{n}{2}} \cdot (\pi/2)^{\frac{n}{2}} \cdot e^{\pi} \cdot \text{Max}(1, \rho^n) \cdot || \ \zeta_0 \ ||_{L^2_{\text{DR}}} \cdot 3^n \\
&\leq || \ \zeta_0 \ ||_{L^2_{\text{DR}}} \cdot e^{\pi} \cdot \text{Max}(1, \rho^n) \cdot \left(\frac{8\pi \cdot n}{\text{Im}(\tau)} \right)^{\frac{n}{2}}
\end{aligned}$$

as desired. $\bigcirc$

§6. Differential Calculus on the Theta-Weighted Circle

In this §, we commence our study of the *function theory of canonical SW zeta functions.* We begin by giving various *estimates* (which will be useful to us in Chapter VIII, as well as in the present §) of the *coefficients of the q-expansions* of several types of canonical SW zeta functions (Lemmas 6.1, 6.2). Next, we introduce what we call *orthogonal canonical SW zeta functions.* This new type of canonical SW zeta function is obtained by applying the orthogonalization process discussed in §1 (cf., especially, Example (4)) to (a slight variant of) the *divided power canonical SW zeta functions* of Chapter III, Theorem 7.4, and Chapter IV, Theorem 3.2. Certain limits (in which the scaling factor has *slope* $\frac{1}{2}$ – cf. the discussion surrounding "Terminology 3.7") of the orthogonal SW zeta functions give rise to canonical SW zeta functions based on the *Hermite polynomials* (Theorem 6.7). Finally, we relate this fact to the discussion of §5 (which may be summarized as stating that the de Rham zeta functions are essentially Hermite polynomial-based canonical SW zeta functions).

Write

$$A \stackrel{\mathrm{def}}{=} \mathbf{C}\{\{q_{\mathrm{sc}}\}\}$$

(i.e., convergent series in q_{sc}). In fact, in this §, we shall think of $q_{\mathrm{sc}} = \exp(2\pi i \tau_{\mathrm{sc}})$ as a complex variable on the *unit disc* $|q_{\mathrm{sc}}| < 1$, i.e., $\tau_{\mathrm{sc}} \in \mathfrak{H} \stackrel{\mathrm{def}}{=} \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$. Let $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A)$ (cf. the notation of Chapter IV, §2,3). We would like to think of q_{sc} as the q-parameter associated to the elliptic curve $\widetilde{E}$ of Chapter IV, §2,3. Thus, q_{sc} is equal to "q^n" in the notation of Chapter IV, §2,3. In the following, we shall write q_{cv} (where "cv" stands for "covering") for the "q" of Chapter IV, §2,3. Thus, in the present complex analytic context, the elliptic curve "E" (respectively, "$\widetilde{E}$") of Chapter IV, §2,3, corresponds to $E_{\mathrm{cv}} \stackrel{\mathrm{def}}{=} \mathbf{G}_{\mathrm{m}}/q_{\mathrm{cv}}^{\mathbf{Z}}$ (respectively, $E_{\mathrm{sc}} \stackrel{\mathrm{def}}{=} \mathbf{G}_{\mathrm{m}}/q_{\mathrm{sc}}^{\mathbf{Z}}$). Also, we shall write U_{cv} for the standard multiplicative coordinate on the copy $(\mathbf{G}_{\mathrm{m}})_{\mathrm{cv}}$ of $\mathbf{G}_{\mathrm{m}}$ which uniformizes $E_{\mathrm{cv}} = (\mathbf{G}_{\mathrm{m}})_{\mathrm{cv}}/q_{\mathrm{cv}}^{\mathbf{Z}}$, and U_{sc} for the multiplicative coordinate on the copy $(\mathbf{G}_{\mathrm{m}})_{\mathrm{sc}}$ of $\mathbf{G}_{\mathrm{m}}$ which uniformizes $E_{\mathrm{sc}} \stackrel{\mathrm{def}}{=} (\mathbf{G}_{\mathrm{m}})_{\mathrm{sc}}/q_{\mathrm{sc}}^{\mathbf{Z}}$. Thus,

$$U_{\mathrm{sc}} = U_{\mathrm{cv}}^n; \quad q_{\mathrm{sc}} = q_{\mathrm{cv}}^n$$

and $E_{\mathrm{sc}} = E_{\mathrm{cv}}/\boldsymbol{\mu}_n$.

Fix a *character*

$$\chi \in \mathrm{Hom}(\Pi_n, (\boldsymbol{\mu}_n)_S)$$

as in Chapter IV, Theorem 3.2, where $n = 2m \in 2\mathbf{Z}$. Let $\sigma_\chi \in \Gamma(C^\infty_{\widehat{S}}, (\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}})^\chi)$, $i_\chi \in \{-m, -m+1, \dots, m-1\}$ be as in Chapter V, Theorem 4.8. Note that the action of

$\mathbf{Z}_{\text{et}} \times \boldsymbol{\mu}_n$ on $(\mathcal{L}^{\otimes n}_{C^\infty_{\widehat{S}}})^\chi$ is compatible with the Galois action of $\mathbf{Z}_{\text{et}} \times \boldsymbol{\mu}_n$ on $C^\infty_{\widehat{S}}$, regarded as a covering space of the elliptic curve $\widetilde{E}$ of Chapter IV, §2,3. Thus, if we descend and then translate into the complex analytic language of the present section, we obtain a natural *line bundle*

$$\mathcal{L}^\chi_{\text{sc}}$$

of degree 1 on E_{sc}. Note that the section "θ^m" may then be thought of as a section of the line bundle $\mathcal{L}^\chi_{\text{sc}}$ over the cover

$$(\mathbf{G}_{\text{m}})_{\text{cv}} \to (\mathbf{G}_{\text{m}})_{\text{sc}} \to E_{\text{sc}}$$

(where $(\mathbf{G}_{\text{m}})_{\text{cv}} \to (\mathbf{G}_{\text{m}})_{\text{sc}}$ is the map $U_{\text{cv}} \mapsto U_{\text{sc}} = U^n_{\text{cv}}$). Thus, if we divide the series for σ_χ in Chapter V, Theorem 4.8, by θ^m, we obtain the (twisted) *theta function* Θ_χ:

$$\sum_{k \in \mathbf{Z}} q_{\text{sc}}^{\frac{1}{2}\cdot k^2 + (i_\chi/n)\cdot k} \cdot U_{\text{cv}}^{2mk+i_\chi} \cdot \chi(k_{\text{et}}) = \sum_{k \in \mathbf{Z}} q_{\text{or}}^{\frac{1}{d}(\frac{1}{2}\cdot k^2 + (i_\chi/n)\cdot k)} \cdot U_{\text{cv}}^{2mk+i_\chi} \cdot \chi(k_{\text{et}})$$

where $q_{\text{or}} \stackrel{\text{def}}{=} q^d_{\text{sc}}$, and d is a *fixed positive integer.* (We shall also write $\tau_{\text{or}} \stackrel{\text{def}}{=} d \cdot \tau_{\text{sc}}$.) The point of introducing these two "q's" (i.e., q_{sc} and q_{or}) is that we want to think of ourselves as being interested in *degree d line bundles on the elliptic curve* $E_{\text{or}} \stackrel{\text{def}}{=} \mathbf{G}_{\text{m}}/q^{\mathbf{Z}}_{\text{or}}$. To study such bundles, it is convenient to take invariants with respect to "$\mathbf{Z}_{\text{et}}/d$" (cf. Chapter IV, Theorem 1.4) and work with degree d line bundles on the quotient $E_{\text{or}}/(\mathbf{Z}_{\text{et}}/d) = E_{\text{sc}}$. Thus, we would like to think of the d as a *scaling factor.* That is to say, q_{or} is our "*original q*," while q_{sc} is our "*scaled q*."

Recall the *differential operator* δ^* of Chapter IV, Theorem 3.2. This operator satisfies $\delta^*(U^N_{\text{cv}}) = N \cdot U^{n\cdot N}_{\text{cv}}$ (for $N \in \mathbf{Z}$). Let us define the "*χ-shifted operator*"

$$\delta^*_\chi \stackrel{\text{def}}{=} \delta^* - \frac{i_\chi}{n}$$

Thus,

$$\delta^*_\chi(U^{n\cdot N+i_\chi}_{\text{cv}}) = N \cdot U^{n\cdot N+i_\chi}_{\text{cv}}$$

(for $N \in \mathbf{Z}$). Our first goal in this § is *to bound the coefficients in the expansion of the "shifted and scaled monomial" canonical SW zeta functions*

$$\zeta^{\text{SS}}_r \cdot \theta^{-m} \stackrel{\text{def}}{=} \left(\frac{\delta^*_\chi}{d}\right)^r (\Theta_\chi) = \sum_{k \in \mathbf{Z}} \left(\frac{k}{d}\right)^r \cdot q_{\text{or}}^{\frac{1}{d}(\frac{1}{2}\cdot k^2 + (i_\chi/n)\cdot k)} \cdot U_{\text{cv}}^{2mk+i_\chi} \cdot \chi(k_{\text{et}})$$

(where r is a nonnegative integer).

Lemma 6.1. *Let a be a positive integer. Then the coefficient C_k^{SS} of $U_{\mathrm{cv}}^{2mk+i_\chi}$ in ζ_r^{SS} satisfies the following:*

(1) *If $|k| \leq a \cdot d$, then $|C_k^{\mathrm{SS}}| \leq e^{-\frac{\pi}{d}\cdot \mathrm{Im}(\tau_{\mathrm{or}})\cdot(k^2+(i_\chi/m)\cdot k)} \cdot a^r \leq a^r$.*

(2) *Let $\epsilon > 0$ be a real number. If $|k| \geq \mathrm{Max}(\frac{1+\epsilon}{\epsilon}, a \cdot d)$, $r \leq d$, and $a \geq \frac{1}{2} \cdot (b+1) \cdot Q^{-1}$, where $b \in \mathbf{R}_{\geq 1}$, and*

$$Q \stackrel{\mathrm{def}}{=} \frac{\pi}{2(1+\epsilon)} \cdot \mathrm{Im}(\tau_{\mathrm{or}})$$

then $|C_k^{\mathrm{SS}}| \leq e^{-b\cdot|k|}$. In particular, in this case, we have:

$$\sum_{|k|=a\cdot d}^{\infty} |C_k^{\mathrm{SS}}| \leq 4 \cdot e^{-a\cdot b\cdot d}; \qquad \sum_{|k|=a\cdot d}^{\infty} |C_k^{\mathrm{SS}}|^2 \leq 4 \cdot e^{-2a\cdot b\cdot d}$$

Proof. The estimates of (1) follow immediately from the facts that $|q_{\mathrm{or}}| \leq 1$; $|k| \leq a\cdot d \implies (|k|/d)^r \leq a^r$.

The estimates of (2) are obtained as follows: First, we observe that since $|k| \geq a \cdot d$, $|k| \geq \frac{1+\epsilon}{\epsilon}$ (so $k^2 - |k| \geq \frac{1}{1+\epsilon} \cdot k^2$), $r \leq d$, and $a \cdot Q \geq \frac{1}{2} \cdot (b+1)$,

$$\begin{aligned}
\log(|C_k^{\mathrm{SS}}|) &= r \cdot \log(|k|/d) - 2\pi \cdot \mathrm{Im}(\tau_{\mathrm{or}}) \cdot \frac{1}{2d}(k^2 + (2 \cdot i_\chi/n) \cdot k) \\
&\leq r \cdot \log(|k|/r) - 2\pi \cdot \mathrm{Im}(\tau_{\mathrm{or}}) \cdot \frac{1}{2d}(k^2 - |k|) \\
&\leq |k| - 2\pi \cdot \mathrm{Im}(\tau_{\mathrm{or}}) \cdot \frac{k^2}{2d(1+\epsilon)} \\
&= |k| - 2Q \cdot (|k|/d) \cdot |k| \\
&\leq |k| - 2Q \cdot a \cdot |k| \\
&\leq -b \cdot |k|
\end{aligned}$$

(where in the third line, we use Lemma 3.6), as desired. The inequalities concerning the sums then follow immediately. Note that here we use the facts that

$$\sum_{k=0}^{\infty} e^{-k} = (1-e^{-1})^{-1} \leq 2; \quad \sum_{k=0}^{\infty} e^{-2k} = (1-e^{-2})^{-1} \leq 2$$

(which follow from $e^2 \geq e \geq 2$). $\bigcirc$

Next, although it will not be logically necessary in the following discussion, we observe that the estimates of Lemma 6.1 allow us to *estimate the coefficients of the congruence canonical SW zeta functions of Chapter V, Theorem 4.8* (cf. Chapter VIII, §3,4,5, for more details on the archimedean properties of these functions):

Lemma 6.2. *The coefficient $C^{\mathrm{CG}}[r]_k$ of $U_{\mathrm{cv}}^{2mk+i_\chi}$ in the series*

$$\zeta_r^{\mathrm{CG}} \cdot \theta^{-m} = \binom{\delta_\chi^* + \lambda_r}{r}(\Theta_\chi) = \sum_{k \in \mathbf{Z}} \binom{k+\lambda_r}{r} \cdot q_{\mathrm{sc}}^{\frac{1}{2}\cdot k^2 + (i_\chi/n)\cdot k} \cdot U_{\mathrm{cv}}^{2mk+i_\chi} \cdot \chi(k_{\mathrm{et}})$$

of Chapter V, Theorem 4.8, satisfies the following:

(1) *If $|k| \leq a \cdot d$, then $|C^{\mathrm{CG}}[r]_k| \leq a^r \cdot e^{3r} \cdot \left(\frac{d}{r}\right)^r \leq a^r \cdot e^{3r+d}$.*

(2) *Let $\epsilon > 0$ be a real number. If $|k| \geq \mathrm{Max}(\frac{1+\epsilon}{\epsilon}, a \cdot d)$, $1 \leq r \leq d$, and $a \geq \frac{1}{2} \cdot (b+5) \cdot Q^{-1}$, where $b \in \mathbf{R}_{\geq 1}$, and*

$$Q \stackrel{\mathrm{def}}{=} \frac{\pi}{2(1+\epsilon)} \cdot \mathrm{Im}(\tau_{\mathrm{or}})$$

then $|C^{\mathrm{CG}}[r]_k| \leq e^{-(b+1)\cdot|k|} \cdot \left(\frac{d}{r}\right)^r \leq e^{-b\cdot|k|}$. In particular, in this case, we have:

$$\sum_{|k|=a\cdot d}^{\infty} |C^{\mathrm{CG}}[r]_k| \leq 4 \cdot e^{-a\cdot b\cdot d}; \quad \sum_{|k|=a\cdot d}^{\infty} |C^{\mathrm{CG}}[r]_k|^2 \leq 4 \cdot e^{-2a\cdot b\cdot d}$$

Proof. To apply Lemma 6.1, it suffices to estimate the coefficients of the polynomial $\binom{T+\lambda_r}{r}$ in terms of $(T/d)^i$. One sees easily (since $|\lambda_r| \leq \frac{r}{2}$; for $N = 0, 1, \ldots, r-1$,

$|-N+\lambda_r| \leq r$) that the estimates of Proposition 3.3 for the coefficients of the polynomial $\binom{T}{r}$ also hold for $\binom{T+\lambda_r}{r}$ (by the same proof as that given for Proposition 3.3). Thus, these coefficients may be bounded by $e^{2r} \cdot \left(\frac{d}{r}\right)^r \leq e^{2r+d}$. Since there are a total of $r+1 \leq e^r$ terms to contend with, we thus see that in order to get estimates for $C^{\mathrm{CG}}[r]_k$, it suffices to multiply the estimates for C_k^{SS} in Lemma 6.1 by $e^{3r} \cdot \left(\frac{d}{r}\right)^r \leq e^{3r+d}$. This takes care of (1). For (2), we note further that if we take the "b" of Lemma 6.1 to be $b+4$ (in the notation of the present Corollary), then using $r \leq d \leq a \cdot d \leq |k|$, we obtain

$$
\begin{aligned}
|C^{\mathrm{CG}}[r]_k| &\leq e^{-(b+4)\cdot|k|+3r} \cdot \left(\frac{d}{r}\right)^r \\
&\leq e^{-(b+4)\cdot|k|+3\cdot|k|} \cdot \left(\frac{d}{r}\right)^r \\
&\leq e^{-(b+1)\cdot|k|} \cdot \left(\frac{d}{r}\right)^r \\
&\leq e^{-(b+1)\cdot|k|+d} \leq e^{-(b+1)\cdot|k|+|k|} = e^{-b\cdot|k|}
\end{aligned}
$$

as desired. ○

Next, we would like to introduce the *orthogonal canonical SW zeta functions*, as follows. First, in the following discussion, we would like to restrict the variable $U_{\mathrm{cv}} \in \mathbf{C}^\times$ to the *unit circle*; we shall write $U_{\mathrm{cv}}|_{(\mathbf{S^1})_{\mathrm{cv}}} = \exp(2\pi i t_{\mathrm{cv}})$ (where $t_{\mathrm{cv}} \in [0, 2\pi)$) for this restricted U_{cv}. Then observe that $\Theta_\chi|_{(\mathbf{S^1})_{\mathrm{cv}}}$ is a smooth function on $(\mathbf{S^1})_{\mathrm{cv}}$ (varying with the parameter q). Recall that $L^2((\mathbf{S^1})_{\mathrm{cv}})$ is equipped with a natural inner product given by

$$
(f, g) \stackrel{\mathrm{def}}{=} \frac{1}{2\pi} \cdot \int_{(\mathbf{S^1})_{\mathrm{cv}}} f \cdot \overline{g} \cdot dt
$$

Now we would like to apply the theory of §1, especially Example (4). That is to say, we take $\mathcal{H} \stackrel{\mathrm{def}}{=} L^2((\mathbf{S^1})_{\mathrm{cv}})$, $X \stackrel{\mathrm{def}}{=} \delta^*$, and (for $r \geq 0$) $F^r(\mathcal{H})$ to be the complex vector subspace of $\mathcal{H}$ generated by functions of the form $P(\delta_\chi^*) \cdot \Theta_\chi$, where $P(-)$ is a polynomial (with complex coefficients) of degree $\leq r$. Put another way, we would like to consider the ($\mathbf{C}$-)linear combinations of the *"standard monomial" canonical SW zeta functions*

$$
\begin{aligned}
\zeta_r^{\mathrm{SM}} \cdot \theta^{-m} \stackrel{\mathrm{def}}{=} \left(\delta^*\right)^r(\Theta_\chi) &= \sum_{k \in \mathbf{Z}} (k + \frac{i_\chi}{n})^r \cdot q_{\mathrm{or}}^{\frac{1}{d}(\frac{1}{2}\cdot k^2 + (i_\chi/n)\cdot k)} \cdot U_{\mathrm{cv}}^{2mk+i_\chi} \cdot \chi(k_{\mathrm{et}}) \\
&= q_{\mathrm{sc}}^{-\frac{1}{2}\cdot(i_\chi/n)^2} \cdot \sum_{k_\chi \in \mathbf{Z}+\frac{i_\chi}{n}} k_\chi^r \cdot q_{\mathrm{sc}}^{\frac{1}{2}k_\chi^2} \cdot U_{\mathrm{sc}}^{k_\chi} \cdot \chi((k_\chi - \frac{i_\chi}{n})_{\mathrm{et}}) \\
&= q_{\mathrm{or}}^{-\frac{1}{2d}\cdot(i_\chi/n)^2} \cdot \sum_{k_\chi \in \mathbf{Z}+\frac{i_\chi}{n}} k_\chi^r \cdot q_{\mathrm{or}}^{\frac{1}{2d}k_\chi^2} \cdot U_{\mathrm{sc}}^{k_\chi} \cdot \chi((k_\chi - \frac{i_\chi}{n})_{\mathrm{et}})
\end{aligned}
$$

(where $U_{\mathrm{sc}} \stackrel{\mathrm{def}}{=} U_{\mathrm{cv}}^n$) whose restrictions to $(\mathbf{S^1})_{\mathrm{cv}}$ are *orthonormal with respect to the inner product just defined on* $L^2((\mathbf{S^1})_{\mathrm{cv}})$. If we normalize the orthonormal system that we deal with by assuming that the leading coefficient (i.e., the coefficient of ζ_r^{SM}) in the degree r member of this system is a *positive real number*, then we see that we obtain uniquely defined functions

$$\zeta_0^{\mathrm{OR},\mathbf{S^1}}, \ldots, \zeta_r^{\mathrm{OR},\mathbf{S^1}}, \ldots$$

(for $r \geq 0$). Because these functions are a sort of *prototype* of the orthogonal functions that we will consider in Chapter VIII(cf., especially, §2), we give them a name:

Definition 6.3. We refer to the $\zeta_r^{\mathrm{OR},\mathbf{S^1}}$ as the *orthogonal canonical Schottky-Weierstrass zeta functions.*

Note that it follows from the general theory of §1 (e.g., Proposition 1.2) that the $\zeta_r^{\mathrm{OR},\mathbf{S^1}}$ are *completely determined by their means and principal submeans.*

In the following discussion, we would like to consider the trivialization θ^m of the line bundle $\mathcal{L}_{\mathrm{sc}}^{\chi}$ over the covering $(\mathbf{G}_{\mathrm{m}})_{\mathrm{cv}} \to (\mathbf{G}_{\mathrm{m}})_{\mathrm{sc}} \to E_{\mathrm{sc}}$, where $(\mathbf{G}_{\mathrm{m}})_{\mathrm{cv}} \to (\mathbf{G}_{\mathrm{m}})_{\mathrm{sc}}$ is the map $U_{\mathrm{cv}} \mapsto U_{\mathrm{sc}} = U_{\mathrm{cv}}^n$. Note that this line bundle $\mathcal{L}_{\mathrm{sc}}^{\chi}$ on the elliptic curve E_{sc} admits a *unique* (up to a positive real multiple) *metric with translation-invariant curvature* (cf. the discussion of the metric $|| \sim ||_{\mathcal{L}}$ in §4). This metric thus defines a function $||\theta^m||$ of U_{cv}. *For simplicity, let us assume that our metric has been normalized so that* $||\theta^m|| = 1$ *at* $U_{\mathrm{cv}} = 1$. Note that since the *natural action* of $\boldsymbol{\mu}_{\infty}$ on "$\mathcal{L}^{\otimes n}_{C^{\infty}_{\widehat{S}}}$" (cf. the discussion at the beginning of Chapter IV, §2) fixes θ^m (as well as the metric $|| \sim ||$), we thus see that $||\theta^m||$ is invariant under translation by elements of $(\mathbf{S^1})_{\mathrm{cv}}$. In particular, it follows that *we may regard* $||\theta^m||$ *as a function of* U_{sc}.

Lemma 6.4. *At* $U_{\mathrm{sc}} = q_{\mathrm{sc}}^{\rho} \cdot \exp(2\pi i t_{\mathrm{cv}})$ *(where* $t_{\mathrm{cv}} \in [0, 2\pi)$, $\rho \in \mathbf{R}_{\geq 0}$*), we have*

$$||\theta^m|| = |q_{\mathrm{sc}}|^{\frac{1}{2}\cdot\rho^2}$$

Proof. First, observe that the section θ^m (of the line bundle $\mathcal{L}_{\mathrm{sc}}^{\chi}$ over the cover $(\mathbf{G}_{\mathrm{m}})_{\mathrm{cv}} \to (\mathbf{G}_{\mathrm{m}})_{\mathrm{sc}} \to E_{\mathrm{sc}}$) is *nonzero* for all values of $U_{\mathrm{cv}} \in \mathbf{C}^{\times}$ (i.e., it really is a "trivialization"). Indeed, this follows, for instance, from the fact that the holomorphic connection defined by θ is *regular* (i.e., has no poles) for all $U_{\mathrm{cv}} \in \mathbf{C}^{\times}$ (cf. Chapter III, Theorem 5.6), which would not be the case if θ had zeroes. Thus, θ^m constitutes a trivialization of the line bundle in question at all values of $U_{\mathrm{cv}} \in \mathbf{C}^{\times}$. In particular, since the curvature of the metric that we are using is translation-invariant (i.e., "constant"), it follows that the *Laplacian* $\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial t^2}$

of the function $\phi(\rho,t) \stackrel{\text{def}}{=} \log(||\theta^m||)$ (where $U_{\text{sc}} = q_{\text{sc}}^{\rho} \cdot \exp(2\pi i t_{\text{cv}})$) is *constant.* Since $\phi(\rho,t) = \phi(\rho)$ is independent of t_{cv}, we thus obtain that $\frac{d^2\phi}{d\rho^2}$ is *constant.* But this implies that $\phi(\rho)$ is a *quadratic function* of ρ, hence determined as soon as it is determined for $U_{\text{sc}} = q_{\text{sc}}^{\mathbf{Z}}$, i.e., for $U_{\text{cv}} = q_{\text{cv}}^{\mathbf{Z}}$. Thus, it suffices to prove Lemma 6.4 for such values of U_{sc}. But for $U_{\text{cv}} = q_{\text{cv}}^{N}$, we may calculate $||\theta^m||$ as the value of $||N_{\text{et}}(\theta^m)|| = ||q_{\text{cv}}^{m\cdot N^2} \cdot U_{\text{cv}}^{n\cdot N} \cdot \theta^m||$ at $U_{\text{cv}} = 1$, which is just $|q_{\text{cv}}|^{m\cdot N^2} = |q_{\text{sc}}|^{\frac{1}{2}\cdot N^2}$, as desired (since $U_{\text{cv}} = q_{\text{cv}}^{N}$ corresponds to $U_{\text{sc}} = q_{\text{sc}}^{N}$). This completes the proof. $\bigcirc$

Now that we have computed the absolute value of our trivialization θ^m, we would like compute various L^2*-norms of canonical zeta functions* as follows. First, let

$$P(\mathbf{s})$$

be a polynomial with *complex coefficients in the indeterminate* $\mathbf{s}$. Then let us write

$$\zeta^P[T]\cdot\theta^{-m} \stackrel{\text{def}}{=} P(\delta^*+T)\cdot(\Theta_\chi) = q_{\text{sc}}^{-\frac{1}{2}\cdot(i_\chi/n)^2}\cdot \sum_{k_\chi \in \mathbf{Z}+\frac{i_\chi}{n}} P(k_\chi+T)\cdot q_{\text{sc}}^{\frac{1}{2}k_\chi^2}\cdot U_{\text{sc}}^{k_\chi}\cdot\chi((k_\chi-\frac{i_\chi}{n})_{\text{et}})$$

(where the indeterminate "T" is as in Chapter IV, §3). We would like to consider *the value of this zeta function at* $U_{\text{cv}} = q_{\text{cv}}^{\rho} \cdot \exp(2\pi i t_{\text{cv}})$ (where $t_{\text{cv}} \in [0,2\pi)$, $\rho \in \mathbf{R}_{\geq 0}$), i.e., $U_{\text{sc}} = U_{\text{cv}}^{n} = q_{\text{sc}}^{\rho} \cdot \exp(2\pi i t_{\text{cv}} \cdot n)$. Since at this value of U_{cv}, the indeterminate T takes on the value ρ (cf. Chapter III, Corollary 5.9, for rational ρ; since T is a *continuous* function, we thus obtain the result for arbitrary real ρ). Thus,

$$\zeta^P[T]\cdot\theta^{-m}|_{U=q_{\text{cv}}^{\rho}\cdot\exp(2\pi i t_{\text{cv}})} = q_{\text{sc}}^{-\frac{1}{2}\cdot(i_\chi/n)^2}\cdot$$
$$\sum_{k_\chi \in \mathbf{Z}+\frac{i_\chi}{n}} P(k_\chi+\rho)\cdot q_{\text{sc}}^{\frac{1}{2}k_\chi^2+k_\chi\cdot\rho}\cdot\exp(2\pi i t_{\text{cv}}\cdot n\cdot k_\chi)\cdot\chi((k_\chi-\frac{i_\chi}{n})_{\text{et}})$$

In particular, by applying Lemma 6.4, for $U_{\text{cv}} = q_{\text{cv}}^{\rho} \cdot \exp(2\pi i t_{\text{cv}})$, we obtain:

$$\begin{aligned}
||\zeta^P[T]|| &= |q_{\text{sc}}|^{-\frac{1}{2}\cdot(i_\chi/n)^2+\frac{1}{2}\cdot\rho^2} \\
&\quad \cdot\Big|\sum_{k_\chi \in \mathbf{Z}+\frac{i_\chi}{n}} P(k_\chi+\rho)\cdot q_{\text{sc}}^{\frac{1}{2}k_\chi^2+k_\chi\cdot\rho}\cdot\exp(2\pi i t_{\text{cv}}\cdot n\cdot k_\chi)\cdot\chi((k_\chi-\frac{i_\chi}{n})_{\text{et}})\Big| \\
&= |q_{\text{sc}}|^{-\frac{1}{2}\cdot(i_\chi/n)^2} \\
&\quad \cdot\Big|\sum_{k_{\chi,\rho} \in \mathbf{Z}+\rho+\frac{i_\chi}{n}} P(k_{\chi,\rho})\cdot q_{\text{sc}}^{\frac{1}{2}k_{\chi,\rho}^2}\cdot\exp(2\pi i t_{\text{cv}}\cdot n\cdot k_{\chi,\rho})\cdot\chi((k_{\chi,\rho}-\rho-\frac{i_\chi}{n})_{\text{et}})\Big|
\end{aligned}$$

Thus, by squaring and integrating over $(\mathbf{S^1})_{\mathrm{cv}}$, we obtain the following:

Lemma 6.5. *Write* $U_{\mathrm{cv}} = q_{\mathrm{cv}}^{\rho} \cdot \exp(2\pi i t_{\mathrm{cv}})$, *where we fix* ρ, *and let* $\exp(2\pi i t_{\mathrm{cv}}) \in (\mathbf{S^1})_{\mathrm{cv}}$ *vary. Then for* $|U_{\mathrm{cv}}| = |q_{\mathrm{cv}}|^{\rho}$, *we have:*

$$||\zeta^P[T]||^2_{L^2((\mathbf{S^1})_{\mathrm{cv}})} = |q_{\mathrm{sc}}|^{-(i_\chi/n)^2} \cdot \sum_{k_{\chi,\rho} \;\in\; \mathbf{Z}+\rho+\frac{i_\chi}{n}} |P(k_{\chi,\rho})|^2 \cdot |q_{\mathrm{sc}}|^{k^2_{\chi,\rho}}$$

If, moreover, we integrate these $L^2((\mathbf{S^1})_{\mathrm{cv}})$ *norms as* ρ *ranges from* 0 *to* 1, *we obtain:*

$$\begin{aligned} ||\zeta^P[T]||^2_{L^2(E_{\mathrm{sc}})} &= |q_{\mathrm{sc}}|^{-(i_\chi/n)^2} \cdot \int_{k_\chi \;\in\; \mathbf{R}+\frac{i_\chi}{n}} |P(k_\chi)|^2 \cdot |q_{\mathrm{sc}}|^{k_\chi^2} \cdot dk_\chi \\ &= e^{\frac{2\pi}{d}\cdot \mathrm{Im}(\tau_{\mathrm{or}})\cdot(i_\chi/n)^2} \cdot \int_{\mathbf{R}} |P(k)|^2 \cdot e^{-\frac{2\pi}{d}\cdot\mathrm{Im}(\tau_{\mathrm{or}})\cdot k^2} \cdot dk \end{aligned}$$

(where the L^2*-norm* $L^2(E_{\mathrm{sc}})$ *is defined by integrating the* $L^2((\mathbf{S^1})_{\mathrm{cv}})$ *norms as a function of* ρ, *where* ρ *ranges from* 0 *to* 1*).*

Next, we would like to see what happens if we *replace* $(\mathbf{S^1})_{\mathrm{cv}}$ *in the above discussion by* $\boldsymbol{\mu}_d$. Let us write $t_{\mathrm{sc}} \stackrel{\mathrm{def}}{=} n \cdot t_{\mathrm{cv}}$, and think of $\boldsymbol{\mu}_d \subseteq (\mathbf{S^1})_{\mathrm{sc}}$ as a subset of the circle $(\mathbf{S^1})_{\mathrm{sc}}$ with coordinate $\exp(2\pi i t_{\mathrm{sc}})$. Then by restricting ζ^P to $\boldsymbol{\mu}_d$ and dividing by a(n) (unnecessary) factor of $\exp(2\pi i \cdot i_\chi \cdot t_{\mathrm{cv}}) \cdot \theta^m$, we obtain

$$\begin{aligned} \zeta^{P,\boldsymbol{\mu}_d} &\stackrel{\mathrm{def}}{=} \exp(-2\pi i \cdot i_\chi \cdot t_{\mathrm{cv}}) \cdot \theta^{-m} \cdot P(\delta^*) \cdot (\Theta_\chi)|_{\boldsymbol{\mu}_d} \\ &= q_{\mathrm{sc}}^{-\frac{1}{2}\cdot(i_\chi/n)^2} \cdot \sum_{k_{\mathrm{mod}} \in \mathbf{Z}/d\mathbf{Z}} \exp(2\pi i t_{\mathrm{sc}} \cdot k_{\mathrm{mod}}) \cdot \sum_{k_\chi \;\in\; k_{\mathrm{mod}}+\frac{i_\chi}{n}} P(k_\chi) \cdot q_{\mathrm{sc}}^{\frac{1}{2}k_\chi^2} \\ &\quad \cdot \chi((k_\chi - \frac{i_\chi}{n})_{\mathrm{et}}) \end{aligned}$$

Let us define the *inner product* on $L^2(\boldsymbol{\mu}_d)$ by

$$(f, g) \stackrel{\mathrm{def}}{=} \frac{1}{d} \cdot \sum_{\alpha \in \boldsymbol{\mu}_d} f(\alpha) \cdot \overline{g}(\alpha)$$

Note that, just as in Lemma 6.5, we have:

$$||\zeta^{P,\boldsymbol{\mu}_d}||^2_{L^2(\boldsymbol{\mu}_d)} = |q_{\mathrm{sc}}|^{-(i_\chi/n)^2} \cdot \sum_{k_{\mathrm{mod}} \in \mathbf{Z}/d\mathbf{Z}} \left| \sum_{k_\chi \;\in\; k_{\mathrm{mod}}+\frac{i_\chi}{n}} P(k_\chi) \cdot q_{\mathrm{sc}}^{\frac{1}{2}k_\chi^2} \cdot \chi((k_\chi - \frac{i_\chi}{n})_{\mathrm{et}}) \right|^2$$

Now assume (just for this discussion of functions on $\boldsymbol{\mu}_d$) that $i_\chi \neq 0$. By Chapter VI, Theorem 3.1, (2), this assumption implies that as $P(-)$ ranges over the polynomials of degree $< d$, the resulting $\zeta^{P,\boldsymbol{\mu}_d}$ form a basis of $L^2(\boldsymbol{\mu}_d)$. Thus, we may orthonormalize these $\zeta^{P,\boldsymbol{\mu}_d}$'s to obtain

$$\zeta_0^{\mathrm{OR},\boldsymbol{\mu}}, \ldots, \zeta_{d-1}^{\mathrm{OR},\boldsymbol{\mu}}$$

Definition 6.6. We refer to the $\zeta_r^{\mathrm{OR},\boldsymbol{\mu}}$ as the *$\boldsymbol{\mu}_d$-orthogonal canonical Schottky-Weierstrass zeta functions.*

We are now ready to state the *main result* of this §: First, let us write $\mathcal{L}_{\mathrm{sc}}^\chi|_{(\mathbf{S}^1)_{\mathrm{cv}}}$ for the restriction of $\mathcal{L}_{\mathrm{sc}}^\chi$ to the circle $(\mathbf{S}^1)_{\mathrm{cv}} \subseteq (\mathbf{G}_{\mathrm{m}})_{\mathrm{cv}}$ which is parametrized by U_{cv}. Note that this bundle $\mathcal{L}_{\mathrm{sc}}^\chi|_{(\mathbf{S}^1)_{\mathrm{cv}}}$ over $(\mathbf{S}^1)_{\mathrm{cv}}$ admits a natural trivialization defined by θ^m. In the following theorem, we would like to consider the analogue for the line bundle $\mathcal{L}_{\mathrm{sc}}^\chi$ on E_{sc} of the real analytic sections ζ_r^{DR} of §4, 5, of the line bundle associated to the origin of an elliptic curve. Since the pair $(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^\chi)$ is *isomorphic* (by translation) to the pair $(E_{\mathrm{sc}}, \mathcal{O}_{E_{\mathrm{sc}}}(e_{E_{\mathrm{sc}}}))$ (where $e_{E_{\mathrm{sc}}}$ is the origin of E), we thus see that, by transport of structure, *the theory of §4,5, also applies to the pair* $(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^\chi)$. Thus, by abuse of notation, we denote by ζ_r^{DR} the resulting real analytic section of $\mathcal{L}_{\mathrm{sc}}^\chi$ on E_{sc} (associated to the choice of ζ_0^{DR} defined by σ_χ). The following theorem concerns the restriction $\zeta_r^{\mathrm{DR}}|_{(\mathbf{S}^1)_{\mathrm{cv}}}$ of ζ_r^{DR} to $(\mathbf{S}^1)_{\mathrm{cv}} \subseteq (\mathbf{G}_{\mathrm{m}})_{\mathrm{cv}}$.

Theorem 6.7. *Let $r \geq 0$ be an integer. Let us* fix q_{or}, *and consider $d \geq 1$ as* variable. *Write*

$$\gamma_d \stackrel{\mathrm{def}}{=} \left\{ \frac{d}{4\pi \cdot \mathrm{Im}(\tau_{\mathrm{or}})} \right\}^{\frac{1}{2}}$$

and $P_{r,d}(\mathbf{s}) \stackrel{\mathrm{def}}{=} H_r(\mathbf{s} \cdot \gamma_d^{-1})$, where $H_r(-)$ is the Hermite polynomial *of Proposition 2.2. Let*

$$\zeta_r^{\mathrm{HM}_d} \stackrel{\mathrm{def}}{=} \zeta^{P_{r,d}}; \quad \zeta_r^{\mathrm{HM}_d,\boldsymbol{\mu}_d} \stackrel{\mathrm{def}}{=} \zeta^{P_{r,d},\boldsymbol{\mu}_d}$$

Then if we regard sections of $\mathcal{L}_{\mathrm{sc}}^\chi|_{(\mathbf{S}^1)_{\mathrm{cv}}}$ as functions on $(\mathbf{S}^1)_{\mathrm{cv}}$ by means of the trivialization θ^m, we have:

$$\lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot (r!)^{\frac{1}{2}} \cdot \zeta_r^{\mathrm{OR},\mathbf{S}^1} = \lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot \zeta_r^{\mathrm{HM}_d} = \lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot \frac{r!}{(2\pi)^{r/2}} \cdot \zeta_r^{\mathrm{DR}}$$

(where the convergence is relative to the sup norm for functions on $(\mathbf{S}^1)_{\mathrm{cv}}$). If, moreover, $i_\chi \neq 0$, then (if we trivialize $\mathcal{L}_{\mathrm{sc}}^\chi|_{\boldsymbol{\mu}_\infty}$ by means of $\exp(2\pi i \cdot i_\chi \cdot t_{\mathrm{cv}}) \cdot \theta^m$*) we have:*

$$\lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot (r!)^{\frac{1}{2}} \cdot \zeta_r^{\mathrm{OR},\boldsymbol{\mu}} = \lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot \zeta_r^{\mathrm{HM}_d,\boldsymbol{\mu}_d} = \lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot \frac{r!}{(2\pi)^{r/2}} \cdot \zeta_r^{\mathrm{DR}}|_{\boldsymbol{\mu}_d}$$

(where the convergence is relative to the sup norm for functions on $\boldsymbol{\mu}_\infty$).

Remark. It is not difficult to see that in the latter part of Theorem 6.7, if χ is of order precisely m, and one takes the limit over d which are *not divisible* by m, then the latter part of Theorem 6.7 holds even if $i_\chi = 0$. Indeed, the only place where we used/will use that $i_\chi \neq 0$ is in the application of Chapter VI, Theorem 3.1, (2).

Proof. Let us first consider the $(\mathbf{S^1})_{\mathrm{cv}}$ case. Recall from Lemma 6.5 that for $|U_{\mathrm{cv}}| = |q_{\mathrm{cv}}|^\rho$, $\rho = 0$, we have:

$$\begin{aligned}
\gamma_d^{-1} \cdot ||\zeta^P||^2_{L^2((\mathbf{S^1})_{\mathrm{cv}})} &= \gamma_d^{-1} \cdot e^{\frac{2\pi}{d}\cdot \mathrm{Im}(\tau_{\mathrm{or}})\cdot(i_\chi/n)^2} \cdot \sum_{k_\chi \in \mathbf{Z}+\frac{i_\chi}{n}} |P(k_\chi)|^2 \cdot e^{-\frac{2\pi}{d}\cdot\mathrm{Im}(\tau_{\mathrm{or}})\cdot k_\chi^2} \\
&= e^{\frac{2\pi}{d}\cdot \mathrm{Im}(\tau_{\mathrm{or}})\cdot(i_\chi/n)^2} \cdot \sum_{k' \in \gamma_d^{-1}(\mathbf{Z}+\frac{i_\chi}{n})} |P_{\gamma_d}(k')|^2 \cdot e^{-\frac{1}{2}\cdot(k')^2} \cdot \gamma_d^{-1}
\end{aligned}$$

where $k' \stackrel{\mathrm{def}}{=} k_\chi \cdot \gamma_d^{-1}$; $P_{\gamma_d}(k') \stackrel{\mathrm{def}}{=} P(k' \cdot \gamma_d)$. Thus, as $d \to \infty$,

$$\lim_{d\to\infty} \gamma_d^{-1} \cdot ||\zeta^P||^2_{L^2((\mathbf{S^1})_{\mathrm{cv}})} = \lim_{d\to\infty} \int_{k'\in\mathbf{R}} |P_{\gamma_d}(k')|^2 \cdot e^{-\frac{1}{2}\cdot(k')^2} \cdot dk'$$

Put another way, this is the L^2-norm of $P_{\gamma_d}(-)$ for the measure on $\mathbf{R}$ that appears in Proposition 2.2. Since all the norms considered here are on the *finite-dimensional* $\mathbf{C}$-vector space of ζ^P for $P(-)$ of degree $\leq r$, we thus see that if we take the basis for this vector space given by letting

$$P(\mathbf{s}) = 1, \ldots, (\mathbf{s}/\gamma_d)^j, \ldots, (\mathbf{s}/\gamma_d)^r$$

and we divide the norms by $\gamma_d^{\frac{1}{2}}$, then this normed vector space converges to the normed vector space spanned by the first $r+1$ Hermite polynomials of Proposition 2.2. This proves $\lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot (r!)^{\frac{1}{2}} \cdot \zeta_r^{\mathrm{OR},\mathbf{S^1}} = \lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot \zeta_r^{\mathrm{HM}_d}$. (Note that here we use the fact that the L^2-norm of the Hermite polynomial H_r is a constant (independent of r) times $(r!)^{\frac{1}{2}}$.) To prove that $\lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot \zeta_r^{\mathrm{HM}_d} = \lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot \frac{r!}{(2\pi)^{r/2}} \cdot \zeta_r^{\mathrm{DR}}$, it suffices to observe that (by the latter part of Lemma 6.5)

$$\lim_{d\to\infty} \int_{k'\in\mathbf{R}} |P_{\gamma_d}(k')|^2 \cdot e^{-\frac{1}{2}\cdot(k')^2} \cdot dk' = \lim_{d\to\infty} \gamma_d^{-1} \cdot ||\zeta^P||^2_{L^2(E_{\mathrm{sc}})}$$

and that the L^2-norm $L^2(E_{\mathrm{sc}})$ of Lemma 6.5 coincides with the norm $|| \sim ||_{L^2_{\mathbf{R}}}$ of §4. (Note that we use here the fact that the L^2-norm of the Hermite polynomial H_r is a constant (independent of r) times $(r!)^{\frac{1}{2}}$, while the $L^2_{\mathbf{R}}$-norm of ζ_r^{DR} is a constant times $\left(\frac{(2\pi)^r}{(r!)}\right)^{\frac{1}{2}}$.)

Thus, it remains to prove the analogous results for $\boldsymbol{\mu}_d$ when $i_\chi \neq 0$. These results follow similarly by applying the formula

$$||\zeta^{P,\boldsymbol{\mu}_d}||^2_{L^2(\boldsymbol{\mu}_d)} = |q_{\mathrm{or}}|^{-\frac{1}{d}\cdot(i_\chi/n)^2} \cdot \sum_{k_{\mathrm{mod}}\in\mathbf{Z}/d\mathbf{Z}} \left| \sum_{k_\chi \,\in\, k_{\mathrm{mod}}+\frac{i_\chi}{n}} P(k_\chi)\cdot q_{\mathrm{or}}^{\frac{1}{2d}k_\chi^2} \cdot \chi((k_\chi - \frac{i_\chi}{n})_{\mathrm{et}}) \right|^2$$

together with the estimates of Lemma 6.1. When we apply Lemma 6.1, we take "d," "r" (notation of Lemma 6.1) to be r (in the present notation), and "a" (notation of Lemma 6.1) to be $[\frac{d}{2r}]$ (in the present notation). This allows us to take "b" (notation of Lemma 6.1) *arbitrarily large* as $d \to \infty$. Also, note that the factor of "d^r" (notation of Lemma 6.1) in ζ_r^{SS} does not bother us here, since in the present notation, this factor corresponds to r^r which is *constant* as $d \to \infty$. Thus, we obtain:

$$\begin{aligned}
\lim_{d\to\infty} & \sum_{k_{\mathrm{mod}}\in\mathbf{Z}/d\mathbf{Z}} \left| \sum_{k_\chi \,\in\, k_{\mathrm{mod}}+\frac{i_\chi}{n}} P(k_\chi)\cdot q_{\mathrm{or}}^{\frac{1}{2d}k_\chi^2} \cdot \chi((k_\chi - \frac{i_\chi}{n})_{\mathrm{et}}) \right|^2 \\
&= \lim_{d\to\infty} \sum_{k_\chi \,\in\, \mathbf{Z}+\frac{i_\chi}{n},\ |k_\chi|\leq\frac{d}{2}} \left| P(k_\chi)\cdot q_{\mathrm{or}}^{\frac{1}{2d}k_\chi^2} \right|^2 \\
&= \lim_{d\to\infty} \sum_{k_\chi \,\in\, \mathbf{Z}+\frac{i_\chi}{n}} \left| P(k_\chi)\cdot q_{\mathrm{or}}^{\frac{1}{2d}k_\chi^2} \right|^2
\end{aligned}$$

which is the same sum as that which appeared in the discussion of the $(\mathbf{S^1})_{\mathrm{cv}}$ case. ○

Remark 1. Stated in words, Theorem 6.7 asserts that:

> *As $d \to \infty$, the orthogonal systems of canonical SW zeta functions obtained from the natural inner products on $L^2((\mathbf{S^1})_{\mathrm{cv}})$ and $L^2(\boldsymbol{\mu}_d)$ converge to: (i) one another; (ii) canonical SW zeta functions modeled on the Hermite polynomials; (iii) the de Rham zeta functions studied in §4,5.*

Thus, in particular, Theorem 6.7 implies that *the de Rham zeta functions ζ_r^{DR} are equal to Hermite polynomial-based canonical Schottky-Weierstrass zeta functions.* In other words, we see that we have obtained another independent proof of the latter part of Theorem 5.3, which, in the notation of the present discussion, reads:

$$\begin{aligned}
\zeta_r^{\mathrm{DR}} &= \sum_{j=0}^{[r/2]} \frac{(-\pi \cdot u)^j \cdot (\delta^*)^{r-2j}(\sigma_\chi)}{L^{r-2j} \cdot j!\ (r-2j)!} \\
&= \frac{(2\pi \cdot u)^{\frac{r}{2}}}{(-1)^r \cdot r!} \cdot (-1)^r \cdot r! \cdot \sum_{j=0}^{[r/2]} \frac{(-2)^{-j} \cdot (\delta^*)^{r-2j}(\sigma_\chi)}{\{(2\pi \cdot u)^{\frac{1}{2}} \cdot L\}^{r-2j} \cdot j!\ (r-2j)!} \\
&= \frac{(2\pi \cdot u)^{\frac{r}{2}}}{(-1)^r \cdot r!} \cdot H_r(\{(2\pi \cdot u)^{\frac{1}{2}} \cdot L\}^{-1} \cdot \delta^*) \cdot \sigma_\chi
\end{aligned}$$

(where $|L| = \{8\pi^2 \cdot \mathrm{Im}(\tau_{\mathrm{sc}})\}^{-\frac{1}{2}}$, $|u| = 1$). Note that the *scaling factor* here, i.e., $\{(2\pi \cdot u)^{\frac{1}{2}} \cdot L\}$, has absolute value equal to $\{\frac{2\pi}{8\pi^2 \cdot \mathrm{Im}(\tau_{\mathrm{sc}})}\}^{\frac{1}{2}} = \gamma_d$. That is to say, Theorems 5.3 and 6.7 are consistent in the sense that they yield the same scaling factor. Note, relative to the discussion surrounding Terminology 3.7, that this implies that (if we think of $\mathrm{Im}(\tau_{\mathrm{or}})$ as fixed), the $\zeta_r^{\mathrm{OR},\boldsymbol{\mu}}$'s (for varying d) have *slope* $\frac{1}{2}$ (as promised in §3).

Remark 2. It is interesting to note that in the discussion of this §, there are *three natural parameters* involved; we would like to think of these parameters as follows:

<u>Holomorphic Variable</u>: τ_{or}

<u>Anti-Holomorphic Variable</u>: d

<u>Arithmetic Variable</u>: r

That is to say, one way to think of the goal of this paper is that *we are trying to descend the* $\mathbf{Z}$*-scheme* $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ *(i.e., the moduli stack of elliptic curves) to some sort of object over the hypothetical/mythical "field of constants"* $\mathbf{F}_1 \subseteq \mathbf{Z}$. This descent may be carried out at different primes; here we are interested in carrying it out at the *infinite prime*. Thus, one expects that $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ should have *three arithmetic/real dimensions* over $\mathbf{F}_1$. Two of these arithmetic dimensions should be geometric and correspond to the holomorphic and anti-holomorphic coordinates on the complex manifold $(\overline{\mathcal{M}}_{1,0})_{\mathbf{C}}$. We feel that these two dimensions are given by τ_{or} (*the holomorphic coordinate*) and the *scaling parameter* d (which plays the role of the *anti-holomorphic coordinate*, and corresponds to powers of the *geometric* Frobenius morphism in the p-adic theory). Note that this interpretation of these parameters is compatible with the fact that the exponent of the exponential appearing in the various theta/zeta functions of this § essentially consists of $\tau_{\mathrm{or}} \cdot d^{-1}$, which is reminiscent of the exponent "$|z|^2 = z \cdot \overline{z}$" of the exponential which appears in the usual complex theory of theta functions (cf., e.g., Remark 2 following Proposition 2.2; [Mumf3], §12). Finally, we have the index "r" which plays the role of the single arithmetic dimension of $\mathbf{F}_1$. In the p-adic theory, this dimension corresponds to *arithmetic* Frobenius, i.e., the single dimension of $\mathrm{Gal}(\mathbf{F}_p)$.

Note that this point of view is compatible with the point of view of the discussion at the end of §3 (see also Remark 1 above). Indeed, the slope as discussed in §3 is obtained precisely by looking at the "action of the scaling parameter d." Also, we observe that this point of view is reminiscent of the point of view of the Introductions of [Mzk1,2] which discuss the *analogy between Frobenius actions and real analytic Kähler metrics*. (The connection here being that orthogonal functions typically arise from some sort of real analytic Kähler metric, as in §4.)

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Chapter VIII: The Hodge-Arakelov Comparison Theorem

§0. Introduction

In Chapter VII, we so-to-speak "introduced the *actors*" (which are systems of orthogonal functions) in the archimedean theory of the Comparison Isomorphism. In the present Chapter, we "present the *play*." Put another way, we complete the "Scheme-Theoretic Comparison Isomorphism" of Chapter VI by computing the behavior of the Comparison Isomorphism at *archimedean primes*. Thus, in summary, our result may be roughly stated as follows (cf. the Introduction to Chapter VI):

> *There is a natural bijection between certain natural types of algebraic functions on the universal extension of an elliptic curve and the "set-theoretic functions" on the torsion points of the elliptic curve. Moreover, this bijection is compatible with certain natural metrics/integral structures defined at all (finite and infinite) primes of a number field, as well as at the divisor at which the elliptic curve in question degenerates.*

We refer to this result as the *Hodge-Arakelov Comparison Isomorphism*. In more precise terms, our result, obtained by combining Theorem 7.4 of the present Chapter with Chapter VI, Theorem 4.1, is as follows:

Theorem A. **(The Hodge-Arakelov Comparison Isomorphism)** *Let $d, m \geq 1$ be integers such that m does* **not** *divide d. Suppose that $S^{\log}$ is a fine noetherian log scheme, and let*

$$C^{\log} \to S^{\log}$$

be a **log elliptic curve** *over $S^{\log}$ such that the divisor at infinity $D \subseteq S$ (i.e., the pull-back of the divisor at infinity of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ via the classifying morphism $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$) is a Cartier divisor on S. Also, let us assume that étale locally on the completion of S along D, the pull-back of the Tate parameter q to this completion admits a d-th root, and that we are given a* **torsion point**

$$\eta \in E_{\infty,S}(S_\infty)$$

of order precisely *m which defines line bundles $\overline{\mathcal{L}}_{\mathrm{st},\eta}$, $\overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta}$ (cf. Chapter V, §1). If d is* **odd** *(respectively,* **even***), then let $\overline{\mathcal{L}} \stackrel{\mathrm{def}}{=} \overline{\mathcal{L}}_{\mathrm{st},\eta}$ (respectively, $\overline{\mathcal{L}} \stackrel{\mathrm{def}}{=} \overline{\mathcal{L}}^{\mathrm{ev}}_{\mathrm{st},\eta}$). Then:*

(1) (Compatibility with Base-Change) *The formation of the push-forward (cf. Chapter VI, Definition 1.3)*

$$(f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<d}\{\infty, \mathrm{et}\}$$

(along with its filtration) commutes with base-change (among bases $S^{\log}$ satisfying the hypotheses given above).

(2) (Zero Locus of the Determinant) *Assume that S is* $\mathbf{Z}$-flat. *The scheme-theoretic zero locus of the determinant* $\det(\Xi\{\infty, \mathrm{et}\})$*, i.e., the determinant of the* **evaluation map** *(cf. Chapter V, Proposition 2.2; Chapter VI, Theorem 3.1, (1))*

$$\Xi\{\infty, \mathrm{et}\} : (f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<d}\{\infty, \mathrm{et}\} \to (f_S)_*(\overline{\mathcal{L}}|_{({}_dE^{\dagger}_{\infty})})$$

is given by the **divisor**

$$d \cdot [\eta \bigcap ({}_dE)]$$

(where ${}_dE$ is the kernel of multiplication by d on E_d). In fact, the divisor of poles of the inverse morphism to $\Xi\{\infty, \mathrm{et}\}$ *is contained in the divisor* $[\eta \bigcap ({}_dE)]$.

(3) (Analytic Torsion at the Divisor at Infinity) *For each ι, there is a sequence of elements*

$$\mathbf{a}_\iota = \{(\mathbf{a}_\iota)_0, \ldots, (\mathbf{a}_\iota)_{d-1}\}; \quad (\mathbf{a}_\iota)_j \approx \frac{j^2}{8d}$$

of $\mathbf{Q}_{\geq 0} \cdot \log(q)$*, where $(\mathbf{a}_\iota)_j$ goes roughly (as a function of j) as $\frac{j^2}{8d}$ (cf. Chapter VI, Theorem 3.1, (2)), such that the subquotients of the natural filtration on the domain of* $\Xi\{\infty, \mathrm{et}\}$ *admits natural isomorphisms:*

$$(F^{j+1}/F^j)((f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<d}\{\infty, \mathrm{et}\}) \longrightarrow \frac{1}{j!} \cdot \exp(-(\mathbf{a}_\iota)_j) \cdot (f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}}) \otimes_{\mathcal{O}_S} \tau_E^{\otimes j}$$

for $j = 0, \ldots, d-1$. Moreover, the sections of $\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}}$ that realize these bijections have **q-expansions** *in a neighborhood of infinity that are given* **explicitly** *in Chapter V, Theorem 4.8.*

(4) **(Integrality Properties at the Infinite Prime)** *Suppose that $D = \emptyset$, and S is of finite type over $\mathbf{C}$. Let us then write $\mathcal{L}$, Ξ, ${}_dE^\dagger$, $E^\dagger_{[d]}$ for $\overline{\mathcal{L}}$, $\Xi\{\infty, \mathrm{et}\}$, ${}_dE^\dagger_\infty$, $E^\dagger_{\infty,[d]}$. Then one may equip $\mathcal{L}$ with a (smooth, i.e., $\mathcal{C}^\infty$) metric $| \sim |_{\mathcal{L}}$ whose* **curvature is translation-invariant** *on the fibers of $E \to S$. Moreover, such a metric is unique up to multiplication by a (smooth) positive function on $S(\mathbf{C})$. Then $| \sim |_{\mathcal{L}}$ defines a metric on the vector bundle $(f_S)_*(\mathcal{L}|_{{}_dE^\dagger})$ (i.e., the range of Ξ), namely, the L^2-metric for "$\mathcal{L}$-valued functions on ${}_dE^\dagger$" (where we assume that the total mass of ${}_dE^\dagger$ is 1). Since Ξ is an isomorphism, this metric thus induces a metric on $(f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d}$ (i.e., the domain of Ξ), which we denote by*

$$|| \sim ||_{\mathrm{et}}$$

and refer to as the **étale metric**. *On the other hand, by using the canonical real analytic splitting of $E^\dagger_{[d]}(\mathbf{C}) \to E(\mathbf{C})$ (i.e., the unique splitting which is a continuous homomorphism), we may split sections of $(f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d}$ into components which are real analytic sections of $\mathcal{L} \otimes \tau_E^{\otimes r}$ (where $r < d$) over $E(\mathbf{C})$. Since τ_E gets a natural metric by square integration over E, these components have natural L^2-norms determined by integrating their $| \sim |^2_{\mathcal{L}}$ over the fibers of $E(\mathbf{C}) \to S(\mathbf{C})$. This defines what we refer to as the* **de Rham metric**

$$|| \sim ||_{\mathrm{DR}}$$

on $(f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d}$. The relationship between the étale and de Rham metrics may be described using three "models":

> **(A.) The Hermite Model:** *This model states that if we fix $r < d$, and let $d \to \infty$, then over any compact subset of $S(\mathbf{C})$, the étale metric $|| \sim ||_{\mathrm{et}}$ on $F^r((f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d})$ converges (up to a factor $\leq e^{\pi+r}$, ≥ 1) to the metric $|| \sim ||_{\mathrm{DR}}$, as well as to a certain metric "$|| \sim ||_{\mathrm{HM}_d}$" defined by considering* **Hermite polynomials scaled by a factor of (constant)$\cdot\sqrt{\mathbf{d}}$** *in the derivatives of the theta functions $\in (f_S)_*(\mathcal{L}|_E) = F^1((f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d})$.*
>
> **(B.) The Legendre Model:** *This model states (roughly) that over any compact subset of $S(\mathbf{C})$, a certain* **average** *– which we denote*

$|| \sim ||_{w,\boldsymbol{\mu}_a}$ – *of translates of the étale metric* $|| \sim ||_{\mathrm{et}}$ *on* $(f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d}$ *is equal (provided* $d \geq 25$*), up to a factor of* $(\mathrm{constant})^d$*, to the de Rham metric* $|| \sim ||_{\mathrm{DR}}$*, as well as to a certain metric "*$|| \sim ||_{\mathrm{Tch}}$*" defined by considering* **discrete Tchebycheff polynomials scaled by a factor of d** *in the derivatives of the theta functions* $\in (f_S)_*(\mathcal{L}|_E) = F^1((f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d})$*. These discrete Tchebycheff polynomials are discrete versions of the Legendre polynomials, and in fact, if we let* $d \to \infty$ *with the said scaling by* d*, then the discrete Tchebycheff polynomials converge uniformly to the* **Legendre polynomials***.*

(C.) The Binomial Model: *This model involves the explicit* q*-expansions (where we write* $E = \mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$*, and* q *is a holomorphic function which is defined, at least locally, on* $S(\mathbf{C})$*) referred to in (3) above, which are essentially* **binomial coefficient polynomials (scaled by 1)** *in the derivatives of the theta functions* $\in (f_S)_*(\mathcal{L}|_E) = F^1((f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d})$*. If we divide these functions by appropriate powers of* q*, then the norm* $|| \sim ||_{\mathrm{qCG}}$ *for which these functions divided by powers of* q *are orthonormal satisfies the following property: If* $d \geq 12$*, and* $\mathrm{Im}(\tau) \geq 200\{\log^2(d) + n \cdot \log(d) + n \cdot \log(n)\}$ *(where* $q = \exp(2\pi i\tau)$*), then:*

$$n^{-1} \cdot e^{-32d} \cdot || \sim ||_{\mathrm{qCG}} \leq || \sim ||_{\mathrm{et}} \leq e^{4d} \cdot || \sim ||_{\mathrm{qCG}}$$

Here, the factor of n^{-1} *that appears is the exact* **archimedean analogue** *of the poles that appeared at finite primes in (2) above.*

Finally, for each of these three models, the **combinatorial/arithmetic portion of the analytic torsion** *(i.e., the portion not arising from letting the elliptic curve* E *degenerate – cf. (3) above for the portion arising from degeneration of the elliptic curve) induced on* $(F^{r+1}/F^r)((f_S)_*(\mathcal{L}|_{E^\dagger_{[d]}})^{<d})$ *by the metrics* $|| \sim ||_{\mathrm{DR}}$*;* $|| \sim ||_{\mathrm{HM}_d}$*;* $|| \sim ||_{\mathrm{Tch}}$*;* $|| \sim ||_{w,\boldsymbol{\mu}_a}$*;* $|| \sim ||_{\mathrm{qCG}}$ *(in their respective domains of applicability) as* $r \to d$*, goes (modulo factors of the order* $(\mathrm{constant})^d$*) as*

$$\approx (r!)^{-1} \approx (d!)^{-1}$$

which is precisely what you would expect by applying the **product formula** *to the computation of the "analytic torsion" in the* **finite prime case***, which consists of a factor of precisely* $(r!)^{-1}$ *(cf. Chapter V, Theorem 3.1; Chapter VI, Theorem 4.1; Chapter VII, Proposition 3.4).*

As discussed in the statement of Theorem A, the key point of the archimedean portion of Theorem A is the comparison of the *étale and de Rham metrics* $|| \sim ||_{\mathrm{et}}$, $|| \sim ||_{\mathrm{DR}}$. Unfortunately, we are unable to prove a simple sharp result that they always coincide. Instead, we choose three natural "domains of investigation" – which we refer to as *models*

– where we compare these two metrics using a particular system of functions which are well-adapted to the domain of investigation in question. One of the most important features of these three models is that they each have natural *scaling factors* associated to them. The three models, along with their natural scaling factors, and natural domains of applicability are as follows:

$$\underline{\text{Hermite Model}}\ (\text{scaling factor} = d^{\frac{1}{2}}) : \text{nondegenerating } E, \text{ fixed } r < d$$

$$\underline{\text{Legendre Model}}\ (\text{scaling factor} = d) : \text{nondegenerating } E, \text{ varying } r < d$$

$$\underline{\text{Binomial Model}}\ (\text{scaling factor} = 1) : \text{degenerating } E$$

It is interesting to observe that the exponents appearing in these scaling factors, i.e., $0, \frac{1}{2}, 1$, which we refer to as *slopes*, are precisely the *same as the slopes that appear when one considers the action of Frobenius on the crystalline cohomology of an elliptic curve at a finite prime* (cf. the discussions at the end of Chapter VII, §3, 6, for more on this analogy).

If we write $E = \mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$, $q = \exp(2\pi i\tau)$, then the Hermite Model (respectively, Legendre Model, Binomial Model) corresponds to the case where/is most useful when $\mathrm{Im}(\tau)$ is fixed (respectively, $\to 0$; $\to \infty$). The Binomial Model is interesting in that it is the *most closely analogous to the situation at the finite primes.* The Hermite model is interesting in that it sheds light on the factors of $q^{\approx \frac{1}{8d} \cdot j^2}$ – which are essentially Gaussians – that appear in Theorem A, (3). Moreover, the Hermite model in some sense makes explicit that *essentially what we are doing throughout this entire paper is simply studying the derivatives of a Gaussian/theta function.* Finally, the Legendre model is interesting in that it shows how the Comparison Isomorphism for general elliptic curves may be thought of as being *essentially a (more complicated) version of the Fundamental Combinatorial Model* (cf. Chapter VII, §3) given by considering integer-valued polynomial functions on a set of d points.

Our main technique for comparing the étale and de Rham metrics in these various models is to reduce the problem to the computation of certain metrics on certain finite-dimensional spaces of q-series. The q-series then tend to have the form:

$$q\text{-series} = \text{“head”} + \text{“tail”}$$

where

$$\text{“head”} = \text{a function of the sort that appears in the “model”}$$

while

$$\Big| \text{“tail”} \Big| \leq \epsilon$$

(i.e., the tail is rather small compared to the head). Such estimates then show that the behavior of the q-series is roughly the same as the behavior as the special functions of the "model." This sort of analysis was already carried out for the Hermite Model, which is technically the simplest of the three models, in Chapter VII, §6. The analogous analysis for the Legendre Model (which is technically the next simplest of the three) is carried out in §1,2, of the present Chapter. Finally, we carry out this analysis for the Binomial Model, which is by far the most technically intricate of the three models, in §3,4,5, of the present Chapter. In §6, we then relate these estimates to the analysis of the de Rham metric given in Chapter VII, §4,5. Finally, in §7, we summarize everything in the form of the main theorem.

In a word, the main point of this sort of "head + tail" estimate is to show that the "tail" is small. The reason that the tail is small is, invariably, because it *decays like a Gaussian*, since the q-series in question are essentially just derivatives of theta functions. Unfortunately, however, because we are ultimately working with the L^2-metric $|| \sim ||_{\mathrm{et}}$ defined by a *discrete set of points* (as opposed to, say, the L^2-metric obtained by integrating over the entire *continuous* elliptic curve in question), numerous technical difficulties arise. These technical difficulties make the resulting analysis rather unpleasant, especially in the case of the binomial model. Because of the cumbersome and unenlightening nature of this analysis, we submit that the experienced mathematician may find it substantially easier to work out the details for himself/herself than to wade through the lengthy estimates of the present Chapter.

Also, we remark in passing, that although in Theorem A, we only give "qualitative estimates" (e.g., of the form "(constant)^d"), in fact, all the results that we derive involve *explicit estimates of the constants that appear*. We refer to §6,7, for such explicit estimates. Nevertheless, despite the fact that we made an effort to give such explicit estimates of the constants that appear, we made *no effort to give "best possible estimates" for these constants*. Thus, throughout the arguments of the present Chapter, the reader will often note that the estimates tend to be rather weak, and that with very little extra effort, one could give somewhat stronger estimates for these constants. Our aim here was not to make any pretense of giving such strong estimates, but only to give *estimates that are representative of the essential technique involved*.

Finally, we remark that although this "head + tail" technique has an unenlightening technical side, it also admits the following intriguing *philosophical interpretation*. Namely, the terms appearing in the q-series are naturally indexed by $\mathbf{Z}$. Roughly speaking, the terms that contribute to the head are those numbered $0, 1, \ldots, N-1$ (where N is a positive integer). The remaining terms then naturally form deformations of the "head terms" in the following fashion: The term numbered by j deforms the term numbered by the unique $\mathrm{Head}(j) \in \{0, \ldots, N-1\}$ such that $\mathrm{Head}(j) \equiv j$ modulo N. Thus, the fact that ultimately only the "head terms" contribute significantly may thus be *interpreted as the datum of the map* (from $\mathbf{Z}$ to $\{0, 1, \ldots, N-1\}$)

$$j \mapsto \mathrm{Head}(j)$$

in the language of function-theory. Put another way, it may be regarded as a *function-theoretic encoding of the splitting*

$$\mathbf{Z} \cong \{0, 1, \ldots, N-1\} \times (N \cdot \mathbf{Z})$$

(given by $j \mapsto (\mathrm{Head}(j), j - \mathrm{Head}(j))$).

This interpretation is interesting for the following reason: As explained in the Introduction to the present paper, the Hodge-Arakelov Comparison Isomorphism was originally envisioned as a sort of Hodge theory-type comparison theorem between de Rham and étale/singular cohomology, in the context of Arakelov theory. By analogy with the case of a geometric base (as opposed to, say, a number field as base) for the elliptic curve in question, such a comparison should result in some sort of *Kodaira-Spencer map*, i.e., a map from the tangent vectors of the base to tangent vectors of the moduli stack of elliptic curves (cf. the Introduction). Thus, in particular, this sort of comparison theorem should have something to do with *absolute differentials of* $\mathbf{Z}$, i.e., with the point of view that $\mathbf{Z}$ contains some sort of "absolute field of constants $\mathbf{F}_1$" over which it acts like a polynomial algebra:

$$\text{“}\mathbf{Z} \cong \mathbf{F}_1[t]\text{”}$$

But now note that this formula "$\mathbf{Z} \cong \mathbf{F}_1[t]$" is strikingly reminiscent of the decomposition "$\mathbf{Z} \cong \{0, 1, \ldots, N-1\} \times (N \cdot \mathbf{Z})$" that appeared above. Thus, in summary, it is as if *the "head + tail" approach is a sort of appeal to the notion* $\mathbf{Z} \cong \mathbf{F}_1[t]$ *encoded in function-theoretic language*.

In fact, it is tempting to take this analogy a step further as follows: The "fundamental symmetries" of a polarized elliptic curve are controlled by the "theta group" and its analogous Lie algebra version (cf. Chapter VII, §4). Moreover, this sort of algebraic structure is exactly what noncommutative geometers refer to as a *noncommutative torus*. Thus, *it is as if the mathematics of the Hodge-Arakelov Comparison Isomorphism is trying to assert that*:

> *The symmetries/twist inherent in the inclusion "$\mathbf{F}_1 \subseteq \mathbf{Z}$" are precisely the symmetries/twist described by the structure known as a noncommutative torus.*

It is the hope of the author that some day future research will allow one to make these ideas more rigorous.

§1. Averages of Metrics

In this §, we discuss certain metrics on the space of functions on $\mathbf{S^1}$ obtained by *averaging the usual L^2-metrics obtained by restricting these functions to translates of a fixed finite subgroup* of $\mathbf{S^1}$. These metrics will play an important role in the theory of §2.

We would like to begin by discussing the *Hilbert space* $L^2(\mathbf{S^1})$ equipped with the standard inner product:

$$(f,g) \stackrel{\text{def}}{=} \frac{1}{2\pi} \cdot \int_{\mathbf{S^1}} f \cdot \overline{g}$$

We shall write $U \stackrel{\text{def}}{=} \exp(2\pi i t)$ (where $t \in [0, 2\pi)$) for the standard coordinate on $\mathbf{S^1}$. Thus, any element $f \in L^2(\mathbf{S^1})$ may be written in the form

$$f = \sum_{k \in \mathbf{Z}} c_k \cdot U^k$$

(where $c_k \in \mathbf{C}$). Then L^2-norm of such an f is then given by

$$||f||^2 \stackrel{\text{def}}{=} \sum_{k \in \mathbf{Z}} |c_k|^2$$

Let us write $\mathcal{C}^\infty(\mathbf{S^1}) \subseteq L^2(\mathbf{S^1})$ for the subspace of *smooth functions* on $\mathbf{S^1}$.

Now let us suppose that we are also given two *positive integers* d, a. Write $\widetilde{d} \stackrel{\text{def}}{=} d \cdot a$. Suppose that we are also given *positive real numbers* ("weights") $w_0, \ldots, w_{d-1}$. If $\phi = \sum_{k=0}^{d-1} \gamma_k \cdot U^k$ (where $\gamma_k \in \mathbf{C}$) is a function on $\boldsymbol{\mu}_d$ ($\subseteq \mathbf{S^1}$), then we would like to consider the norm

$$||\phi||^2_w \stackrel{\text{def}}{=} \frac{1}{d} \cdot \sum_{k=0}^{d-1} w_k \cdot |\gamma_k|^2$$

on $L^2(\boldsymbol{\mu}_d)$. Thus, for smooth functions $f = \sum_{k \in \mathbf{Z}} c_k \cdot U^k \in \mathcal{C}^\infty(\mathbf{S^1})$, we have:

$$||f||^2_w \stackrel{\text{def}}{=} ||(f|_{\boldsymbol{\mu}_d})||^2_w = \frac{1}{d} \cdot \sum_{k=0}^{d-1} w_k \cdot \Big| \sum_{k' \in k + d \cdot \mathbf{Z}} c_{k'} \Big|^2$$

Moreover, if $\alpha \in \mathbf{S^1}$, then we may *translate* $|| \sim ||_w$ as follows:

$$||(T_\alpha(f)|_{\boldsymbol{\mu}_d})||^2_w = \frac{1}{d} \cdot \sum_{k=0}^{d-1} w_k \cdot \Big| \sum_{k' \in k + d \cdot \mathbf{Z}} c_{k'} \cdot \alpha^{k'} \Big|^2$$

(where $T_\alpha(f)(U) \stackrel{\text{def}}{=} f(\alpha \cdot U)$). Note that $||(T_\alpha(f)|_{\boldsymbol{\mu}_d})||_w^2$ only depends on the image of α in $\mathbf{S^1}/\boldsymbol{\mu}_d$. Thus, if we write $\alpha_{\boldsymbol{\mu}}$ for the image of α in $\mathbf{S^1}/\boldsymbol{\mu}_d$, it makes sense to define

$$||f||_{w,\alpha_{\boldsymbol{\mu}}}^2 \stackrel{\text{def}}{=} ||(T_\alpha(f)|_{\boldsymbol{\mu}_d})||_w^2$$

Now we have the following result:

Proposition 1.1. *For smooth functions $f = \sum_{k\in\mathbf{Z}} c_k \cdot U^k \in \mathcal{C}^\infty(\mathbf{S^1})$, we have:*

$$\begin{aligned}
||f||_{w,\boldsymbol{\mu}_a}^2 &\stackrel{\text{def}}{=} \frac{1}{a} \cdot \sum_{\alpha_{\boldsymbol{\mu}} \in \boldsymbol{\mu}_a} ||f||_{w,\alpha_{\boldsymbol{\mu}}}^2 \\
&= \frac{1}{d} \cdot \sum_{k=0}^{d-1} w_k \cdot \sum_{k'=0}^{a-1} \Big| \sum_{k'' \in k + d\cdot k' + \widetilde{d}\cdot\mathbf{Z}} c_{k''} \Big|^2 \geq \frac{1}{d} \cdot \sum_{k=0}^{d-1} w_k \cdot \Big| \sum_{k' \in k + \widetilde{d}\cdot\mathbf{Z}} c_{k'} \Big|^2
\end{aligned}$$

(where we regard $\boldsymbol{\mu}_a = \boldsymbol{\mu}_{\widetilde{d}}/\boldsymbol{\mu}_d \subseteq \mathbf{S^1}/\boldsymbol{\mu}_d$).

Proof. Indeed, from the definitions, we have:

$$\begin{aligned}
\frac{1}{a} \cdot \sum_{\alpha_{\boldsymbol{\mu}} \in \boldsymbol{\mu}_a} ||f||_{w,\alpha_{\boldsymbol{\mu}}}^2 &= \frac{1}{a} \cdot \sum_{\alpha_{\boldsymbol{\mu}} \in \boldsymbol{\mu}_a} \frac{1}{d} \cdot \sum_{k=0}^{d-1} w_k \cdot \Big| \sum_{k' \in k + d\cdot\mathbf{Z}} c_{k'} \cdot \alpha^{k'} \Big|^2 \\
&= \frac{1}{a} \cdot \sum_{\alpha_{\boldsymbol{\mu}} \in \boldsymbol{\mu}_a} \frac{1}{d} \cdot \sum_{k=0}^{d-1} w_k \cdot \Big| \sum_{k'=0}^{a-1} \sum_{k'' \in k + d\cdot k' + \widetilde{d}\cdot\mathbf{Z}} c_{k''} \cdot \alpha^{k''} \Big|^2 \\
&= \frac{1}{a} \cdot \sum_{\alpha_{\boldsymbol{\mu}} \in \boldsymbol{\mu}_a} \frac{1}{d} \cdot \sum_{k=0}^{d-1} w_k \cdot \Big| \sum_{k'=0}^{a-1} \alpha^{d\cdot k'} \sum_{k'' \in k + d\cdot k' + \widetilde{d}\cdot\mathbf{Z}} c_{k''} \Big|^2
\end{aligned}$$

(where $\alpha \in \mathbf{S^1}$ denotes any element that maps to $\alpha_{\boldsymbol{\mu}} \in \mathbf{S^1}/\boldsymbol{\mu}_d$). Thus, (readjusting the notation) we see that it suffices to prove that if $\lambda_0, \ldots, \lambda_{a-1} \in \mathbf{C}$, then

$$\frac{1}{a} \cdot \sum_{\alpha \in \boldsymbol{\mu}_a} \Big| \sum_{k=0}^{a-1} \alpha^k \cdot \lambda_k \Big|^2 = \sum_{k=0}^{a-1} \Big| \lambda_k \Big|^2$$

To prove this, it suffices to observe that

$$\Big|\sum_{k=0}^{a-1} \ \alpha^k \cdot \lambda_k\Big|^2 = \Big(\sum_{k=0}^{a-1} \alpha^k \cdot \lambda_k\Big)\Big(\sum_{j=0}^{a-1} \alpha^{-j} \cdot \overline{\lambda}_j\Big)$$
$$= \sum_{k=0}^{a-1} \sum_{j=0}^{a-1} \ \alpha^{k-j} \cdot \lambda_k \cdot \overline{\lambda}_j$$

(where $\overline{\lambda}_j$ denotes the complex conjugate of λ_j). But if we average over $\alpha \in \boldsymbol{\mu}_a$ in this last sum, we see that the only terms that survive are those for which $k = j$. Moreover, the sum of these terms is precisely $\sum_{k=0}^{a-1} \Big|\lambda_k\Big|^2$, as desired. ◯

§2. The Legendre Model

In this §, we study the *space of canonical SW zeta functions equipped with a certain natural metric, arising from restriction to torsion points.* In particular, we show that when equipped with this metric, the space of canonical SW zeta functions becomes "approximately isomorphic" to the space of *discrete Tchebycheff polynomials* studied in Chapter VII, §3.

We begin by making certain *estimates*, based on the estimates of Chapter VII, Lemma 6.1. In the following discussion, we use the notation of Chapter VII, §6; also, we assume (for simplicity) that $d \geq 2$. First, we recall the *theta function* Θ_χ

$$\sum_{k \in \mathbf{Z}} q_{\mathrm{sc}}^{\frac{1}{2}\cdot k^2 + (i_\chi/n)\cdot k} \cdot U_{\mathrm{cv}}^{2mk+i_\chi} \cdot \chi(k_{\mathrm{et}}) = \sum_{k \in \mathbf{Z}} q_{\mathrm{or}}^{\frac{1}{d}(\frac{1}{2}\cdot k^2 + (i_\chi/n)\cdot k)} \cdot U_{\mathrm{cv}}^{2mk+i_\chi} \cdot \chi(k_{\mathrm{et}})$$

In Chapter VII, §6, we also considered the *derivatives*

$$\zeta_r^{\mathrm{SS}} \ \stackrel{\mathrm{def}}{=} \ \Big(\frac{\delta_\chi^*}{d}\Big)^r(\Theta_\chi) \ = \ \sum_{k \in \mathbf{Z}} \ \Big(\frac{k}{d}\Big)^r \cdot q_{\mathrm{or}}^{\frac{1}{d}(\frac{1}{2}\cdot k^2 + (i_\chi/n)\cdot k)} \cdot U_{\mathrm{cv}}^{2mk+i_\chi} \cdot \chi(k_{\mathrm{et}})$$

where (to simplify the notation) we omit the "$\cdot\theta^{-m}$" in the definition of ζ_r^{SS}.

To prepare for the discussion below, we would like to introduce various objects related to the coefficients of this expansion. If we think of the integers $0, \ldots, d-1$ as representing the elements of $\mathbf{Z}/d\mathbf{Z}$, then for $j = 0, \ldots, d-1$, let us write

$$e_j \stackrel{\mathrm{def}}{=} \mathrm{Min}\Big\{\frac{1}{2} \cdot k^2 + (i_\chi/n) \cdot k)\Big\}$$

(where the minimum is over k such that $k \equiv j$ modulo d) and $w_j \stackrel{\text{def}}{=} |q_{\text{sc}}|^{-2 \cdot e_j} \geq 1$. Next, recall the *linear function* "$L_d(T) = T + \lambda_d - \frac{i_\chi}{2m}$" of Chapter V, Theorem 4.8. Let us write

$$l_d \stackrel{\text{def}}{=} \frac{\lambda_d}{d}$$

Thus, $0 \leq l_d \leq \frac{1}{2}$ (since $d \geq 2$). It is easy to check that λ_d has the following property (cf. the proof of Chapter V, Theorem 4.8): the minimum appearing in the definition of e_j is attained for any k such that $0 \leq k + \lambda_d \leq d - 1$. Let us write

$$K_{\text{Crit}} \stackrel{\text{def}}{=} \{0 - \lambda_d, 1 - \lambda_d, \ldots, d - 1 - \lambda_d\}$$

for the set of such "critical" k. It thus follows that e_j is attained for some k satisfying $|k| \leq d$. In particular, it follows that $2|e_j| \leq |k^2 + (i_\chi/m) \cdot k| \leq \frac{3}{2} \cdot d^2$, hence that

$$0 \leq \log(w_j) \leq 3\pi \cdot \text{Im}(\tau_{\text{or}}) \cdot d \leq 6(1 + \epsilon)Q \cdot d$$

(in the notation of Chapter VII, Lemma 6.1, (2)). Thus, *if, for instance, one is considering a situation in which* $\text{Im}(\tau_{\text{or}})$ *varies in a compact subset of the upper half-plane, then it follows that on that compact subset,* w_j *is bounded above and below by constants* C_1^d, C_2^d, where $C_1, C_2 \in \mathbf{R}_{>0}$ depend only on the compact set.

Now that $w_0, \ldots, w_{d-1}$ have been defined, we may apply the theory of §1. In particular, if we fix a *positive integer* $a \geq 2$, then it follows from Proposition 1.1 that for smooth functions $f = \sum_{k \in \mathbf{Z}} \phi_k \cdot U^k \in \mathcal{C}^\infty((\mathbf{S^1})_{\text{sc}})$, we have the following "averaged" L^2-norm:

$$||f||^2_{w,\boldsymbol{\mu}_a} \stackrel{\text{def}}{=} \frac{1}{d} \cdot \sum_{j=0}^{d-1} w_j \cdot \sum_{j'=0}^{a-1} \left| \sum_{k \in j + d \cdot j' + \widetilde{d} \cdot \mathbf{Z}} \phi_k \right|^2$$

In this §, we would like to consider two related norms:

$$||f||^2_{\text{Tch}_\epsilon} \stackrel{\text{def}}{=} \frac{1}{d} \cdot \sum_{j \in K_{\text{Crit}}} w_j \cdot \left| \sum_{k \in j + \widetilde{d} \cdot \mathbf{Z}} \phi_k \right|^2 \leq ||f||^2_{w,\boldsymbol{\mu}_a}$$

(where, for $j < 0$ or $> d-1$, w_j is defined as $w_{j'}$ for the unique $j' \in \{0, \ldots, d-1\}$ such that $j' \equiv j$; and the inequality follows from the fact that $||f||^2_{\text{Tch}_\epsilon}$ is *defined* to be the sum of a *subset* of the same collection of nonnegative numbers whose sum constitutes the definition of $||f||^2_{w,\boldsymbol{\mu}_a}$) and

$$||f||^2_{\text{Tch}} \stackrel{\text{def}}{=} \frac{1}{d} \cdot \sum_{j \in K_{\text{Crit}}} w_j \cdot \left| \phi_j \right|^2$$

In fact, in the following discussion, we would like to consider functions $f \in \mathcal{C}^\infty((\mathbf{S^1})_{\text{cv}})$ which are of the form $f_1 \cdot U_{\text{cv}}^{i_\chi}$, for some $f_1 \in \mathcal{C}^\infty((\mathbf{S^1})_{\text{sc}})$. For such f, we write $||f||_{??} \stackrel{\text{def}}{=} ||f_1||_{??}$, where $??= w, \boldsymbol{\mu}_a$; Tch_ϵ; or Tch.

Next, let us observe that when $f = \Theta_\chi$, then it follows from the above discussion concerning l_d that the resulting "ϕ_j" satisfy $w_j \cdot |\phi_j|^2 = 1$, for $j \in K_{\text{Crit}}$. Similarly, if δ_χ^* is the differential operator of Chapter VII, §6, and $P(-)$ is a polynomial with complex coefficients, then

$$||P(\delta_\chi^*) \cdot \Theta_\chi||^2_{\text{Tch}} = \frac{1}{d} \cdot \sum_{j \in K_{\text{Crit}}} \left|P(j)\right|^2$$

that is to say, the expression on the right-hand side of this equation is essentially (i.e., up to a factor of $\frac{1}{d}$ in front, and a shift in the variable j by the number λ_d of Chapter V, Theorem 4.8) the (squared) L^2-norm $|| \sim ||^2$ considered in our discussion of *discrete Tchebycheff polynomials* (Chapter VII, §3). In the following, we would like to show that in the situation that we are interested in, the norms subscripted $w, \boldsymbol{\mu}_a$; Tch_ϵ; and Tch are *very close to another* and hence that *the theory of orthogonal functions with respect to the* $w, \boldsymbol{\mu}_a$*-norm is very close to the theory of discrete Tchebycheff polynomials* of Chapter VII, §3.

This observation motivates the following definitions: First, we introduce some indeterminates

$$\mathbf{t} \stackrel{\text{def}}{=} \frac{T}{d}; \quad \mathbf{s} \stackrel{\text{def}}{=} \mathbf{t} + l_d$$

Thus, as T ranges over K_{Crit}, $\mathbf{s}$ ranges over $0, \frac{1}{d}, \ldots, \frac{d-1}{d}$. In particular, we have

$$||P(\delta_\chi^*) \cdot \Theta_\chi||^2_{\text{Tch}} = \frac{1}{d} \cdot \sum_{j \in K_{\text{Crit}}} \left|P(-)|_{T=j}\right|^2 = \frac{1}{d} \cdot \sum_{j=0}^{d-1} \left|P(j)|_{\mathbf{s}=\frac{j}{d}}\right|^2$$

This makes the relationship between the norms considered here and the norms of Chapter VII, §3, explicit. Recall that the "*normalized discrete Tchebycheff polynomials*" $\widetilde{t}_r(\mathbf{s})$ of Chapter VII, Proposition 3.2, are *orthonormal* with respect to the (square L^2-) norm $Q(\mathbf{s}) \mapsto \frac{1}{d} \cdot \sum_{j=0}^{d-1} \left|Q(\frac{j}{d})\right|^2$. Now we define the *(discrete) Tchebycheff canonical SW zeta functions* as follows:

$$\zeta_r^{\text{TCH}} \stackrel{\text{def}}{=} \widetilde{t}_r(\frac{\delta_\chi^*}{d} + l_d) \cdot (\Theta_\chi) = \sum_{k \in \mathbf{Z}} \widetilde{t}_r(\frac{k}{d} + l_d) \cdot q_{\text{or}}^{\frac{1}{d}(\frac{1}{2} \cdot k^2 + (i_\chi/n) \cdot k)} \cdot U_{\text{cv}}^{2mk + i_\chi} \cdot \chi(k_{\text{et}})$$

where $0 \le r \le d-1$, $\widetilde{t}_r(\mathbf{s})$ is the "normalized discrete Tchebycheff polynomial" of Chapter VII, Proposition 3.2. Thus,

> The ζ_r^{TCH} are *orthonormal* with respect to the inner product defined by $|| \sim ||_{\mathrm{Tch}}$.

Now we would like to estimate the difference between the various norms defined above:

Lemma 2.1. *Let A be an integer which satisfies $A \ge \mathrm{Max}(a, 1 + Q^{-1})$, where $Q \stackrel{\mathrm{def}}{=} \frac{\pi}{4} \cdot \mathrm{Im}(\tau_{\mathrm{or}})$. Then we have:*

$$||P(\delta_\chi^*) \cdot \Theta_\chi||_{\mathrm{Tch}_\epsilon} \le ||P(\delta_\chi^*) \cdot \Theta_\chi||_{w,\boldsymbol{\mu}_a} \le (r+1)^{\frac{9}{2}} \cdot e^{4r+4} \cdot A^{2r+1} \cdot ||P(\delta_\chi^*) \cdot \Theta_\chi||_{\mathrm{Tch}}$$

for any polynomial with complex coefficients $P(-)$ of degree $\le r < d$.

Proof. The first inequality was already noted in the discussion above. Thus, let us prove the second inequality. We may assume without loss of generality that $||P(\delta_\chi^*) \cdot \Theta_\chi||_{\mathrm{Tch}}^2 = 1$. Thus, it follows from the above discussion that

$$P(T) = \sum_{j=0}^{r} \gamma_j \cdot \widetilde{t}_j(\mathbf{t} + l_d)$$

where $\gamma_j \in \mathbf{C}$, $\sum_{j=0}^{r} |\gamma_j|^2 = 1$.

Next, we would like to bound the coefficients of $P(T)$ as a *polynomial in* $\mathbf{t}$. To do this, first we observe that the coefficients of $\widetilde{t}_j(-)$ (for $j \le r$) may be bounded by $(2r+1)^{\frac{1}{2}} \cdot e^{3r+1}$ (by Chapter VII, Proposition 3.2, (ii.)). Moreover, if one expands $(\mathbf{t} + l_d)^j$ (for $j \le r$) as a polynomial in $\mathbf{t}$, the absolute values of the coefficients are bounded by $\binom{j}{j'} \cdot |l_d|^{j'} \le 2^j \le 2^r$ (where $0 \le j' \le j$, and we note that $|l_d| \le \frac{1}{2}$). Since $(\mathbf{t} + l_d)^j$ (for $j \le r$) has $\le r+1$ terms as a polynomial in $\mathbf{t}$, it thus follows that as a polynomial in $\mathbf{t}$, $\widetilde{t}_j(\mathbf{t} + l_d)$ has coefficients of absolute value $\le (r+1)^{\frac{5}{2}} \cdot e^{3r+2} \cdot 2^r \le (r+1)^{\frac{5}{2}} \cdot e^{4r+2}$. Since $|\gamma_j| \le 1$ (for $j \le r$), we thus obtain that *as a polynomial in* $\mathbf{t}$, *$P(T) = P(\mathbf{t} + l_d)$ has coefficients of absolute value*

$$\le (r+1)^{\frac{7}{2}} \cdot e^{4r+2}$$

Thus, we obtain

$$\begin{aligned} ||P(\delta_\chi^*) \cdot \Theta_\chi||_{w,\boldsymbol{\mu}_a} &\le (r+1)^{\frac{7}{2}} \cdot e^{4r+2} \cdot \sum_{j=0}^{r} ||\left(\frac{\delta_\chi^*}{d}\right)^j \cdot \Theta_\chi||_{w,\boldsymbol{\mu}_a} \\ &\le (r+1)^{\frac{7}{2}} \cdot e^{4r+2} \cdot \sum_{j=0}^{r} ||\zeta_j^{\mathrm{SS}}||_{w,\boldsymbol{\mu}_a} \end{aligned}$$

On the other hand, the terms $||\zeta_j^{\mathrm{SS}}||_{w,\boldsymbol{\mu}_a}$ may be bounded by what was already done in Chapter VII, Lemma 6.1, as follows. To simplify the notation (more precisely, to free up the letter "j" for use in future summations), we consider the case $j = r$. When $j < r$, a similar argument applies. Now recall the general formula:

$$||f||^2_{w,\boldsymbol{\mu}_a} = \frac{1}{d} \cdot \sum_{j=0}^{d-1} w_j \cdot \sum_{j'=0}^{a-1} \Big| \sum_{k \in j + d\cdot j' + \widetilde{d}\cdot \mathbf{Z}} \phi_k \Big|^2$$

In the case of $f = \zeta_r^{\mathrm{SS}}$, we would like to divide up the task of bounding the $|\phi_k|$ into *two cases*, the case where $|k| < A \cdot d$, and the case where $|k| \geq A \cdot d$:

> *Case (i):* First, we consider the case where $|k| < A \cdot d$. In this case, it follows from Chapter VII, Lemma 6.1, (1), (plus the definition of w_k) that $w_j \cdot |\phi_k|^2 \leq A^{2r}$. Note that the number of k in a given residue class modulo $\widetilde{d} = a \cdot d$ that fall under the present Case (i) is $\leq 2 \cdot A \cdot a^{-1}$.
>
> *Case (ii):* Next, we consider the case where $|k| \geq A \cdot d$. Let k_0 denote the unique integer $\in K_{\mathrm{Crit}}$ such that $k \equiv k_0$ modulo d. Let σ_k be the sign of k, i.e., 1 (respectively, -1) if k is > 0 (respectively, < 0). Write $k_\delta \stackrel{\mathrm{def}}{=} \sigma_k \cdot (|k| - |k_0|)$. Note that since $|k_0| \leq d$, we obtain that $|k_\delta| \geq (A-1) \cdot d \geq 1$ (since $A \geq 2$). Thus,

$$\begin{aligned}(k^2 + (i_\chi/m)\cdot k) - (k_0^2 + (i_\chi/m)\cdot k_0) &= (|k_0| + |k_\delta|)^2 - k_0^2 + (i_\chi/m)\cdot(k_\delta + k - k_0 - k_\delta) \\ &= k_\delta^2 + (i_\chi/m)\cdot k_\delta + 2\cdot|k_0|\cdot|k_\delta| + (i_\chi/m)\cdot(\sigma_k\cdot|k_0| - k_0) \\ &\geq k_\delta^2 + (i_\chi/m)\cdot k_\delta + 2\cdot|k_0|\cdot(|k_\delta| - 1) \\ &\geq k_\delta^2 + (i_\chi/m)\cdot k_\delta\end{aligned}$$

> But note that, if we raise these inequalities to the base q_{sc}, we obtain that
>
> $$\begin{aligned} w_j \cdot |\phi_k|^2 = w_{k_0} \cdot |\phi_k|^2 &\leq \left(\frac{|k|}{|k_\delta|}\right)^{2r} \cdot |\phi_{k_\delta}|^2 \\ &\leq \left(\frac{A}{A-1}\right)^{2r} \cdot |\phi_{k_\delta}|^2 \leq 4^r \cdot |\phi_{k_\delta}|^2 \end{aligned}$$
>
> Thus, by Chapter VII, Lemma 6.1, if $A-1 \geq Q^{-1}$, where $Q \stackrel{\mathrm{def}}{=} \frac{\pi}{4}\cdot\mathrm{Im}(\tau_{\mathrm{or}})$ (and we take "b," "ϵ" in *loc. cit.* to be 1; "a" in *loc. cit.* to be $A-1$; "k" in *loc. cit.* to be k_δ), then
>
> $$\begin{aligned} \sum_{|k| \geq A\cdot d} w_j^{\frac{1}{2}} \cdot |\phi_k| &\leq 2^{r+2} \cdot e^{-(A-1)\cdot d} \leq 2^{d+2} \cdot e^{-(A-1)\cdot d} \\ &\leq 4 \cdot e^{-(A-2)\cdot d} \leq 4 \end{aligned}$$

If we then add up the contributions from these two cases, we obtain

$$\begin{aligned}\frac{1}{d}\cdot\sum_{j=0}^{d-1} w_j\cdot\sum_{j'=0}^{a-1}\Big|\sum_{k\in j+d\cdot j'+\widetilde{d}\cdot\mathbf{Z}}\phi_k\Big|^2 &\leq \frac{1}{d}\cdot\sum_{j=0}^{d-1}\sum_{j'=0}^{a-1}\Big(\sum_{k\in j+d\cdot j'+\widetilde{d}\cdot\mathbf{Z}} w_j^{\frac{1}{2}}\cdot\Big|\phi_k\Big|\Big)^2\\ &\leq \frac{1}{d}\cdot\sum_{j=0}^{d-1}\sum_{j'=0}^{a-1}\Big(2\cdot A\cdot a^{-1}\cdot A^{2r}+4\Big)^2\\ &\leq a\cdot\Big(6\cdot a^{-1}\cdot A^{2r+1}\Big)^2\leq\Big(6\cdot A^{2r+1}\Big)^2\end{aligned}$$

(where we use that $A \geq a$ in the third inequality). Thus, we have $||\zeta_r^{\mathrm{SS}}||_{w,\boldsymbol{\mu}_a} \leq 6\cdot A^{2r+1}$. Combining this with the previous inequalities concerning $||P(\delta_\chi^*)\cdot\Theta_\chi||^2_{w,\boldsymbol{\mu}_a}$, we thus obtain

$$||P(\delta_\chi^*)\cdot\Theta_\chi||_{w,\boldsymbol{\mu}_a} \leq (r+1)^{\frac{9}{2}}\cdot e^{4r+2}\cdot 6\cdot A^{2r+1} \leq (r+1)^{\frac{9}{2}}\cdot e^{4r+4}\cdot A^{2r+1}$$

as desired. ○

Next, we would like to *compare* $||\sim||_{\mathrm{Tch}_\epsilon}$ *and* $||\sim||_{\mathrm{Tch}}$.

Lemma 2.2. *Suppose that* $a \geq \mathrm{Max}(2+Q^{-1}, 8)$*, where* $Q \stackrel{\mathrm{def}}{=} \frac{\pi}{4}\cdot\mathrm{Im}(\tau_{\mathrm{or}})$*. Then we have:*

$$\Big\{1-(d+1)^{\frac{9}{2}}\cdot e^{4-d}\Big\}\cdot||P(\delta_\chi^*)\cdot\Theta_\chi||_{\mathrm{Tch}} \leq ||P(\delta_\chi^*)\cdot\Theta_\chi||_{\mathrm{Tch}_\epsilon} \leq \Big\{1+(d+1)^{\frac{9}{2}}\cdot e^{4-d}\Big\}\cdot||P(\delta_\chi^*)\cdot\Theta_\chi||_{\mathrm{Tch}}$$

for any polynomial with complex coefficients $P(-)$ *of degree* $\leq r < d$*. Here, the first expression in large brackets, i.e.,* $1-(d+1)^{\frac{9}{2}}\cdot e^{4-d}$*, is* $\geq \frac{1}{2}$ *if* $d \geq 25$*.*

Proof. The argument is similar to the argument given for "Case (ii)" in the proof of Lemma 2.1 (except that we take "A" to be $a-1$). Indeed, the difference between the two norms

$$||f||^2_{\mathrm{Tch}_\epsilon} \stackrel{\mathrm{def}}{=} \frac{1}{d}\cdot\sum_{j\in K_{\mathrm{Crit}}} w_j\cdot\Big|\sum_{k\in j+\widetilde{d}\cdot\mathbf{Z}}\phi_k\Big|^2;\quad\text{and}\quad ||f||^2_{\mathrm{Tch}} \stackrel{\mathrm{def}}{=} \frac{1}{d}\cdot\sum_{j\in K_{\mathrm{Crit}}} w_j\cdot\Big|\phi_j\Big|^2$$

is that the sum in the definition of $||f||^2_{\mathrm{Tch}_\epsilon}$ involves the ϕ_k for $k \in (K_{\mathrm{Crit}}+\widetilde{d}\cdot\mathbf{Z})\backslash K_{\mathrm{Crit}}$, i.e., (by the triangle inequality)

$$(||f||_{\mathrm{Tch}_\epsilon} - ||f||_{\mathrm{Tch}})^2 \leq ||f||^2_{\mathrm{Tch}_\Delta} \stackrel{\mathrm{def}}{=} \frac{1}{d} \cdot \sum_{j \in K_{\mathrm{Crit}}} w_j \cdot \Big| \sum_{k \in (j + \widetilde{d} \cdot \mathbf{Z}) \setminus \{j\}} \phi_k \Big|^2$$

Just as in the proof of Lemma 2.1, we may assume without loss of generality that $||P(\delta^*_\chi) \cdot \Theta_\chi||^2_{\mathrm{Tch}} = 1$. This implies (just as in the proof of Lemma 2.1) that the coefficients of $P(-)$ have absolute value $\leq (r+1)^{\frac{7}{2}} \cdot e^{4r+2}$, hence that

$$||P(\delta^*_\chi) \cdot \Theta_\chi||_{\mathrm{Tch}_\Delta} \leq (r+1)^{\frac{7}{2}} \cdot e^{4r+2} \cdot \sum_{j=0}^{r} ||\zeta^{\mathrm{SS}}_j||_{\mathrm{Tch}_\Delta}$$

Thus, it suffices to bound $||\zeta^{\mathrm{SS}}_r||_{\mathrm{Tch}_\Delta}$. But (as remarked at the beginning of this proof) this was essentially already done in "Case (ii)" of the proof of Lemma 2.1. More precisely, let $k_0 \in K_{\mathrm{Crit}}$, $k \equiv k_0$ modulo $\widetilde{d}$; assume, moreover, that $k \notin K_{\mathrm{Crit}}$. We would like to consider the coefficient ϕ_k in the expansion of $f = \zeta^{\mathrm{SS}}_r$. Note that since $|k_0| \leq d$, $k \notin K_{\mathrm{Crit}}$, it follows that $|k| \geq \widetilde{d} - d = (a-1)d$. Thus, if we substitute $a - 1$ ($\geq 1 + Q^{-1}$) for "A" in the treatment of Case (ii) in the proof of Lemma 2.1, we obtain

$$\sum_{|k| \geq (a-1) \cdot d} w^{\frac{1}{2}}_{k_0} \cdot |\phi_k| \leq 4 \cdot e^{-(a-3) \cdot d}$$

hence

$$\begin{aligned} ||\zeta^{\mathrm{SS}}_r||^2_{\mathrm{Tch}_\Delta} &= \frac{1}{d} \cdot \sum_{k_0 \in K_{\mathrm{Crit}}} w_{k_0} \cdot \Big| \sum_{k \in (k_0 + \widetilde{d} \cdot \mathbf{Z}) \setminus \{k_0\}} \phi_k \Big|^2 \\ &\leq \frac{1}{d} \cdot \sum_{k_0 \in K_{\mathrm{Crit}}} 16 \cdot e^{-2(a-3) \cdot d} = 16 \cdot e^{-2(a-3) \cdot d} \end{aligned}$$

Thus, we have

$$\begin{aligned} ||P(\delta^*_\chi) \cdot \Theta_\chi||_{\mathrm{Tch}_\Delta} &\leq (r+1)^{\frac{9}{2}} \cdot e^{4r+2} \cdot 4 \cdot e^{-(a-3) \cdot d} \\ &\leq (d+1)^{\frac{9}{2}} \cdot e^4 \cdot e^{-(a-7) \cdot d} \leq (d+1)^{\frac{9}{2}} \cdot e^{4-d} \end{aligned}$$

(where in the last inequality we use that $a \geq 8$). Moreover, $(d+1)^{\frac{9}{2}} \cdot e^{4-d} \leq \frac{1}{2}$ if $d \geq 25$. This completes the proof. ◯

We are now ready to state the main result of this §:

Theorem 2.3. *Let $\mathcal{H}^d_\chi$ be the vector space of functions on $\mathbf{S^1}$ of the form $P(\delta^*_\chi) \cdot \Theta_\chi$, where $P(-)$ is a polynomial of degree $< d$ with complex coefficients, and Θ_χ and δ^*_χ are as* *defined in Chapter VII, §6. For $r \in \mathbf{Z}$, let $F^r(\mathcal{H}^d_\chi) \subseteq \mathcal{H}^d_\chi$ be the subspace of such functions for which the degree of $P(-)$ is $\leq r$ (cf. the notation of Chapter VII, §1). Let $a \geq 8$ be an integer. Let $|| \sim ||_{w,\boldsymbol{\mu}_a}$ be the L^2-norm on $\mathcal{H}^d_\chi$ given by averaging the L^2-norms on $\alpha \cdot \boldsymbol{\mu}_d \subseteq (\mathbf{S^1})_{\mathrm{sc}}$ with weights $w_0, \ldots, w_{d-1}$ (as defined at the beginning of this §) as α varies in $\boldsymbol{\mu}_{a\cdot d}/\boldsymbol{\mu}_d$ (see the above discussion for more details). Let $|| \sim ||_{\mathrm{Tch}}$ be the L^2-norm on $\mathcal{H}^d_\chi$ given by:*

$$||P(\delta^*_\chi) \cdot \Theta_\chi||^2_{\mathrm{Tch}} = \frac{1}{d} \cdot \sum_{j=0}^{d-1} \left|P(j - \lambda_d)\right|^2$$

where l_d is $\frac{1}{d}$ times the constant "λ_d" of Chapter V, Theorem 4.8.

Then as r varies over $0, \ldots, d-1$, the functions

$$\zeta^{\mathrm{TCH}}_r \stackrel{\mathrm{def}}{=} \widetilde{t}_r(\frac{\delta^*_\chi}{d}+l_d)\cdot(\Theta_\chi) = \sum_{k\in\mathbf{Z}} \widetilde{t}_r(\frac{k}{d}+l_d)\cdot q_{\mathrm{or}}^{\frac{1}{d}(\frac{1}{2}\cdot k^2+(i_\chi/n)\cdot k)}\cdot U_{\mathrm{cv}}^{2mk+i_\chi}\cdot\chi(k_{\mathrm{et}}) \quad \in F^r(\mathcal{H}^d_\chi)$$

(where $\widetilde{t}_r(-)$ is the "normalized discrete Tchebycheff polynomial" of Chapter VII, Proposition 3.2) form an orthonormal basis *of $\mathcal{H}^d_\chi$ with respect to the inner product associated to norm $|| \sim ||_{\mathrm{Tch}}$. Moreover, if $a \geq \mathrm{Max}(2 + Q^{-1}, 8)$ (where $Q \stackrel{\mathrm{def}}{=} \frac{\pi}{4} \cdot \mathrm{Im}(\tau_{\mathrm{or}})$ – cf. Chapter VII, Lemma 6.1), and $d \geq 25$, then we have*

$$\frac{1}{2} \cdot ||P(\delta^*_\chi) \cdot \Theta_\chi||_{\mathrm{Tch}} \leq ||P(\delta^*_\chi) \cdot \Theta_\chi||_{w,\boldsymbol{\mu}_a} \leq (r+1)^{\frac{9}{2}} \cdot e^{4r+4} \cdot a^{2r+1} \cdot ||P(\delta^*_\chi) \cdot \Theta_\chi||_{\mathrm{Tch}}$$

In particular, if $\zeta^{w,\boldsymbol{\mu}_a}_0, \ldots, \zeta^{w,\boldsymbol{\mu}_a}_{d-1}$ form an orthonormal basis of $\mathcal{H}^d_\chi$ with respect to the inner product associated to $|| \sim ||_{w,\boldsymbol{\mu}_a}$, and we write $\zeta^{w,\boldsymbol{\mu}_a}_r = \sum_{j=0}^r \gamma_{r,j} \cdot \zeta^{\mathrm{TCH}}_j$, then $\sum_{j=0}^r |\gamma_{r,j}|^2 \leq 4$, and

$$\left\{(r+1)^{\frac{9}{2}} \cdot e^{4r+4} \cdot a^{2r+1}\right\}^{-1} \leq |\gamma_{r,r}| \leq 2$$

Finally, if $\mathrm{Im}(\tau_{\mathrm{or}})$ varies in a compact subset of the upper half-plane, then on that compact subset, the w_j are bounded above and below, i.e., there exist constants $C_1, C_2 \in \mathbf{R}_{>0}$ (depending only on the compact set) such that

$$C^d_1 \cdot || \sim ||_{1,\boldsymbol{\mu}_a} \leq || \sim ||_{w,\boldsymbol{\mu}_a} \leq C^d_2 \cdot || \sim ||_{1,\boldsymbol{\mu}_a}$$

where $|| \sim ||_{1,\boldsymbol{\mu}_a}$ is the norm defined in the same way as $|| \sim ||_{w,\boldsymbol{\mu}_a}$ except with all the "weights" (i.e., "w_j") equal to 1.

Proof. Most of the assertions of Theorem 2.3 have already been proven. Thus, it remains only to check the following: The inequalities relating $|| \sim ||_{w,\boldsymbol{\mu}_a}$ and $|| \sim ||_{\mathrm{Tch}}$ follow from Lemmas 2.1, 2.2 (where in Lemma 2.1, we take "A" to be a). The resulting bound $\sum_{j=0}^{r} |\gamma_{r,j}|^2 \leq 4$ follows immediately. This implies $|\gamma_{r,r}| \leq 2$. The lower bound for $|\gamma_{r,r}|$ results from the elementary geometry of Hilbert spaces, together with the inequality bounding $|| \sim ||_{w,\boldsymbol{\mu}_a}$ in terms of $|| \sim ||_{\mathrm{Tch}}$. ○

Remark. Thus, *stated in words*, Theorem 2.3 asserts the following:

> *Up to a factor of order C^r on the degree $\leq r$ portion of*
>
> $$\mathcal{H}_\chi^d = \{P(\delta_\chi^*) \cdot \Theta_\chi \mid \deg(P(-)) < d\}$$
>
> *the L^2-norm on $\mathcal{H}_\chi^d$ given by averaging the L^2-norms on the $\alpha \cdot \boldsymbol{\mu}_d$ (as α varies over $\boldsymbol{\mu}_{a \cdot d}/\boldsymbol{\mu}_a$, and $a \geq 8$, $d \geq 25$ are sufficiently large) with weights $w_0, \ldots, w_{d-1}$ induces orthonormal polynomials which are essentially the discrete Tchebycheff polynomials. Thus, as $d \to \infty$, the orthonormal polynomials for this averaged L^2-norm are essentially the Legendre polynomials (cf. Chapter VII, Propositions 2.1, 3.1). Moreover, if the elliptic curve in question varies on a compact set, then, up to a factor of order C^d, we may even assume that the weights (i.e., the "w_j") are all equal to* 1.

Stated in this way, we feel that this theorem justifies the terminology "*Legendre model*" used in the title of this §. Also, note that, as expected, the Legendre limit involves a *scaling factor* of d, i.e., this limit is a *slope* 1 *limit* – cf. the discussions at the end of Chapter VII, §3, 6. Finally, we remark that that if, on the other hand, one fixes d and lets $a \to \infty$, then one verifies easily that the norm with weights equal to 1, i.e.,

$$||f||^2_{1,\boldsymbol{\mu}_a} = \frac{1}{d} \cdot \sum_{j=0}^{d-1} \sum_{j'=0}^{a-1} \Big| \sum_{k \in j + d \cdot j' + \widetilde{d \cdot \mathbf{Z}}} \phi_k \Big|^2$$

converges (up to a constant factor) to the *usual L^2-norm on* $L^2(\mathbf{S^1})$. Thus, in this limit, the resulting orthonormal functions converge (up to a constant factor) to the *orthogonal canonical Schottky-Weierstrass zeta functions* $\zeta_r^{\mathrm{OR},\mathbf{S^1}}$ of Chapter VII, Definition 6.3.

§3. The Truncated Binomial Model

In this §, we discuss the analogue of the "truncated" (i.e., by comparison to $|| \sim ||_{\mathrm{Tch}_\epsilon}$, $|| \sim ||_{w,\boldsymbol{\mu}_a}$) norm $|| \sim ||_{\mathrm{Tch}}$ of §2 for what we call the *binomial model.* In the case of the Legendre model that we studied in §2, the truncated version was essentially isomorphic to the system of discrete Tchebycheff polynomials (cf. Chapter VII, §3), which is already well-known. In the present binomial case, however, the truncated version is not so well-known, so we give an independent treatment in the present §. This will prepare us for the following two §'s, in which we study the function space of derivatives of the theta function by regarding it as a *deformation of the truncated binomial model* (cf. what we did in §2 – i.e., we studied this same function space by regarding it as a deformation of the discrete Tchebycheff polynomials).

We maintain the notation of §2. In this §, we would like to consider a *filtration* on the set K_{Crit}, defined as follows. Recall the function "$L_r(T) = T + \lambda_r - i_\chi/n$" of Chapter V, Theorem 4.8. Then we define (for $r = 0, 1, \ldots, d$)

$$F^r(K_{\mathrm{Crit}}) \stackrel{\mathrm{def}}{=} \{0 - \lambda_r, 1 - \lambda_r, \ldots, r - 1 - \lambda_r\}$$

Thus, $F^r(K_{\mathrm{Crit}})$ has r elements; $F^0(K_{\mathrm{Crit}}) = \emptyset$; $F^d(K_{\mathrm{Crit}}) = K_{\mathrm{Crit}}$. Moreover, one checks easily from the definitions that (for $r = 1, \ldots, d$) $F^{r-1}(K_{\mathrm{Crit}}) \subseteq F^r(K_{\mathrm{Crit}})$. Thus, for $r = 0, \ldots, d-1$

$$F^{r+1}(K_{\mathrm{Crit}}) \backslash F^r(K_{\mathrm{Crit}})$$

consists of precisely one element, which we denote by $k[r]$. Moreover, the filtration $F^r(K_{\mathrm{Crit}})$ induces a *total ordering "$\leq_{\mathrm{Crit}}$"* on the set K_{Crit} defined by $k \leq_{\mathrm{Crit}} k'$ (for $k, k' \in K_{\mathrm{Crit}}$) if $k' \in F^r(K_{\mathrm{Crit}}) \Longrightarrow k \in F^r(K_{\mathrm{Crit}})$ (for all $r = 1, \ldots, d$). Put another way,

$$k[r_1] \leq_{\mathrm{Crit}} k[r_2] \quad \Longleftrightarrow \quad r_1 \leq r_2$$

When the chosen character belongs to Case I (respectively, Case II; Case III), and r is even (respectively, odd; either even or odd) $F^r(K_{\mathrm{Crit}})$ is the set of exponents of "U" (cf. Chapter V, Schola 4.1) of *the first r special monomials considered* in Chapter V, Schola 4.1, where "first" is relative to the ordering induced by the exponent of q.

Stated in a word, the purpose of the present and the next two §'s, is to study *the congruence canonical SW zeta functions* (cf. Chapter V, Theorem 4.8)

$$\zeta_r^{\mathrm{CG}} \stackrel{\mathrm{def}}{=} \binom{L_r(\delta^*)}{r}(\zeta_0^{\mathrm{CG}}) = \sum_{k \in \mathbf{Z}} \binom{k + \lambda_r}{r} \cdot q_{\mathrm{sc}}^{\frac{1}{2} \cdot k^2 + (i_\chi/n) \cdot k} \cdot U_{\mathrm{cv}}^{2mk + i_\chi} \cdot \chi(k_{\mathrm{et}})$$

at the infinite prime (where, for simplicity, we omit mention of the "trivialization" θ^m). In the present §, we would like to consider the following *truncated versions* of the ζ_r^{CG}:

$$Z_r^{\mathrm{CG}} \stackrel{\mathrm{def}}{=} q_{\mathrm{sc}}^{-\frac{1}{2}\cdot k[r]^2 - (i_\chi/n)\cdot k[r]} \cdot \sum_{k \in K_{\mathrm{Crit}}} \binom{k+\lambda_r}{r} \cdot q_{\mathrm{sc}}^{\frac{1}{2}\cdot k^2 + (i_\chi/n)\cdot k} \cdot U_{\mathrm{sc}}^k \cdot \chi(k_{\mathrm{et}})$$

where $U_{\mathrm{cv}}^n = U_{\mathrm{sc}}$ (note that we also divide by the unnecessary factor of $U_{\mathrm{cv}}^{i_\chi}$). Note that *the only nonzero terms of this sum over* K_{Crit} *are those for which* $k \notin F^r(K_{\mathrm{Crit}})$. Thus, the exponents of q_{sc} are all *nonnegative*. In the following discussion, we shall write

$$\Psi(k) \stackrel{\mathrm{def}}{=} \frac{1}{2} \cdot k^2 + \frac{i_\chi}{n} k$$

for the function of k appearing in the exponent of q_{sc}. In this §, we would like to think of the space of linear combinations of the Z_r^{CG} as equipped with the *usual* L^2*-norm* $|| \sim ||$ *on* $(\mathbf{S^1})_{\mathrm{sc}}$ (i.e., the norm for which the U_{sc}^k ($k \in \mathbf{Z}$) form an *orthonormal system* – cf. §1).

The purpose of the present § is to consider the following problem: Let

$$Z \stackrel{\mathrm{def}}{=} \sum_{r=0}^{d-1} \gamma_r \cdot Z_r^{\mathrm{CG}}$$

be a $\mathbf{C}$-linear combination (i.e., the $\gamma_r \in \mathbf{C}$) of the Z_r^{CG}.

> *Suppose that* $||Z|| = 1$. *Then to what extent can one bound the* $g_r \stackrel{\mathrm{def}}{=} |\gamma_r|$*?*

Since $||Z|| = 1$, and the U_{sc}^k ($k \in \mathbf{Z}$) form an orthonormal system, it follows that the coefficients of the U_{sc}^k (in the "Fourier expansion" of Z) are ≤ 1. In particular, the coefficient of $U_{\mathrm{sc}}^{k[r]}$ is ≤ 1, hence:

$$\left| \sum_{j=0}^{r} \gamma_j \cdot \binom{k[r]+\lambda_j}{j} \cdot q_{\mathrm{sc}}^{\Psi(k[r]) - \Psi(k[j])} \cdot \chi(k[r]_{\mathrm{et}}) \right| \leq 1$$

Thus, if we write

$$C[r,j] \stackrel{\mathrm{def}}{=} \binom{k[r]+\lambda_j}{j} \cdot |q_{\mathrm{sc}}|^{\Psi(k[r]) - \Psi(k[j])}$$

then we obtain

$$g_r \leq 1 + \sum_{r_1=0}^{r-1} C[r, r_1] \cdot g_{r_1}$$

(so in particular, $g_0 \leq 1$). If we then substitute into this last inequality the *analogous* inequality for g_{r_1}, we obtain

$$g_r \leq 1 + \sum_{r > r_1 \geq 0} C[r, r_1] + \sum_{r > r_1 > r_2 \geq 0} C[r, r_1] \cdot C[r_1, r_2] \cdot g_{r_2}$$

Continuing in this fashion, we obtain

$$g_r \leq 1 + \sum_{j=1}^{r} \sum_{r > r_1 > r_2 > \ldots > r_j \geq 0} C[r, r_1] \cdot C[r_1, r_2] \cdot \ldots \cdot C[r_{j-1}, r_j]$$

(where we use that $g_0 \leq 1$, and that if $r > r_1 > \ldots > r_j \geq 0$, then $j \leq r$). Next, we would like to analyze this double summation in more detail. First, we would like to estimate the *number of terms* appearing in this summation. By using the transformation $(r_1, r_2, \ldots, r_j) \mapsto (r - r_1, r_1 - r_2, r_2 - r_3, \ldots, r_{j-1} - r_j)$, one verifies easily that the number of terms in this double summation is $\leq$ the coefficient of $\mathbf{t}^r$ in the power series $(1 - \mathbf{t})^{-r}$ (where $\mathbf{t}$ is an indeterminate). By taking the r-th derivative of $(1 - \mathbf{t})^{-r}$ (and dividing by $r!$), we obtain that this coefficient is equal to $\binom{2r-1}{r} \leq 2^{2r-1} \leq 4^r - 1$. Thus, we obtain that

The number of terms in the double summation above is $\leq 4^r - 1$.

Next, we would like to *bound the* $C[r, r']$*'s* (where $r' < r$). This consists of two parts, i.e., bounding the *binomial coefficient portion* of $C[r, r']$, and bounding the *power of* $|q_{\mathrm{sc}}|$ appearing in $C[r, r']$.

Let us begin with the *binomial coefficient portion*, i.e.,

$$\binom{k[r] + \lambda_{r'}}{r'}$$

of $C[r, r']$, where $0 \leq r' < r \leq d - 1$. First, let us observe that $k[r' - 1] + \lambda_{r'}$ is either 0 or $r' - 1$. Now one may check easily from the definitions that the interval of integers $F^{r+1}(K_{\mathrm{Crit}})$ is obtained from interval of integers $F^r(K_{\mathrm{Crit}})$ by adjoining one more integer to $F^r(K_{\mathrm{Crit}})$, where this last integer lies either immediately to the left or immediately to the right of $F^r(K_{\mathrm{Crit}})$. Moreover, whether this last integers lies to the left or to the right depends only on the parity of r, i.e., as r increases, "left" and "right" occur alternately, one after the other. Thus, it follows that $k[r] + \lambda_{r'}$ is either $\in \{-\frac{1}{2}(r - r' + 2), \ldots, -1\}$ or $\in \{r', \ldots, r' - 1 + \frac{1}{2}(r - r' + 2)\}$. But this implies that $|\binom{k[r]+\lambda_{r'}}{r'}|$ is $\leq$ a product of

$\frac{1}{2}(r - r' + 2) \leq 2(r - r')$ positive integers, each of which is $\leq r' - 1 + \frac{1}{2}(r - r' + 2) = \frac{1}{2}(r + r') \leq r$. Thus, we obtain that:

> *The binomial coefficient portion of $C[r, r']$ is $\leq r^{2(r-r')}$. In particular, the binomial coefficient portion of the term*
>
> $$C[r, r_1] \cdot C[r_1, r_2] \cdot \ldots \cdot C[r_{j-1}, r_j]$$
>
> *in the double summation above is $\leq r^{2\{(r-r_1)+(r_1-r_2)+\ldots+(r_{j-1}-r_j)\}} \leq r^{2(r-r_j)}$.*

Next, we would like to bound the *power of* $|q_{\mathrm{sc}}|$, i.e.,

$$|q_{\mathrm{sc}}|^{\Psi(k[r])-\Psi(k[r'])}$$

appearing in $C[r, r']$ (where $0 \leq r' < r \leq d - 1$). Since $|q_{\mathrm{sc}}| < 1$, this means that we would like to bound $\Psi(k[r]) - \Psi(k[r'])$ from *below*. Note that we always have $\Psi(k[r]) - \Psi(k[r']) \geq 0$. In the following, we would like to obtain a stronger bound from below *under the assumption that* $r \geq r' + 6$.

Lemma 3.1. *Suppose that $r' + 6 \leq r$. Then $\Psi(k[r]) - \Psi(k[r']) \geq \frac{1}{16}(r + r')(r - r')$.*

Proof. The proof consists of a case by case analysis, depending on which of the three Cases I, II, III of Chapter V, Schola 4.1, the character χ belong to.

We begin with *Case I*. In this Case, r may be written as $2k$ or $2k + 1$ (where $k \geq 0$ is an integer). Then $\Psi(k[r]) = \frac{1}{2}(k^2 + k)$ (cf. Chapter V, Schola 4.1). Let us write k' for the "k" associated to r'. Then

$$\begin{aligned}
\frac{\Psi(k[r]) - \Psi(k[r'])}{r - r'} &= \left\{\frac{k - k'}{2(r - r')}\right\} \cdot (k + k' + 1) \geq \left\{\frac{r - 1 - r'}{4(r - r')}\right\} \cdot \left(\frac{r + r'}{2}\right) \\
&\geq \left\{\frac{r - r'}{8(r - r')}\right\} \cdot \left(\frac{r + r'}{2}\right) = \frac{1}{16}(r + r')
\end{aligned}$$

as desired.

Next, we consider *Case II*. In this Case, r may be written as $2k$ or $2k - 1$ (where $k \geq 0$ is an integer). Then $\Psi(k[r]) = \frac{1}{2} \cdot k^2$ (cf. Chapter V, Schola 4.1). Let us write k' for the "k" associated to r'. Then

$$
\begin{aligned}
\frac{\Psi(k[r]) - \Psi(k[r'])}{r - r'} &= \left\{\frac{k - k'}{2(r - r')}\right\} \cdot (k + k') \geq \left\{\frac{r - 1 - r'}{4(r - r')}\right\} \cdot \left(\frac{r + r'}{2}\right) \\
&\geq \left\{\frac{r - r'}{8(r - r')}\right\} \cdot \left(\frac{r + r'}{2}\right) = \frac{1}{16}(r + r')
\end{aligned}
$$

as desired.

Finally, we consider *Case III*. In this Case, r may be written as $2k$ or $2k - 1$ (where $k \geq 0$ is an integer). Then $\Psi(k[r])$ is $\geq \frac{1}{2}(k^2 - k)$, $\leq \frac{1}{2}(k^2 + k)$ (cf. Chapter V, Schola 4.1). Let us write k' for the "k" associated to r'. Note that since $r \geq r' + 6$, we have $r - r' - 3 \geq \frac{1}{2} \cdot (r - r')$, $k \geq k' + 2$. Thus,

$$
\begin{aligned}
\frac{\Psi(k[r]) - \Psi(k[r'])}{r - r'} &\geq \left\{\frac{k - k' - 1}{2(r - r')}\right\} \cdot \left(\frac{k^2 - k - (k')^2 - k'}{k - k' - 1}\right) \geq \left\{\frac{r - r' - 3}{4(r - r')}\right\} \cdot (k + k') \\
&\geq \left\{\frac{r - r'}{8(r - r')}\right\} \cdot \left(\frac{r + r'}{2}\right) = \frac{1}{16}(r + r')
\end{aligned}
$$

as desired. ○

Thus, in particular, it follows that

The power of $|q_{\mathrm{sc}}|$ appearing in the term $C[r, r_1] \cdot C[r_1, r_2] \cdot \ldots \cdot C[r_{j-1}, r_j]$ in the double summation above is

$$
\begin{aligned}
=&\{\Psi(k[r]) - \Psi(k[r_1])\} + \{\Psi(k[r_1]) - \Psi(k[r_2])\} + \\
&\ldots + \{\Psi(k[r_{j-1}]) - \Psi(k[r_j])\} \\
=&\Psi(k[r]) - \Psi(k[r_j])
\end{aligned}
$$

which is ≥ 0 always and $\geq \frac{1}{16}(r + r_j)(r - r_j)$ if $r \geq r_j + 6$.

Lemma 3.2. *We have*

$$
r \cdot \left\{\log(r) - 2r \cdot \left(\frac{\log^2(d)}{d}\right)\right\} \ \leq \ 12 \cdot d
$$

for any integer r satisfying $0 \leq r \leq d$.

Proof. Lemma 3.2 is clear if $r = 0, 1, 2$ (note that if $r = 1, 2$, then the expression in brackets $\{,\}$ is ≤ 1), or if $r = d$. Thus, we may assume that $r \geq 3$ (so, in particular, $d \geq 3$). Write

$$f(r) \stackrel{\text{def}}{=} r \cdot \Big\{\log(r) - 2r \cdot \Big(\frac{\log^2(d)}{d}\Big)\Big\}$$

Since we have checked that $f(r) \leq 12 \cdot d$ for $r = 2, d$, it suffices (by elementary calculus) to show that $f(\rho) \leq 12 \cdot d$ for any real number $\rho \in [2, d]$ such that $f'(\rho) = 0$. For such a ρ, we have

$$3 \cdot \log(\rho) \geq \log(\rho) + 1 = 4\rho \cdot \Big(\frac{\log^2(d)}{d}\Big)$$

(where we use that $\rho \geq 2$ implies $2 \cdot \log(\rho) \geq \log(4) \geq 1$) hence

$$f(\rho) = \rho \cdot \Big\{\log(\rho) - \frac{1}{2} \cdot (\log(\rho) + 1)\Big\} \leq \frac{1}{2} \cdot \rho \cdot \log(\rho)$$

Next, let us apply the function $\log^2(-)$ to both sides of the inequality $\frac{3d}{4 \cdot \log^2(d)} \geq \frac{\rho}{\log(\rho)}$ (≥ 1). Since both sides of this equality are ≥ 1, it thus follows that we obtain an inequality

$$\log^2(d) \geq \log^2\Big\{\frac{3d}{4 \cdot \log^2(d)}\Big\} \geq \log^2\Big(\frac{\rho}{\log(\rho)}\Big) \geq \frac{1}{16} \cdot \log^2(\rho)$$

(where we use that $\log(d) \geq 1$ (since $d \geq 3$), and $\frac{\rho}{\log(\rho)} \geq \rho^{\frac{1}{4}}$ (since $\rho \geq 2$)). If we then multiply the inequality $\log^2(d) \geq \frac{1}{16} \cdot \log^2(\rho)$ by $\frac{3d}{4 \cdot \log^2(d)} \geq \frac{\rho}{\log(\rho)}$, we thus obtain

$$\frac{3}{4} \cdot d \geq \log^2(d) \cdot \frac{3d}{4 \cdot \log^2(d)} \geq \frac{1}{16} \cdot \log^2(\rho) \cdot \frac{\rho}{\log(\rho)} = \frac{1}{16} \cdot \rho \cdot \log(\rho)$$

i.e., $f(\rho) \leq \rho \cdot \log(\rho) \leq 12 \cdot d$, as desired. ○

Now let us return to the inequality

$$g_r \leq 1 + \sum_{j=1}^{r} \sum_{r > r_1 > r_2 > \ldots > r_j \geq 0} C[r, r_1] \cdot C[r_1, r_2] \cdot \ldots \cdot C[r_{j-1}, r_j]$$

Suppose that q_{sc} satisfies:

$$-\log|q_{\text{sc}}| = 2\pi \cdot \frac{1}{d} \cdot \text{Im}(\tau_{\text{or}}) \geq \frac{64 \cdot \log^2(d)}{d}$$

Then by what we have done above (cf. especially Lemmas 3.1, 3.2), each term in the double summation satisfies:

$$\begin{aligned}\log(C[r,r_1]\cdot C[r_1,r_2]\cdot\ldots\cdot C[r_{j-1},r_j]) &\leq 2(r-r_j)\cdot\log(r)+\frac{1}{16}\cdot(r+r_j)(r-r_j)\cdot\log|q_{\mathrm{sc}}|\\ &\leq 2(r-r_j)\cdot(\log(r)+\frac{1}{32}\cdot r\cdot\log|q_{\mathrm{sc}}|)\\ &\leq 2(r-r_j)\cdot(\log(r)-2\cdot r\cdot d^{-1}\cdot\log^2(d))\\ &\leq 2(r-r_j)\cdot 12\cdot\frac{d}{r}\leq 24\cdot d\end{aligned}$$

when $r \geq r_j + 6$ and

$$\log(C[r,r_1]\cdot C[r_1,r_2]\cdot\ldots\cdot C[r_{j-1},r_j])\leq 2(r-r_j)\cdot\log(r)\leq 12\cdot\log(r)\leq 12\cdot\log(d)$$

otherwise. Thus, since (as we saw above) there are $4^r - 1$ such terms in the above double summation, (if we add in the additional "1+" at the beginning, so that we get a total of 4^r terms, then) it follows that

$$g_r \leq 4^r \cdot e^{24\cdot d} \leq e^{26\cdot d}$$

Now we are ready to state the *main result* of the present §:

Theorem 3.3. *Suppose that*

$$\mathrm{Im}(\tau_{\mathrm{or}}) \geq \frac{32}{\pi}\cdot\log^2(d)$$

and let $Z \stackrel{\mathrm{def}}{=} \sum_{r=0}^{d-1} \gamma_r \cdot Z_r^{\mathrm{CG}}$ *be a* $\mathbf{C}$*-linear combination (i.e., the* $\gamma_r \in \mathbf{C}$*) of the* Z_r^{CG}*. Then*

(1) *If the* $|\gamma_r| \leq 1$ *for* $r = 0, \ldots, d-1$*, then the* $L^2((\mathbf{S^1})_{\mathrm{sc}})$*-norm of* Z *satisfies:* $||Z|| \leq d^2 \cdot e^d \leq e^{3d}$*.*

(2) *If* $||Z|| \leq 1$*, then the* $|\gamma_r| \leq e^{26\cdot d}$*, for* $r = 0, \ldots, d-1$*.*

In particular, if we orthonormalize the Z_r^{CG} *to form*

$$Z_0^{\mathrm{OC}}, \ldots, Z_{d-1}^{\mathrm{OC}}$$

– *which are unique if we stipulate that the* leading coefficient of Z_r^{OC} *(i.e., the coefficient of* Z_r^{CG} *in the linear combination of* $Z_0^{\mathrm{CG}}, \ldots, Z_r^{\mathrm{CG}}$ *that forms* Z_r^{OC}*) be* positive *– then the coefficients of the* Z_r^{CG} *in each* $Z_{r'}^{\mathrm{OC}}$ *have absolute value* $\leq e^{26d}$*, and the absolute values of the leading coefficients are* $\geq d^{-2} \cdot e^{-d} \geq e^{-3d}$*.*

Proof. The assertion concerning the Z_r^{OC} at the end of Theorem 3.3 follows formally from Assertions (1) and (2) (cf. the proof of Theorem 2.3). Assertion (2) is precisely what we have just proven in the above discussion. Thus, it remains to verify Assertion (1). As we saw above, the absolute value of the coefficient of $U_{\mathrm{sc}}^{k[r]}$ (for $k[r] \in K_{\mathrm{Crit}}$) in Z is

$$\Big| \sum_{j=0}^{r} \gamma_j \cdot \binom{k[r]+\lambda_j}{j} \cdot q_{\mathrm{sc}}^{\Psi(k[r])-\Psi(k[j])} \cdot \chi(k[r]_{\mathrm{et}}) \Big| \leq \sum_{j=0}^{r} \Big| \binom{k[r]+\lambda_j}{j} \Big|$$

Thus, since the cardinality of K_{Crit} is d, it suffices to show that $\sum_{j=0}^{r} |\binom{k[r]+\lambda_j}{j}| \leq d \cdot e^d$, for all $k[r] \in K_{\mathrm{Crit}}$. Moreover, this last inequality will follow as soon as we show that

$$\Big| \binom{k[r]+\lambda_j}{j} \Big| \leq 2^d \leq e^d$$

But this follows from the analysis of the "binomial coefficient portion of $C[r, r']$" in the discussion above: Indeed, this analysis shows that there exists an integer N satisfying $j \leq N \leq j - 1 + \frac{1}{2}(r - j + 2) \leq r$ such that

$$\Big| \binom{k[r]+\lambda_j}{j} \Big| = \binom{N}{j} \leq 2^N \leq 2^r \leq e^r \leq e^d$$

as desired. ○

Remark. Thus, stated in words, Theorem 3.3 asserts that:

> *If* $\mathrm{Im}(\tau_{\mathrm{or}}) \geq \frac{32}{\pi} \cdot \log^2(d)$*, then up to a factor of order* C^d *(for some constant* C*), the* $L^2((\mathbf{S^1})_{\mathrm{sc}})$*-norm on the complex vector space generated by* $Z_0^{\mathrm{CG}}, \ldots, Z_{d-1}^{\mathrm{CG}}$*, is the same as the norm for which the functions* $Z_0^{\mathrm{CG}}, \ldots, Z_{d-1}^{\mathrm{CG}}$ *are orthonormal.*

Note that unlike the case with Theorem 2.3, in the present situation, one does not have the stronger result that the discrepancy is of order C^r on the subspace generated by $Z_0^{\mathrm{CG}}, \ldots, Z_r^{\mathrm{CG}}$. Indeed, typically results such as Theorem 3.3, (2), (i.e., where one must show that the coefficients are bounded whenever the $L^2((\mathbf{S^1})_{\mathrm{sc}})$-norm is bounded) are much more difficult than results such as Theorem 3.3, (1) (i.e., where one must show that if the coefficients are bounded, then the $L^2((\mathbf{S^1})_{\mathrm{sc}})$-norm is bounded). But it is not difficult to check that *even in this easier direction* (i.e., the direction of Theorem 3.3, (1)) *the binomial coefficients that one must bound do not satisfy* $\leq C^r$ *type inequalities.*

§4. The Combinatorics of the Full Binomial Model

In this and the following §'s, we discuss what we call the (full) *binomial model.* This "full" binomial model is a *deformation* of the truncated binomial model studied in §3 (cf. especially Theorem 3.3) in essentially the same way as the "Legendre model" studied in §2 (cf. especially Theorem 2.3) is a deformation of (what are essentially) the discrete Tchebycheff polynomials. One difference, however, between what we do in the present and following §'s and what was done in §3 is that here, we *do not use the averaging process* of §1. As a result, there are *two types* of deformation term that occur. The first type results from the fact that we do not use the averaging process of §1. This type will be dealt with in the discussion of the *semi-truncated binomial model* below, and is the more difficult to handle (of the two types). The second type is exactly the same as the deformation terms that occurred in "Case (ii)" of the proof of Lemma 2.1, and may be handled in exactly the same way as in the proof of Lemma 2.1. Unfortunately, since our estimates of the first type of deformation term are very involved, we treat the *combinatorial aspects of these estimates* in the present §. In the following §, we apply the results of the present § to study the orthogonal system which constitutes the "full binomial model."

We maintain the notation of §3. Also, for simplicity, we assume in the following discussion that

$$d \geq 12$$

Let us write

$$K_{\mathrm{Semi}} \stackrel{\mathrm{def}}{=} \{k \in \mathbf{Z} \mid {}^{\exists}k_0 \in K_{\mathrm{Crit}} \text{ s.t. } |k - k_0| = d,\ |k_0 + (i_\chi/n)| \geq \frac{d}{4},\ k \cdot k_0 < 0\}$$

Thus, $K_{\mathrm{Semi}} \bigcap K_{\mathrm{Crit}} = \emptyset$, $0 \notin K_{\mathrm{Semi}}$. In general, for $k \in \mathbf{Z}$, let us write

$$\mathrm{Crit}(k)$$

for the (unique) $k_0 \in K_{\mathrm{Crit}}$ such that $k \equiv k_0$ modulo d. Also, let us write

$$\{k[d]\} \stackrel{\mathrm{def}}{=} \{0 - \lambda_{d+1}, 1 - \lambda_{d+1}, \ldots, d - \lambda_{d+1}\} \backslash K_{\mathrm{Crit}}; \quad F^{d+1}(K_{\mathrm{Crit}}) \stackrel{\mathrm{def}}{=} \{k[d]\} \bigcup K_{\mathrm{Crit}}$$

In the following discussion, if F is a finite subset of $\mathbf{R}$, and $\lambda \in \mathbf{R}$, then we shall write

$$\mathrm{Dist}(\lambda, F) \stackrel{\mathrm{def}}{=} \mathrm{Min}_{f \in F} \ \{|\lambda - f|\}$$

for the "distance" between λ and F. Finally, we shall write

$$\widetilde{\Psi}(\lambda) \stackrel{\mathrm{def}}{=} \Psi(\lambda) + \frac{1}{2} \cdot (i_\chi/n)^2 = \frac{1}{2} \cdot (\lambda + (i_\chi/n))^2$$

for $\lambda \in \mathbf{R}$. Note that if $r_1 \leq r_2$ (where r_1, r_2 are nonnegative integers $\leq d$), then $\Psi(k[r_1]) \leq \Psi(k[r_2])$, $\widetilde{\Psi}(k[r_1]) \leq \widetilde{\Psi}(k[r_2])$.

We would like to begin by introducing the "*semi-truncated version*" of the ζ_r^{CG}, i.e., a function which is a sort of intermediate step between ζ_r^{CG} and Z_r^{CG}:

$$\begin{aligned}
\mathfrak{Z}_r^{\mathrm{CG}} &\stackrel{\mathrm{def}}{=} q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot \sum_{k \in K_{\mathrm{Crit}} \bigcup K_{\mathrm{Semi}}} \binom{k+\lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(k)} \cdot U_{\mathrm{sc}}^{\mathrm{Crit}(k)} \cdot \chi(k_{\mathrm{et}}) \\
&= Z_r^{\mathrm{CG}} + q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot \sum_{k \in K_{\mathrm{Semi}}} \binom{k+\lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(k)} \cdot U_{\mathrm{sc}}^{\mathrm{Crit}(k)} \cdot \chi(k_{\mathrm{et}})
\end{aligned}$$

The key to understanding $\mathfrak{Z}_r^{\mathrm{CG}}$ is the following *analysis of the combinatorics of the sets* K_{Crit}, K_{Semi}:

Lemma 4.1. *If $A, B \in \mathbf{R}$, let us write $[A, B]_{\mathrm{Int}}$ for the set of integers in the closed interval $[A, B] \subseteq \mathbf{R}$. Let $r \in [2, d]_{\mathrm{Int}}$. Then*

$$\left[-\frac{(r-2)}{2}, \frac{(r-2)}{2}\right]_{\mathrm{Int}} \subseteq F^r(K_{\mathrm{Crit}}) = [-\lambda_r, r-1-\lambda_r]_{\mathrm{Int}} \subseteq \left[-\frac{r}{2}, \frac{r}{2}\right]_{\mathrm{Int}}$$

In particular, for any $k_0 \in F^r(K_{\mathrm{Crit}})$, we have $|k_0| \leq \frac{r}{2}$, and $\frac{r}{2} \geq |k[r-1]| \geq \frac{r-2}{2}$. Finally, the average (or "center of mass") $\mathrm{Avg}(F^r(K_{\mathrm{Crit}}))$ of the elements of $F^r(K_{\mathrm{Crit}})$ is $\frac{1}{2}(r-1) - \lambda_r \in \{0, \pm\frac{1}{2}\}$.

Proof. Indeed, the number λ_r of Chapter V, Theorem 4.8, satisfies $\frac{r}{2} \geq \lambda_r \geq \frac{r-2}{2} \geq 0$ (since $r \geq 2$). Thus, $F^r(K_{\mathrm{Crit}}) = [-\lambda_r, r-1-\lambda_r]_{\mathrm{Int}} \subseteq [-\frac{r}{2}, \frac{r}{2}]$; $F^r(K_{\mathrm{Crit}}) = [-\lambda_r, r-1-\lambda_r]_{\mathrm{Int}} \supseteq [-\frac{r-2}{2}, \frac{r-2}{2}]$. Since $k[r-1] \in \{-\lambda_r, r-1-\lambda_r\}$, it thus follows that $|k[r-1]| \geq \mathrm{Min}(\lambda_r, r-1-\lambda_r) \geq \mathrm{Min}(\frac{r-2}{2}, \frac{r-2}{2}) = \frac{r-2}{2}$. Finally, the statement concerning the average follows immediately. ○

Lemma 4.2. *Suppose that $k \in \mathbf{Z} \backslash (K_{\mathrm{Crit}} \bigcup K_{\mathrm{Semi}})$. Then $|k + (i_\chi/n)| \geq \frac{3}{4}d$; $|k| \geq \frac{1}{2}d$. Moreover, we have $\Psi(k) - \Psi(k[d-1]) \geq \frac{1}{16} \cdot d \cdot |k|$.*

Proof. Write $k_0 \stackrel{\mathrm{def}}{=} \mathrm{Crit}(k)$. Since $k \notin K_{\mathrm{Semi}}, \notin K_{\mathrm{Crit}}$, it follows from the definitions that at least one of the following holds: $|k - k_0| \geq 2d$, *or* $|k_0 + (i_\chi/n)| < \frac{d}{4}$, *or* $k \cdot k_0 \geq 0$. Suppose that $|k - k_0| \geq 2d$. By Lemma 4.1 above, we have $|k_0| \leq \frac{d}{2}$. Thus, it follows that $|k + (i_\chi/n)| \geq 2d - |k_0| - \frac{1}{2} \geq 2d - \frac{1}{2}d - \frac{1}{2} = \frac{3}{2}d - \frac{1}{2} \geq \frac{3}{4}d$ (since $d \geq 1$), as desired.

Next, let us assume that $|k_0 + (i_\chi/n)| < \frac{d}{4}$. Write $k = k_0 + A \cdot d$, where $A \in \mathbf{Z}$, $A \neq 0$. Then $|k + (i_\chi/n)| \geq |A| \cdot d - |k_0 + (i_\chi/n)| \geq d - \frac{1}{4}d = \frac{3}{4}d$, as desired.

Finally, let us assume that $k \cdot k_0 \geq 0$. Write $k = k_0 + A \cdot d$, where $A \in \mathbf{Z}$, $A \neq 0$. Then since $|k_0| \leq \frac{d}{2} \leq d - 1$ (by Lemma 4.1, $d \geq 2$), it follows that $|A| \cdot d \geq d > |k_0|$. Thus, $k \cdot k_0 \geq 0$ implies that $A \cdot k_0 \cdot d \geq -k_0^2$, hence that $A \cdot k_0 \geq 0$. In particular, $|k + (i_\chi/n)| \geq |A \cdot d + k_0| - \frac{1}{2} \geq |A| \cdot d - \frac{1}{2} \geq d - \frac{1}{2} \geq \frac{3}{4}d$ (since $d \geq 2$), as desired.

This completes the proof of the assertion $|k + (i_\chi/n)| \geq \frac{3}{4}d$. This implies that $|k| \geq |k + (i_\chi/n)| - \frac{1}{2} \geq \frac{3}{4}d - \frac{1}{2} \geq \frac{1}{2}d$ (since $d \geq 2$). Now we compute (using this assertion, together with the estimates of Lemma 4.1):

$$
\begin{aligned}
2(\Psi(k) - \Psi(k[d-1])) &= 2(\widetilde{\Psi}(k) - \widetilde{\Psi}(k[d-1])) = (k + (i_\chi/n))^2 - (k[d-1] + (i_\chi/n))^2 \\
&= \{|k + (i_\chi/n)| + |k[d-1] + (i_\chi/n)|\} \\
&\quad \cdot \{|k + (i_\chi/n)| - |k[d-1] + (i_\chi/n)|\} \\
&\geq \frac{3}{4}d \cdot (|k + (i_\chi/n)| - \frac{1}{2} - |k[d-1]|) \\
&\geq \frac{3}{4}d \cdot (|k + (i_\chi/n)| - \frac{1}{2}(d+2) + \frac{1}{2}) \\
&\geq \frac{3}{4}d \cdot (|k + (i_\chi/n)| - \frac{5}{8}d + \frac{1}{2}) \geq \frac{3}{4}d \cdot (\frac{1}{6}|k + (i_\chi/n)| + \frac{1}{2}) \\
&\geq \frac{1}{8}d \cdot (|k + (i_\chi/n)| + 3) \geq \frac{1}{8}d \cdot (|k| - \frac{1}{2} + 3) \geq \frac{1}{8} \cdot d \cdot |k|
\end{aligned}
$$

(where we use $d \geq 8$), as desired. ◯

Lemma 4.3. *Let $k \in K_{\mathrm{Semi}}$. Suppose that there exists some $k[r] \in K_{\mathrm{Crit}}$ (where $r \in [0, d-1]_{\mathrm{Int}}$) such that $\Psi(k) = \Psi(k[r])$. Then $r = d-1$, $k = k[d]$, and the character χ belongs (in the language of Chapter V, §4) to Cases I or II. Moreover, under these hypotheses,* $\mathrm{Crit}(k) = k[d-1]$, *and (for any $r' < d-1$) we have $\Psi(k[r']) < \Psi(k[r]) = \Psi(k[d-1])$. We shall refer to such k as* exceptional.

Proof. Indeed, this follows from the discussion of Chapter V, Schola 4.1. More precisely, if χ belongs to Case III, then Ψ is injective on $\mathbf{Z}$. Thus, χ must belong to Cases I or II.

It follows from the discussion of Chapter V, Schola 4.1, that the map $r' \mapsto \Psi(k[r'])$ (for $r' = 0, \ldots, d$) is *monotone increasing* even in Cases I or II, and that the fibers of the map $k' \mapsto \Psi(k')$ (for $k' \in \mathbf{Z}$) contain at most two elements. Moreover, since $k \notin K_{\mathrm{Crit}}$, it follows that $\Psi(k[d]) \leq \Psi(k)$.

Now I *claim* that $\Psi(k[d-1]) = \Psi(k[d])$. Indeed, if $\Psi(k[d-1]) < \Psi(k[d])$, then it follows that $\Psi(k[r]) \leq \Psi(k[d-1]) < \Psi(k[d]) \leq \Psi(k)$ (which contradicts the hypothesis $\Psi(k[r]) = \Psi(k)$). This proves the claim.

Thus, since the fibers of the map $r' \mapsto \Psi(k[r'])$ (for $r' = 0, \ldots, d$) contain at most two elements, it follows that for $r' = 0, \ldots, d-2$, we have $\Psi(k[r']) < \Psi(k[d-1]) = \Psi(k[d]) \leq \Psi(k)$. Thus, we conclude that $r = d-1$. If $k[d] \neq k$, then the fact that the fibers of the map $k' \mapsto \Psi(k')$ (for $k' \in \mathbf{Z}$) contain at most two elements implies that $\Psi(k[r]) = \Psi(k[d-1]) = \Psi(k[d]) < \Psi(k)$, which is absurd. Thus, we conclude that $k[d] = k$.

Thus, it remains to prove that $\mathrm{Crit}(k) = k[d-1]$. But this follows from the explicit analysis of Chapter V, Schola 4.1. Indeed, in Case I, since $\Psi(k[d-1]) = \Psi(k[d])$, it follows that d is *odd*, and that $|k[d] - k[d-1]| = 2 \cdot \{\frac{1}{2}(d-1)\} + 1 = d$. Similarly, in Case II, since $k[d-1] = k[d]$, it follows that d is *even*, and that $|k[d]| = |k[d-1]| = \frac{1}{2}d$, so $|k[d] - k[d-1]| = d$, as desired. This completes the proof. ○

Lemma 4.4. *Suppose that k is* exceptional *(cf. Lemma 4.3). Write $k_0 \stackrel{\mathrm{def}}{=} \mathrm{Crit}(k)$. Then the coefficient of $U_{\mathrm{sc}}^{k_0}$ in $\mathfrak{Z}_{d-1}^{\mathrm{CG}}$ is a complex number whose absolute value is ≤ 2 and $\geq \frac{\pi}{2m} = \frac{\pi}{n}$.*

Proof. Recall the original nontruncated series

$$\zeta_r^{\mathrm{CG}} \stackrel{\mathrm{def}}{=} \binom{L_r(\delta^*)}{r}(\zeta_0^{\mathrm{CG}}) = \sum_{k' \in \mathbf{Z}} \binom{k' + \lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(k')} \cdot U_{\mathrm{cv}}^{2mk' + i_\chi} \cdot \chi((k')_{\mathrm{et}})$$

Now I *claim* that as k' ranges over $k_0 + d \cdot \mathbf{Z}$, the only k' (other than $k_0 = k[d-1]$) for which $\Psi(k') = \Psi(k_0)$ is the current exceptional k under consideration. Indeed, if $k' \in K_{\mathrm{Crit}}$, this follows from Lemma 4.3. If $k' \in K_{\mathrm{Semi}}$, then by Lemma 4.3, $k' = k[d] = k$. Finally, if $k' \notin K_{\mathrm{Crit}}, \notin K_{\mathrm{Semi}}$, then by Lemma 4.2, $\Psi(k') > \Psi(k[d-1]) = \Psi(k_0)$. This completes the proof of the claim.

Next, recall from Lemma 4.3 that $\Psi(k[d-1]) > \Psi(k[r])$ for $r < d-1$. It thus follows that *if we restrict the $q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot \zeta_r^{\mathrm{CG}}$ (for $r = 0, \ldots, d-1$) to $\boldsymbol{\mu}_d \subseteq (\mathbf{S^1})_{\mathrm{sc}}$, and let $q_{\mathrm{sc}} \to 0$*, then the coefficient of $U_{\mathrm{cv}}^{2mk_0 + i_\chi}$ will be 0 for $r < d-1$, and it will be equal (by the claim of the preceding paragraph) to

$$\sum_{k' = k_0, k} \binom{k' + \lambda_{d-1}}{d-1} \cdot \chi((k')_{\mathrm{et}})$$

for $r = d-1$. Note that for $k' = k_0, k$, we have $k' + \lambda_{d-1} \in \{-1, d-1\}$, so $\binom{k'+\lambda_{d-1}}{d-1} = \pm 1$. Thus, the coefficient in question will be a sum of *two n-th roots of unity*. This shows that this coefficient has absolute value ≤ 2. It also shows that if this coefficient is nonzero, then it has absolute value $\geq \sin(2\pi/n) = \sin(\pi/m) \geq \frac{\pi}{2m}$ (where the inequality may be proven by elementary calculus, using the fact that $m \geq 2$). Moreover, it follows immediately from the definitions that this coefficient is the coefficient referred to in the statement of Lemma 4.4.

Thus, it suffices to show that this coefficient is *nonzero*. But since the corresponding coefficients in $q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot \zeta_r^{\mathrm{CG}}$ are 0 for $r < d-1$, to say that the present coefficient in question is zero would imply that the *comparison isomorphism* (Chapter VI, Theorem 4.1 – in fact, we only need it in characteristic zero, i.e., Chapter VI, Theorem 3.1) is *false*. This contradiction completes the proof of Lemma 4.4. ○

Remark. The proof of Lemma 4.4 is interesting because it may be turned around and *regarded as an explicit way to verify the comparison theorem* (Chapter VI, Theorem 4.1) in a neighborhood of infinity. That is to say, the fact that the coefficient in question is nonzero may also be proven by a *direct computation* (which is not difficult, and is left as an exercise for the reader). This direct computation may then be regarded as an alternate proof of Chapter VI, Theorem 4.1, in a neighborhood of infinity. Put another way, this portion of the treatment of the archimedean theory in the present Chapter may be regarded as the *archimedean analogue* of the description of the scheme-theoretic *zero locus of the determinant* given in Chapter VI, Theorem 4.1, (2).

Lemma 4.5. *Let $k \in K_{\mathrm{Semi}}$. Then $|k| \geq \frac{1}{2}d$.*

Proof. Write $k_0 \stackrel{\mathrm{def}}{=} \mathrm{Crit}(k)$. Since $|k - k_0| = d$, we have $|k| \geq d - |k_0| \geq d - \frac{1}{2}d = \frac{1}{2}d$ (by Lemma 4.1). ○

Lemma 4.6. *Let X, A, and B be nonnegative real numbers such that $B \geq A$, $B > 0$. Then $(X + A) \geq (A/B) \cdot (X + B)$.*

Proof. Indeed, $B(X + A) \geq B \cdot X + A \cdot B \geq A \cdot X + A \cdot B = A(X + B)$. ○

Lemma 4.7. *Let $k \in K_{\mathrm{Semi}}$. Write $k_0 = k[r_0] = \mathrm{Crit}(k)$. Let r be an integer satisfying $4 \leq r \leq d-1$. Then*

$$\Psi(k) - \Psi(k[r]) \geq \frac{1}{32n} \cdot d \cdot \mathrm{Dist}(k, F^r(K_{\mathrm{Crit}}))$$

so long as either k is non-exceptional *(cf. Lemma 4.3) or $r < d-1$.*

Proof. Write $\sigma_k \stackrel{\text{def}}{=} k/|k|$ (note that $|k| \geq 1 > 0$, by Lemma 4.5), $k[r]^* \stackrel{\text{def}}{=} -(i_\chi/n) + \sigma_k \cdot |k[r] + (i_\chi/n)|$. By Lemma 4.1, $|k[r]| \geq \frac{r+1}{2} - 1 \geq 1 + \frac{1}{2}$. It thus follows that $k[r]^*$ has the *same sign* as k, and that $|k[r]^*| \geq \frac{r+1}{2} - 1 - 1 = \frac{r-3}{2}$.

Write $\mathbf{R} \cdot F^r(K_{\text{Crit}})$ for the closed convex hull (in $\mathbf{R}$) of the points in $F^r(K_{\text{Crit}})$. Note that $k \notin F^r(K_{\text{Crit}})$, so $k \notin \mathbf{R} \cdot F^r(K_{\text{Crit}})$; $0 \in \mathbf{R} \cdot F^r(K_{\text{Crit}})$ (by Lemma 4.1). Observe that k divides the real line into two *open rays.* Now I *claim* that $k[r]^*$ lies in the same open ray as $\mathbf{R} \cdot F^r(K_{\text{Crit}})$. Indeed, if this were not the case, then since $0 \in \mathbf{R} \cdot F^r(K_{\text{Crit}})$, it would then follow that $|k[r]^*| \geq |k| \geq 1$. Since $k[r]^*$ and k have the same sign, it thus follows that $|k[r]^* + (i_\chi/n)| - |k[r]^*| = |k + (i_\chi/n)| - |k|$, hence that $|k[r]^* + (i_\chi/n)| \geq |k + (i_\chi/n)|$. But this implies that

$$2 \cdot \widetilde{\Psi}(k[r]) = (k[r] + (i_\chi/n))^2 = (k[r]^* + (i_\chi/n))^2 \geq (k + (i_\chi/n))^2 = 2 \cdot \widetilde{\Psi}(k)$$

i.e., that $\Psi(k[r]) \geq \Psi(k)$. On the other hand, since $k[r] \in K_{\text{Crit}}$, $k \notin K_{\text{Crit}}$, we have $\Psi(k[r]) \leq \Psi(k)$. Thus, we conclude that $\Psi(k[r]) = \Psi(k)$, so k is *exceptional* and $r = d - 1$, contrary to the hypotheses of Lemma 4.7. This completes the proof of the claim.

Thus, $k[r]^* \neq k$ lies to the same side of k as $\mathbf{R} \cdot F^r(K_{\text{Crit}})$. Next, *let us prove the inequality*

$$|k - k[r]^*| \geq \frac{1}{4n} \cdot \text{Dist}(k, F^r(K_{\text{Crit}}))$$

Indeed, if $|k - k[r]^*| \geq \text{Dist}(k, F^r(K_{\text{Crit}}))$, then this inequality follows immediately, so we may assume that $|k - k[r]^*| < \text{Dist}(k, F^r(K_{\text{Crit}}))$, i.e., that $k[r]^*$ lies *between* k and $\mathbf{R} \cdot F^r(K_{\text{Crit}})$. Since $k \neq k[r]^*$ and $n \cdot k, n \cdot k[r]^* \in \mathbf{Z}$, it follows that $|k - k[r]^*| \geq \frac{1}{n}$. On the other hand, $\text{Dist}(k[r]^*, F^r(K_{\text{Crit}})) = |k[r]^* - k[r']|$ for some $r' \in \{r-2, r-1\}$ (i.e., $k[r']$ is the closest element $\in F^r(K_{\text{Crit}})$ to $k[r]^*$). Moreover, by Lemma 4.1, we have $\frac{r+1}{2} + 1 \geq |k[r]^*| \geq \frac{r-3}{2}$; $\frac{r}{2} \geq |k[r-1]|, |k[r-2]| \geq \frac{r-3}{2} > 0$. Thus, in particular, it follows that $k[r']$ and $k[r]^*$ have the *same sign*, so we obtain

$$\begin{aligned}\text{Dist}(k[r]^*, F^r(K_{\text{Crit}})) &= |k[r]^* - k[r']| \\ &\leq \text{Max}\Big(\frac{1}{2}(r+3) - \frac{1}{2}(r-3), \frac{1}{2}r - \frac{1}{2}(r-3)\Big) = 3\end{aligned}$$

Thus, by Lemma 4.6 (where we take the quantity "$X + A$" (respectively, "$X + B$") of Lemma 4.6 to be $|k - k[r]^*|$ (respectively, $\text{Dist}(k, F^r(K_{\text{Crit}}))$)), we have

$$|k - k[r]^*| \geq \frac{1}{n} \cdot \left(\frac{1}{n} + 3\right)^{-1} \cdot \text{Dist}(k, F^r(K_{\text{Crit}})) \geq \frac{1}{4n} \cdot \text{Dist}(k, F^r(K_{\text{Crit}}))$$

This completes the proof of the inequality $|k - k[r]^*| \geq \frac{1}{4n} \cdot \text{Dist}(k, F^r(K_{\text{Crit}}))$.

Now we are ready to complete the proof of Lemma 4.7. Note that since either k is *non-exceptional* or $r < d - 1$, we have $\Psi(k) > \Psi(k[r])$ (cf. Lemma 4.3). Also, by the definition of $k[r]^*$, it follows that $k[r]^* + (i_\chi/n)$ and $k + (i_\chi/n)$ have the same sign. Thus, we compute (using the inequality of the preceding paragraph, together with Lemma 4.5):

$$\begin{aligned}
2(\Psi(k) - \Psi(k[r])) &= 2(\widetilde{\Psi}(k) - \widetilde{\Psi}(k[r])) = 2(\widetilde{\Psi}(k) - \widetilde{\Psi}(k[r]^*)) \\
&= (k + (i_\chi/n))^2 - (k[r]^* + (i_\chi/n))^2 \\
&= (|k + (i_\chi/n)| + |k[r]^* + (i_\chi/n)|) \cdot (|k + (i_\chi/n)| - |k[r]^* + (i_\chi/n)|) \\
&\geq |k + (i_\chi/n)| \cdot |k + (i_\chi/n) - k[r]^* - (i_\chi/n)| \\
&\geq (\frac{1}{2}d - \frac{1}{2}) \cdot |k - k[r]^*| \\
&\geq (\frac{1}{4}d) \cdot \frac{1}{4n} \cdot \text{Dist}(k, F^r(K_{\text{Crit}})) = \frac{1}{16n} \cdot d \cdot \text{Dist}(k, F^r(K_{\text{Crit}}))
\end{aligned}$$

(where we use that $d \geq 2$) as desired. $\bigcirc$

Finally, we need one more lemma in the style of Lemma 4.7, which is a sort of analogue (in the present context) of Lemma 3.1 of §3:

Lemma 4.8. *Let $r > r'$ be nonnegative integers $\leq d - 1$ such that*

$$|k[r] + (i_\chi/n)|, |k[r'] + (i_\chi/n)| \geq \frac{d}{4}$$

and $\Psi(k[r]) \neq \Psi(k[r'])$. Then $\Psi(k[r]) - \Psi(k[r']) \geq \frac{1}{32n} \cdot d \cdot \text{Dist}(k[r], F^{r'}(K_{\text{Crit}}))$.

Proof. Write $k \stackrel{\text{def}}{=} k[r]$, $k' \stackrel{\text{def}}{=} k[r']$, $\sigma_k \stackrel{\text{def}}{=} k/|k|$ (note that $|k| \geq |k + (i_\chi/n)| - \frac{1}{2} \geq \frac{d}{4} - \frac{1}{2} \geq 1 > 0$, since $d \geq 6$), $(k')^* \stackrel{\text{def}}{=} -(i_\chi/n) + \sigma_k \cdot |k' + (i_\chi/n)|$. Since $|k + (i_\chi/n)|, |k' + (i_\chi/n)| \geq \frac{d}{4} > 1$, it follows that k, $(k')^*$, $k + (i_\chi/n)$, and $(k')^* + (i_\chi/n)$ have the *same sign*. Thus,

$$|k+(i_\chi/n)| - |(k')^* + (i_\chi/n)| = \sigma_k(k + (i_\chi/n) - (k')^* - (i_\chi/n)) = \sigma_k(k - (k')^*) = |k| - |(k')^*|$$

On the other hand, since $\Psi(k) > \Psi(k') = \Psi((k')^*)$, it follows that this difference of absolute values is > 0, hence that

$$|k| > |(k')^*|$$

Next, observe that $\lambda_1 = 0$ (cf. Chapter V, Theorem 4.8). Thus, $k[0] = 0$, hence $r, r' \geq 1$ (since $|k|, |k'| \geq \frac{d}{4} - \frac{1}{2} \geq 1$). In particular, $0 \in F^1(K_{\text{Crit}}) \subseteq F^{r'}(K_{\text{Crit}})$. On the other hand, since $r > r'$, $k = k[r] \notin F^{r'}(K_{\text{Crit}})$. Thus, we conclude that $(k')^*$ and $F^{r'}(K_{\text{Crit}})$ lie to the *same side* of k (i.e., in the same connected component of $\mathbf{R}\backslash\{k\}$).

Note that the fact that $0 \in F^{r'}(K_{\text{Crit}})$ also implies that $F^{r'}(K_{\text{Crit}})$ is *nonempty*. Let us write $k[r'']$ for the unique element of $F^{r'}(K_{\text{Crit}})$ such that $|k - k[r'']| = \text{Dist}(k, F^{r'}(K_{\text{Crit}}))$. Then $r'' \in \{r' - 1, r' - 2\}$. Thus, by Lemma 4.1 (plus the fact that $k[0] = 0$), we have:

$$\frac{r''+1}{2} \geq |k[r'']| \geq \frac{r''-1}{2}$$

On the other hand, again by Lemma 4.1, we have

$$\frac{r'+1}{2} \geq |k'| = |k[r']| \geq \frac{r'-1}{2}$$

which implies that

$$\frac{r'+3}{2} \geq |(k')^*| \geq \frac{r'-3}{2}$$

hence (since $k[r''] \cdot k \geq 0$, $(k')^* \cdot k > 0$) that

$$\begin{aligned} |(k')^* - k[r'']| &\leq \text{Max}(\frac{r''+1}{2} - \frac{r'-3}{2}, \frac{r'+3}{2} - \frac{r''-1}{2}) \\ &\leq \frac{1}{2} \cdot \text{Max}(r'' - r' + 4, r' - r'' + 4) \leq 3 \end{aligned}$$

Now we are ready to prove the *inequality*:

$$|k - (k')^*| \geq \frac{1}{4n} \cdot \text{Dist}(k, F^{r'}(K_{\text{Crit}}))$$

Indeed, without loss of generality, we may assume that $|k - (k')^*| < \text{Dist}(k, F^{r'}(K_{\text{Crit}}))$, i.e., that $(k')^*$ *lies strictly between* k and $F^{r'}(K_{\text{Crit}})$. Since $k \neq (k')^*$ and $n \cdot k, n \cdot (k')^* \in \mathbf{Z}$, it follows that $|k - (k')^*| \geq \frac{1}{n}$. On the other hand, $|(k')^* - k[r'']| \leq 3$, so by Lemma 4.6, we have

$$|k - (k')^*| \geq \frac{1}{n} \cdot (\frac{1}{n} + 3)^{-1} \cdot |k - k[r'']| \geq \frac{1}{4n} \cdot |k - k[r'']| = \frac{1}{4n} \cdot \text{Dist}(k, F^{r'}(K_{\text{Crit}}))$$

as desired.

Thus, just as in Lemma 4.7, we have:

$$
\begin{aligned}
2(\Psi(k)-\Psi(k')) &= 2(\widetilde{\Psi}(k)-\widetilde{\Psi}(k')) = 2(\widetilde{\Psi}(k)-\widetilde{\Psi}((k')^*)) \\
&= (k+(i_\chi/n))^2 - ((k')^* + (i_\chi/n))^2 \\
&= (|k+(i_\chi/n)| + |(k')^* + (i_\chi/n)|) \cdot (|k+(i_\chi/n)| - |(k')^* + (i_\chi/n)|) \\
&\geq (|k+(i_\chi/n)|) \cdot |k+(i_\chi/n) - (k')^* - (i_\chi/n)| \\
&\geq \frac{1}{4} d \cdot |k-(k')^*| \\
&\geq \frac{1}{4} d \cdot \frac{1}{4n} \cdot \mathrm{Dist}(k, F^{r'}(K_{\mathrm{Crit}})) = \frac{1}{16n} \cdot d \cdot \mathrm{Dist}(k, F^{r'}(K_{\mathrm{Crit}}))
\end{aligned}
$$

as desired. ○

§5. The Full Binomial Model

In this §, we apply the estimates of the preceding § to study the *orthogonal system which constitutes the full binomial model*.

We maintain the notation of §4. We begin by considering the *binomial coefficients* that appear in the sum defining $\mathfrak{Z}_r^{\mathrm{CG}}$.

Lemma 5.1. *Let* $k \in \mathbf{Z}$, $r \in [0, d-1]_{\mathrm{Int}}$ *(where* $[0, d-1]_{\mathrm{Int}} \stackrel{\mathrm{def}}{=} [0, d-1] \bigcap \mathbf{Z}$ *is as in Lemma 4.1). Then*

$$
\left| \binom{k+\lambda_r}{r} \right| = \binom{\mathrm{Dist}(k, F^r(K_{\mathrm{Crit}})) + r - 1}{r}
$$

In particular, $\log(\left|\binom{k+\lambda_r}{r}\right|) \leq \mathrm{Dist}(k, F^r(K_{\mathrm{Crit}})) \cdot \log(|k|+d)$.

Proof. First, let us observe that in general, for any $N \in \mathbf{Z}$, we have the *identity*

$$
\left| \binom{N}{r} \right| = \binom{\mathrm{Dist}(N, [0, r-1]_{\mathrm{Int}}) + r - 1}{r}
$$

Indeed, this is clear for $N \in [0, r-1]_{\mathrm{Int}}$ (since both sides are zero). For $N < 0$, one checks easily that $|\binom{N}{r}| = \binom{r-1-N}{r}$, $\mathrm{Dist}(N, [0, r-1]_{\mathrm{Int}}) = \mathrm{Dist}(r-1-N, [0, r-1]_{\mathrm{Int}})$, so it suffices to prove the identity when $N \geq r$. But for $N \geq r$, we have $\mathrm{Dist}(N, [0, r-1]_{\mathrm{Int}}) = N-(r-1)$, so the identity follows immediately.

Applying this identity, we obtain

$$\begin{aligned}\left|\binom{k+\lambda_r}{r}\right| &= \binom{\mathrm{Dist}(k+\lambda_r, [0, r-1]_{\mathrm{Int}}) + r - 1}{r} \\ &= \binom{\mathrm{Dist}(k, [0, r-1]_{\mathrm{Int}} - \lambda_r) + r - 1}{r} = \binom{\mathrm{Dist}(k, F^r(K_{\mathrm{Crit}})) + r - 1}{r}\end{aligned}$$

by the definition of $F^r(K_{\mathrm{Crit}})$.

Finally, we note that the binomial coefficient $\binom{\mathrm{Dist}(k, F^r(K_{\mathrm{Crit}}))+r-1}{r}$ may be bounded by the product of $\mathrm{Dist}(k, F^r(K_{\mathrm{Crit}})) - 1$ ($\leq \mathrm{Dist}(k, F^r(K_{\mathrm{Crit}}))$) integers all of which have absolute value $\leq |\mathrm{Dist}(k, F^r(K_{\mathrm{Crit}})) + r - 1| \leq |k - \mathrm{Avg}(F^r(K_{\mathrm{Crit}}))| + |r-1| \leq |k| + \frac{1}{2} + (d-2) \leq |k| + d$ (by Lemma 4.1). ◯

Let us write

$$K^{\mathrm{Semi}}_{\mathrm{Crit}} \stackrel{\mathrm{def}}{=} \mathrm{Crit}(K_{\mathrm{Semi}}) \subseteq K_{\mathrm{Crit}}$$

Thus, (by the definition of K_{Semi}) every $k_0 \in K^{\mathrm{Semi}}_{\mathrm{Crit}}$ satisfies $|k_0 + (i_\chi/n)| \geq \frac{d}{4} \geq 3$ (so $|k_0| \geq \frac{5}{2}$), where we use that $d \geq 12$. Also, if $k[r] \in K^{\mathrm{Semi}}_{\mathrm{Crit}}$, then $r \geq 4$ (indeed, if $r \leq 3$, then by Lemma 4.1, $|k[r]| \leq \frac{r+1}{2} \leq 2$). Let us write

$$R^{\mathrm{Semi}}_{\mathrm{Crit}} \stackrel{\mathrm{def}}{=} \{r \in [0, d-1]_{\mathrm{Int}} \mid k[r] \in K^{\mathrm{Semi}}_{\mathrm{Crit}}\}$$

Next, let us *observe* that

$$K^{\mathrm{Semi}}_{\mathrm{Crit}} = \{k_0 \in K_{\mathrm{Crit}} \mid |k_0 + (i_\chi/n)| \geq \frac{d}{4}\}$$

Indeed, "$\subseteq$" follows from the definition of K_{Semi}. On the other hand, if $|k_0 + (i_\chi/n)| \geq \frac{d}{4}$, and $\sigma_{k_0} \stackrel{\mathrm{def}}{=} k_0/|k_0|$, then (since $|k_0| \leq \frac{d}{2} < d$ by Lemma 4.1, $d \geq 1$) the integer $k \stackrel{\mathrm{def}}{=} k_0 - \sigma_{k_0} \cdot d$ satisfies $k \cdot k_0 < 0$, hence $\in K_{\mathrm{Semi}}$. This completes the proof of the above observation. In the following, we shall write

$$\mathrm{Semi}(k_0) \stackrel{\mathrm{def}}{=} k_0 - \sigma_{k_0} \cdot d$$

In particular, since (for $r \in [0, d-1]_{\mathrm{Int}}$) $r \mapsto |k[r] + (i_\chi/n)| = \{2 \cdot \widetilde{\Psi}(k[r])\}^{\frac{1}{2}}$ is (not necessarily strictly) *monotone increasing*, it follows that $R^{\mathrm{Semi}}_{\mathrm{Crit}}$ is an *interval of integers*, i.e., of the form $[r, d-1]_{\mathrm{Int}}$ (where $|k[d-1] + (i_\chi/n)| \geq \frac{d-2}{2} - \frac{1}{2} \geq \frac{d}{4}$ (by Lemma 4.1, $d \geq 6$) implies $d - 1 \in R^{\mathrm{Semi}}_{\mathrm{Crit}}$) for some integer r.

Next, we would like to define a *matrix* $\mathbf{M} \stackrel{\mathrm{def}}{=} \{M_{r,r'}\}$ (where $r, r' \in R^{\mathrm{Semi}}_{\mathrm{Crit}}$) by:

$$\begin{aligned} M_{r,r'} &\stackrel{\mathrm{def}}{=} \mathrm{Coeff}_{U_{\mathrm{sc}}^{k[r']}}(\mathfrak{Z}_r^{\mathrm{CG}}) \\ &= q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot \Big\{ \binom{k[r'] + \lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(k[r'])} \cdot \chi(k[r']_{\mathrm{et}}) \\ &\quad + \binom{\mathrm{Semi}(k[r']) + \lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(\mathrm{Semi}(k[r']))} \cdot \chi(\mathrm{Semi}(k[r'])_{\mathrm{et}}) \Big\} \end{aligned}$$

Let us write

$$\mathbf{M} \stackrel{\mathrm{def}}{=} \mathbf{M}_0 + \mathbf{M}_{\mathrm{Semi}}$$

where $(M_0)_{r,r'}$ is defined to be the sum of all the terms in $M_{r,r'}$ in which *the exponent of* q_{sc} *is zero*; $\mathbf{M}_{\mathrm{Semi}} \stackrel{\mathrm{def}}{=} \mathbf{M} - \mathbf{M}_0$. More precisely, (by Lemma 4.3) $(M_0)_{r,r'}$ can be nonzero only when $\Psi(k[r]) = \Psi(k[r'])$. Moreover, when $\Psi(k[r]) = \Psi(k[r'])$, we have

$$(M_0)_{r,r'} = \binom{k[r'] + \lambda_r}{r} \cdot \chi(k[r']_{\mathrm{et}}) + \epsilon \cdot \binom{\mathrm{Semi}(k[r']) + \lambda_r}{r} \cdot \chi(\mathrm{Semi}(k[r'])_{\mathrm{et}})$$

where $\epsilon \stackrel{\mathrm{def}}{=} 1$ if $r = r' = d - 1$ and $\mathrm{Semi}(k[d-1]) = k[d]$ is *exceptional* (cf. Lemma 4.3); and $\epsilon \stackrel{\mathrm{def}}{=} 0$ otherwise. In the following discussion, we would like to show that (under certain conditions) $\mathbf{M}$ *admits a bounded inverse*. We will do this by regarding $\mathbf{M}_{\mathrm{Semi}}$ as a *deformation* of $\mathbf{M}_0$, i.e., we will first study the inverse of $\mathbf{M}_0$ and then show that $\mathbf{M}_{\mathrm{Semi}}$ is rather small.

We begin by considering $\mathbf{M}_0$. Note that since the fibers of the map $r \mapsto \Psi(k[r])$ have at most two (necessarily adjacent) elements, it follows that $(M_0)_{r,r'}$ can be nonzero only in the cases $r = r'$ and $r = r' \pm 1$.

Now *suppose that* $r = r'$. Then we have $k[r] + \lambda_r \in \{-1, r\}$, so $\binom{k[r]+\lambda_r}{r} = \pm 1$. Similarly, if $r = r' = d - 1$ and $\mathrm{Semi}(k[d-1]) = k[d]$ is exceptional, then we have $\mathrm{Semi}(k[r]) + \lambda_r \in \{-1, d-1\}$, so $\binom{\mathrm{Semi}(k[r'])+\lambda_r}{r} = \pm 1$. Thus,

$$(M_0)_{r,r} = \pm\chi(k[r]_{\mathrm{et}}) \pm \epsilon \cdot \chi(\mathrm{Semi}(k[r])_{\mathrm{et}})$$

(where the two "$\pm$'s" are not necessarily the same sign). Note that when $r = r' = d - 1$ and $k[d]$ is exceptional, $(M_0)_{r,r}$ is precisely the coefficient discussed in Lemma 4.4.

Next, *suppose that* $r = r' - 1$. In this case, $\epsilon = 0$. Moreover, $k[r'] = k[r+1] \in F^{r+2}(K_{\mathrm{Crit}})$, so $k[r'] + \lambda_r \in \{-1, r\}$. Thus, $\binom{k[r']+\lambda_r}{r} = \pm 1$, so we have:

$$(M_0)_{r,r+1} = \pm\chi(k[r+1]_{\mathrm{et}})$$

Note, moreover, that in this case, since $\Psi(k[r]) = \Psi(k[r+1])$, and the fibers of the map $r \mapsto \Psi(k[r])$ have at most two elements, it follows that $\Psi(k[r+1]) < \Psi(k[r+2])$, i.e., that $(M_0)_{r+1,r+2} = 0$.

Finally, *suppose that* $r = r' + 1$. In this case, $\epsilon = 0$. Moreover, $k[r'] = k[r-1] \in F^r(K_{\mathrm{Crit}})$, so $k[r'] + \lambda_r \in [0, r-1]_{\mathrm{Int}}$. Thus, $\binom{k[r']+\lambda_r}{r} = 0$, so we have:

$$(M_0)_{r,r-1} = 0$$

In particular, we conclude that $(M_0)_{r,r'} = 0$ if $r > r'$ or $r < r' - 1$. In fact, this analysis shows the following:

Lemma 5.2. *The matrix* $\mathbf{M}_0$ *is of the form*

$$\begin{pmatrix} * & 0 & 0 & \dots & 0 \\ 0 & * & 0 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & * \end{pmatrix}$$

that is to say, zeroes away from the "diagonal" (i.e., the "$*$'s"), where each "$*$" in the diagonal is either *a one-by-one matrix* or *an upper triangular two-by-two matrix. In particular,* $\mathbf{M}_0$ *itself is upper triangular. Moreover, the diagonal matrix elements of each submatrix "$*$" are either n-th roots of unity, or (in the case of $(M_0)_{d-1,d-1}$, when $k[d]$ is exceptional) the coefficient of Lemma 4.4. The off-diagonal elements of each submatrix "$*$" are all n-th roots of unity. Finally,* $\mathbf{M}_0$ *is* invertible, *and the components of its inverse* $\mathbf{M}_0^{-1}$ *are complex numbers of absolute value* $\leq \frac{n}{\pi}$.

Proof. All the statements except for the "Finally,..." follow immediately from the above analysis. The "Finally,..." follows from the fact that each of the $*$'s are invertible, with inverses bounded as stated. (Note that here we use the estimates of Lemma 4.4.) ○

Next, we would like to *bound* $\mathbf{M}_{\mathrm{Semi}}$. By definition, $\mathbf{M}_{\mathrm{Semi}}$ consists of the terms of

$$M_{r,r'} = q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot \Big\{ \binom{k[r'] + \lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(k[r'])} \cdot \chi(k[r']_{\mathrm{et}}) + \binom{\mathrm{Semi}(k[r']) + \lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(\mathrm{Semi}(k[r']))} \cdot \chi(\mathrm{Semi}(k[r'])_{\mathrm{et}}) \Big\}$$

for which the exponent of q_{sc} is *nonzero*. Thus, there are two cases (corresponding to the two terms that appear in $M_{r,r'}$) to consider: (i) $\Psi(k[r']) \neq \Psi(k[r])$; (ii) $\Psi(\mathrm{Semi}(k[r'])) \neq \Psi(k[r])$. We begin by considering *case (ii)*. In this case, either $\mathrm{Semi}(k[r'])$ is non-exceptional, or $r < d-1$. Moreover, (since $r \in R_{\mathrm{Crit}}^{\mathrm{Semi}}$) $r \geq 4$. Thus, Lemma 4.7 implies that

$$\Psi(\mathrm{Semi}(k[r'])) - \Psi(k[r]) \geq \frac{1}{32n} \cdot d \cdot \mathrm{Dist}(\mathrm{Semi}(k[r']), F^r(K_{\mathrm{Crit}}))$$

Thus, since $d + |\mathrm{Semi}(k[r'])| \leq 2d + |k[r']| \leq \frac{5}{2} \cdot d \leq d^2$ (by the definition of "Semi$(-)$," Lemma 4.1, $d \geq 3$), we have (by Lemma 5.1):

$$\begin{aligned} \Big| \binom{\mathrm{Semi}(k[r']) + \lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(\mathrm{Semi}(k[r'])) - \Psi(k[r])} \cdot \chi(\mathrm{Semi}(k[r'])_{\mathrm{et}}) \Big| \\ \leq e^{2 \cdot \log(d) \cdot \mathrm{Dist}(\mathrm{Semi}(k[r']), F^r(K_{\mathrm{Crit}}))} \cdot |q_{\mathrm{sc}}|^{\frac{d}{32n} \cdot \mathrm{Dist}(\mathrm{Semi}(k[r']), F^r(K_{\mathrm{Crit}}))} \\ = e^{(2 \cdot \log(d) - \mathrm{Im}(\tau_{\mathrm{or}}) \cdot \frac{\pi}{16n}) \cdot \mathrm{Dist}(\mathrm{Semi}(k[r']), F^r(K_{\mathrm{Crit}}))} \end{aligned}$$

(where we use that $\log|q_{\mathrm{sc}}| = -\frac{2\pi}{d} \cdot \mathrm{Im}(\tau_{\mathrm{or}})$). Also, let us observe that (since $\mathrm{Semi}(k[r']) \notin F^r(K_{\mathrm{Crit}})$), we have $\mathrm{Dist}(\mathrm{Semi}(k[r']), F^r(K_{\mathrm{Crit}})) \geq 1$.

Next, we consider *case (i)*, i.e., the first term in $M_{r,r'}$ when $\Psi(k[r']) \neq \Psi(k[r])$. Note first of all that this condition implies that $r \neq r'$. Moreover, if $r > r'$, then $k[r'] \in F^r(K_{\mathrm{Crit}})$, i.e., $k[r'] + \lambda_r \in [0, r-1]_{\mathrm{Int}}$, so $\binom{k[r']+\lambda_r}{r} = 0$. Thus, *it suffices to bound this term when* $r < r'$. Note that since $r, r' \in R_{\mathrm{Crit}}^{\mathrm{Semi}}$, we have $|k[r] + (i_\chi/n)|, |k[r'] + (i_\chi/n)| \geq \frac{d}{4}$. Thus, by applying Lemma 4.8 (note that "r," "r'" in Lemma 4.8 are the *reverse* of what they are in the present discussion), we obtain

$$\Psi(k[r']) - \Psi(k[r]) \geq \frac{1}{32 \cdot n} \cdot d \cdot \mathrm{Dist}(k[r'], F^r(K_{\mathrm{Crit}}))$$

Thus, using Lemma 5.1 (and the fact that $d + |k[r']| \leq \frac{3}{2} \cdot d \leq d^2$, by Lemma 4.1, $d \geq 2$) as in the preceding paragraph, we obtain:

$$\begin{aligned} \Big| \binom{k[r'] + \lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(k[r']) - \Psi(k[r])} \cdot \chi(k[r']_{\mathrm{et}}) \Big| \\ \leq e^{2 \cdot \log(d) \cdot \mathrm{Dist}(k[r'], F^r(K_{\mathrm{Crit}}))} \cdot |q_{\mathrm{sc}}|^{\frac{d}{32n} \cdot \mathrm{Dist}(k[r'], F^r(K_{\mathrm{Crit}}))} \\ = e^{(2 \cdot \log(d) - \mathrm{Im}(\tau_{\mathrm{or}}) \cdot \frac{\pi}{16n}) \cdot \mathrm{Dist}(k[r'], F^r(K_{\mathrm{Crit}}))} \end{aligned}$$

Also, let us observe that since $r < r'$ (so $k[r'] \notin F^r(K_{\text{Crit}}) \subseteq F^{r'}(K_{\text{Crit}})$), we have $\text{Dist}(k[r'], F^r(K_{\text{Crit}})) \geq 1$.

Now *let us assume that*

$$\text{Im}(\tau_{\text{or}}) \geq \frac{\{5 + (\log(n)/\log(d))\} \cdot 16}{\pi} \cdot \log(d) \cdot n$$

Then it follows that

$$\begin{aligned} 2 \cdot \log(d) - \text{Im}(\tau_{\text{or}}) \cdot \frac{\pi}{16n} &\leq -\{3 + (\log(n)/\log(d))\} \cdot \log(d) \\ &\quad + \left(\{5 + (\log(n)/\log(d))\} \cdot \log(d) - \text{Im}(\tau_{\text{or}}) \cdot \frac{\pi}{16n}\right) \\ &\leq -\{3 + (\log(n)/\log(d))\} \cdot \log(d) = -3 \cdot \log(d) - \log(n) \end{aligned}$$

In particular, it follows that *both of the terms* (cf. the treatment of cases (i), (ii) above) *which appear in the components of* $\mathbf{M}_{\text{Semi}}$ *have absolute value* $\leq e^{-3\cdot\log(d)-\log(n)} = (n \cdot d^3)^{-1}$. Thus, since there are *two* terms involved, we obtain the following result:

Lemma 5.3. *If the condition*

$$\text{Im}(\tau_{\text{or}}) \geq \frac{\{5 + (\log(n)/\log(d))\} \cdot 16}{\pi} \cdot \log(d) \cdot n$$

is satisfied, then the components of the matrix $\mathbf{M}_{\text{Semi}}$ *have absolute value* $\leq \frac{2}{n\cdot d^3}$. *In particular, the components of the matrix* $\mathbf{M}_0^{-1} \cdot \mathbf{M}_{\text{Semi}}$ *have absolute value* $\leq d \cdot \frac{n}{\pi} \cdot \frac{2}{n\cdot d^3} \leq d^{-2}$.

Proof. The statement concerning the absolute values of the components of $\mathbf{M}_{\text{Semi}}$ was proven in the above discussion. The statement concerning the absolute values of the components of $\mathbf{M}_0^{-1} \cdot \mathbf{M}_{\text{Semi}}$ follows from this estimate, plus the estimates of Lemma 5.2 (together with the fact that these matrices are square matrics of order equal to the cardinality $|K_{\text{Crit}}^{\text{Semi}}| \leq |K_{\text{Crit}}| = d$ of $K_{\text{Crit}}^{\text{Semi}}$). ○

Now we come to the *invertibility of the original matrix* $\mathbf{M}$. Note that

$$\mathbf{M}_0^{-1} \cdot \mathbf{M} \stackrel{\text{def}}{=} 1 + \mathbf{M}_0^{-1} \cdot \mathbf{M}_{\text{Semi}}$$

Write $\mathbf{M}' \stackrel{\text{def}}{=} -\mathbf{M}_0^{-1} \cdot \mathbf{M}_{\text{Semi}}$. Then by Lemma 5.3, the components of $\mathbf{M}'$ have absolute value $\leq d^{-2}$. Thus, for any nonnegative integer N, the components of $(\mathbf{M}')^{N+1}$ will have absolute value $\leq d^{-2} \cdot (d \cdot d^{-2})^N = d^{-N-2}$ (where we again use the fact that $|K_{\text{Crit}}^{\text{Semi}}| \leq$

$|K_{\mathrm{Crit}}| = d$). Thus, the series $\sum_{N\geq 0} (\mathbf{M}')^N$ *converges* to a matrix whose components have absolute value $\leq (1 - d^{-1})^{-1} \leq 2$. Moreover,

$$\mathbf{M}^{-1} = \Big(\sum_{N\geq 0} (\mathbf{M}')^N \Big) \cdot \mathbf{M}_0^{-1}$$

Thus, (applying the estimates of Lemma 5.2) we see that we have proven the following:

Lemma 5.4. *Under the hypothesis of Lemma 5.3, the matrix* $\mathbf{M}$ *is invertible, and the components of its inverse have absolute value* $\leq 2 \cdot d \cdot \frac{n}{\pi} \leq d \cdot n$.

Now we would like to consider the *analogous problem for the* $\mathfrak{Z}_r^{\mathrm{CG}}$ to the problem that we solved in §3 for the Z_r^{CG}: Namely, let

$$\mathfrak{Z} \stackrel{\mathrm{def}}{=} \sum_{r=0}^{d-1} \gamma_r \cdot \mathfrak{Z}_r^{\mathrm{CG}}$$

be a $\mathbf{C}$-linear combination (i.e., the $\gamma_r \in \mathbf{C}$) of the $\mathfrak{Z}_r^{\mathrm{CG}}$.

Suppose that $||\mathfrak{Z}|| = 1$. *Then to what extent can one bound the* $g_r \stackrel{\mathrm{def}}{=} |\gamma_r|$*?*

Since $||\mathfrak{Z}|| = 1$, and the U_{sc}^k ($k \in \mathbf{Z}$) form an orthonormal system in $L^2((\mathbf{S^1})_{\mathrm{sc}})$, it follows that the coefficients of the U_{sc}^k (in the "Fourier expansion" of $\mathfrak{Z}$) are ≤ 1. In particular, *for each* $r' = 0, \ldots, d-1$, *the coefficient of* $U_{\mathrm{sc}}^{k[r']}$ *is* ≤ 1. Thus, we would like to analyze these coefficients in a fashion similar to §3. Unfortunately, however, the present situation is somewhat more complicated than in §3, so we must break the situation down into *two cases.*

First, we consider the *case* $r' \notin R_{\mathrm{Crit}}^{\mathrm{Semi}}$. This case is the simpler case since it may be reduced to the results of §3. More precisely, let us observe that for $r' \notin R_{\mathrm{Crit}}^{\mathrm{Semi}}$, there do not exist any $k \in K_{\mathrm{Semi}}$ such that $\mathrm{Crit}(k) = k[r']$. Thus, if we just look at the coefficient of $U_{\mathrm{sc}}^{k[r']}$, $\mathfrak{Z}_r^{\mathrm{CG}}$ is *exactly the same as* Z_r^{CG}. Thus, we obtain exactly the same inequalities

$$\Big| \sum_{r=0}^{r'} \gamma_r \cdot \binom{k[r'] + \lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(k[r']) - \Psi(k[r])} \cdot \chi(k[r]_{\mathrm{et}}) \Big| \leq 1$$

as in §3. Note that since $r' \in R_{\mathrm{Crit}}^{\mathrm{Semi}}$, any $r \leq r'$ will also $\in R_{\mathrm{Crit}}^{\mathrm{Semi}}$. Thus, for the r appearing in this sum, we also get similar inequalities (obtained by considering the coefficient of $U_{\mathrm{sc}}^{k[r]}$). In particular, by the analysis of §3 (more precisely: by Theorem 3.3, (2)), we obtain the following result:

Lemma 5.5. *Suppose that* $\mathrm{Im}(\tau_{\mathrm{or}}) \geq \frac{32}{\pi} \cdot \log^2(d)$*. Then* $|\gamma_r| \leq e^{26 \cdot d}$*, for all* $r \notin R^{\mathrm{Semi}}_{\mathrm{Crit}}$.

Next, we consider the *case* $r' \in R^{\mathrm{Semi}}_{\mathrm{Crit}}$. This case is the more difficult of the two cases. First, let us observe that by looking at the coefficient of $U_{\mathrm{sc}}^{k[r']}$, we obtain an inequality

$$\Big| \sum_{r=0}^{d-1} \gamma_r \cdot \mathrm{Coeff}_{U_{\mathrm{sc}}^{k[r']}}(\mathfrak{Z}_r^{\mathrm{CG}}) \Big| \leq 1$$

analogous to the one considered in the "$\notin R^{\mathrm{Semi}}_{\mathrm{Crit}}$ case." Thus, *under the hypothesis of Lemma 5.5*, we obtain

$$\begin{aligned}
\Big| \sum_{r \in R^{\mathrm{Semi}}_{\mathrm{Crit}}} \gamma_r \cdot M_{r,r'} \Big| &= \Big| \sum_{r \in R^{\mathrm{Semi}}_{\mathrm{Crit}}} \gamma_r \cdot \mathrm{Coeff}_{U_{\mathrm{sc}}^{k[r']}}(\mathfrak{Z}_r^{\mathrm{CG}}) \Big| \\
&\leq 1 + \Big| \sum_{r \notin R^{\mathrm{Semi}}_{\mathrm{Crit}}} \gamma_r \cdot \mathrm{Coeff}_{U_{\mathrm{sc}}^{k[r']}}(\mathfrak{Z}_r^{\mathrm{CG}}) \Big| \\
&\leq 1 + e^{26 \cdot d} \cdot \sum_{r \notin R^{\mathrm{Semi}}_{\mathrm{Crit}}} \Big| \mathrm{Coeff}_{U_{\mathrm{sc}}^{k[r']}}(\mathfrak{Z}_r^{\mathrm{CG}}) \Big|
\end{aligned}$$

Now we have the following result:

Lemma 5.6. *Let* $r \in [0, d-1]_{\mathrm{Int}}$, $r' \in R^{\mathrm{Semi}}_{\mathrm{Crit}}$*. Then* $\Big| \mathrm{Coeff}_{U_{\mathrm{sc}}^{k[r']}}(\mathfrak{Z}_r^{\mathrm{CG}}) \Big| \leq 2^{3d-1}$.

Proof. Note that $\Psi(\mathrm{Semi}(k[r'])) \geq \Psi(k[r])$ always, and $\Psi(k[r']) \geq \Psi(k[r])$ if $r \leq r'$. On the other hand, $\mathrm{Coeff}_{U_{\mathrm{sc}}^{k[r']}}(\mathfrak{Z}_r^{\mathrm{CG}})$ is given by

$$\begin{aligned}
&\binom{k[r'] + \lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(k[r'])-\Psi(k[r])} \cdot \chi(k[r']_{\mathrm{et}}) \\
&\qquad + \binom{\mathrm{Semi}(k[r']) + \lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(\mathrm{Semi}(k[r']))-\Psi(k[r])} \cdot \chi(\mathrm{Semi}(k[r'])_{\mathrm{et}})
\end{aligned}$$

where the first term is *zero* if $r > r'$ (since then $k[r'] \in F^{r'+1}(K_{\mathrm{Crit}}) \subseteq F^r(K_{\mathrm{Crit}})$, so $k[r'] + \lambda_r \in [0, r-1]_{\mathrm{Int}}$). Since the exponents of q_{sc} appearing in the nonzero term(s) are ≥ 0, we thus obtain that this coefficient has absolute value $\leq$

$$\begin{aligned}
&\Big|\binom{k[r'] + \lambda_r}{r}\Big| + \Big|\binom{\mathrm{Semi}(k[r']) + \lambda_r}{r}\Big| \\
&\qquad \leq \binom{\mathrm{Dist}(k[r'], F^r(K_{\mathrm{Crit}})) + r - 1}{r} + \binom{\mathrm{Dist}(\mathrm{Semi}(k[r']), F^r(K_{\mathrm{Crit}})) + r - 1}{r} \\
&\qquad \leq 2^{\mathrm{Dist}(k[r'], F^r(K_{\mathrm{Crit}}))+d-2} + 2^{\mathrm{Dist}(\mathrm{Semi}(k[r']), F^r(K_{\mathrm{Crit}}))+d-2} \leq 2^{2d-2} + 2^{3d-2} \\
&\qquad \leq 2 \cdot 2^{3d-2} = 2^{3d-1}
\end{aligned}$$

(where we use Lemma 5.1, $d \geq 1$, and the estimates

$$\mathrm{Dist}(k[r'], F^r(K_{\mathrm{Crit}})) \leq |k[r'] - \mathrm{Avg}(F^r(K_{\mathrm{Crit}}))| \leq \frac{d}{2} + \frac{1}{2} \leq d$$

and

$$\begin{aligned}\mathrm{Dist}(\mathrm{Semi}(k[r']), F^r(K_{\mathrm{Crit}})) \leq |\mathrm{Semi}(k[r']) - \mathrm{Avg}(F^r(K_{\mathrm{Crit}}))| &\leq |\mathrm{Semi}(k[r'])| + \frac{1}{2} \\ &\leq d + |k[r']| + \frac{1}{2} \leq d + \frac{d}{2} + \frac{1}{2} \leq 2d\end{aligned}$$

– cf. Lemma 4.1). ○

Thus, applying Lemma 5.6 in the above discussion, we obtain:

$$\Big| \sum_{r \in R^{\mathrm{Semi}}_{\mathrm{Crit}}} \gamma_r \cdot M_{r,r'} \Big| \leq 1 + e^{26 \cdot d} \cdot |[0, d-1]_{\mathrm{Int}} \backslash R^{\mathrm{Semi}}_{\mathrm{Crit}}| \cdot 2^{3d-1} \leq d \cdot e^{26 \cdot d} \cdot 2^{3d} \leq d \cdot e^{29 \cdot d}$$

On the other hand, by Lemma 5.4, $\mathbf{M}$ is invertible, and the components of $\mathbf{M}^{-1}$ have absolute value $\leq d \cdot n$. Thus, we conclude that, *under the hypothesis of Lemma 5.3*, we have:

$$|\gamma_r| \leq |R^{\mathrm{Semi}}_{\mathrm{Crit}}| \cdot (d \cdot n) \cdot d \cdot e^{29 \cdot d} \leq n \cdot d^3 \cdot e^{29 \cdot d} \leq n \cdot e^{30 \cdot d}$$

(since $d \leq e^{\frac{1}{3} \cdot d}$ for $d \geq 6$) for $r \in R^{\mathrm{Semi}}_{\mathrm{Crit}}$. That is to say, we have proven the following *analogue for the* $\mathfrak{Z}^{\mathrm{CG}}_r$ of Theorem 3.3:

Theorem 5.7. *Suppose that* $d \geq 12$, *and*

$$\mathrm{Im}(\tau_{\mathrm{or}}) \geq \frac{32}{\pi} \cdot \log^2(d) + \frac{80}{\pi} \cdot n \cdot \log(d) + \frac{16}{\pi} \cdot n \cdot \log(n)$$

and let $\mathfrak{Z} \stackrel{\mathrm{def}}{=} \sum_{r=0}^{d-1} \gamma_r \cdot \mathfrak{Z}^{\mathrm{CG}}_r$ *be a* $\mathbf{C}$*-linear combination (i.e., the* $\gamma_r \in \mathbf{C}$*) of the* $\mathfrak{Z}^{\mathrm{CG}}_r$*. Then*

(1) *If the* $|\gamma_r| \leq 1$ *for* $r = 0, \ldots, d-1$, *then the* $L^2((\mathbf{S}^1)_{\mathrm{sc}})$*-norm of* $\mathfrak{Z}$ *satisfies:* $||\mathfrak{Z}|| \leq d^2 \cdot e^{3d} \leq e^{4d}$.

(2) *If* $||\mathfrak{Z}|| \leq 1$, *then the* $|\gamma_r| \leq n \cdot e^{30 \cdot d}$, *for all* $r = 0, \ldots, d-1$, *and* $|\gamma_r| \leq e^{26 \cdot d}$ *if* $r \notin R^{\mathrm{Semi}}_{\mathrm{Crit}}$.

In particular, if we orthonormalize the $\mathfrak{Z}^{\mathrm{CG}}_r$ *to form*

$$\mathfrak{Z}^{\mathrm{OC}}_0, \ldots, \mathfrak{Z}^{\mathrm{OC}}_{d-1}$$

– *which are unique if we stipulate that the* leading coefficient of $\mathfrak{Z}^{\mathrm{OC}}_r$ *(i.e., the coefficient of* $\mathfrak{Z}^{\mathrm{CG}}_r$ *in the linear combination of* $\mathfrak{Z}^{\mathrm{CG}}_0, \ldots, \mathfrak{Z}^{\mathrm{CG}}_r$ *that forms* $\mathfrak{Z}^{\mathrm{OC}}_r$*) be* positive – *then the coefficients of the* $\mathfrak{Z}^{\mathrm{CG}}_r$ *in each* $\mathfrak{Z}^{\mathrm{OC}}_{r'}$ *have absolute value* $\leq n \cdot e^{30d}$, *and the absolute values of the leading coefficients are* $\geq d^{-2} \cdot e^{-3d} \geq e^{-4d}$.

Proof. Note that the hypothesis of Theorem 5.7 implies the hypotheses of Lemmas 5.3, 5.4, 5.5. The final portion concerning the $\mathfrak{Z}^{\mathrm{OC}}_r$ follows formally from (1), (2) (cf. the proof of Theorem 3.3). Moreover, (2) is precisely what was proven above (cf. Lemma 5.5 for the last part of (2)). Thus, it suffices to prove (1). To prove (1), it suffices to bound the absolute value of the coefficient of each $U^{k[r']}_{\mathrm{sc}}$ in $\mathfrak{Z}^{\mathrm{CG}}_r$ (for $r, r' = 0, \ldots, d-1$) by e^{3d}. For $r' \notin R^{\mathrm{Semi}}_{\mathrm{Crit}}$, (as discussed above) the $\mathfrak{Z}^{\mathrm{CG}}_r$ look exactly the same as the Z^{CG}_r, so we obtain the estimate "$\leq e^d$ ($\leq e^{3d}$)" as in the proof of Theorem 3.3. For $r' \in R^{\mathrm{Semi}}_{\mathrm{Crit}}$, the coefficient of $U^{k[r']}_{\mathrm{sc}}$ has absolute value $\leq e^{3d}$ by Lemma 5.6 (cf. the proof of Theorem 3.3). This completes the proof of (1). ○

It thus remains to *deal with the discrepancy between* $\mathfrak{Z}^{\mathrm{CG}}_r$ *and the original* ζ^{CG}_r. This discrepancy may be dealt with exactly as in the proof of Lemma 2.1 in §2 (where we dealt with deformations of the discrete Tchebycheff polynomials). Indeed, let

$$\begin{aligned}
\zeta^{\mathrm{CG},\boldsymbol{\mu}}_r &\stackrel{\mathrm{def}}{=} \sum_{k \in \mathbf{Z}} \binom{k+\lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(k)-\Psi(k[r])} \cdot U_{\mathrm{sc}}^{\mathrm{Crit}(k)} \cdot \chi(k_{\mathrm{et}}) \\
&= \mathfrak{Z}^{\mathrm{CG}}_r + \sum_{k \in \mathbf{Z} \setminus (K_{\mathrm{Crit}} \bigcup K_{\mathrm{Semi}})} \binom{k+\lambda_r}{r} \cdot q_{\mathrm{sc}}^{\Psi(k)-\Psi(k[r])} \cdot U_{\mathrm{sc}}^{\mathrm{Crit}(k)} \cdot \chi(k_{\mathrm{et}})
\end{aligned}$$

be the series obtained by restricting $q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot U_{\mathrm{cv}}^{-i_\chi} \cdot \zeta^{\mathrm{CG}}_r$ to $\boldsymbol{\mu}_d \subseteq (\mathbf{S^1})_{\mathrm{sc}}$. Let

$$\zeta^{\boldsymbol{\mu}} \stackrel{\mathrm{def}}{=} \sum_{r=0}^{d-1} \gamma_r \cdot \zeta^{\mathrm{CG},\boldsymbol{\mu}}_r$$

be a $\mathbf{C}$-linear combination (i.e., the $\gamma_r \in \mathbf{C}$) of the $\zeta^{\mathrm{CG},\boldsymbol{\mu}}_r$. Moreover, let us assume that

$$\mathfrak{Z} \stackrel{\text{def}}{=} \sum_{r=0}^{d-1} \gamma_r \cdot \mathfrak{Z}_r^{\text{CG}}$$

satisfies $||\mathfrak{Z}|| = 1$. Then we would like to *bound* $||\zeta^{\boldsymbol{\mu}}||$ *from above and below*. In the following discussion, we assume that the *hypotheses of Theorem 5.7 are satisfied*. Then by Theorem 5.7, (2), we obtain

$$|\gamma_r| \leq n \cdot e^{30 \cdot d}$$

for $r = 0, \ldots, d-1$. On the other hand, by Lemma 4.2, for $k \in \mathbf{Z} \backslash (K_{\text{Crit}} \bigcup K_{\text{Semi}})$, we have $|k| \geq \frac{1}{2}d$, $\Psi(k) - \Psi(k[d-1]) \geq \frac{1}{16} \cdot d \cdot |k|$. Thus, (by Lemmas 4.1, 5.1, and the estimate $\text{Dist}(k, F^r(K_{\text{Crit}})) \leq |k - \text{Avg}(F^r(K_{\text{Crit}}))| \leq |k| + \frac{1}{2}$) we have:

$$\begin{aligned}
\left| \binom{k+\lambda_r}{r} \cdot q_{\text{sc}}^{\Psi(k)-\Psi(k[r])} \right| &\leq \binom{\text{Dist}(k, F^r(K_{\text{Crit}})) + r - 1}{r} \cdot |q_{\text{sc}}|^{\Psi(k)-\Psi(k[d-1])} \\
&\leq 2^{\text{Dist}(k, F^r(K_{\text{Crit}})) + r - 1} \cdot |q_{\text{sc}}|^{\frac{1}{16} \cdot d \cdot |k|} \leq 2^{|k| + \frac{1}{2} + d - 2} \cdot |q_{\text{or}}|^{\frac{1}{16} \cdot |k|} \\
&\leq 2^{3 \cdot |k|} \cdot |q_{\text{or}}|^{\frac{1}{16} \cdot |k|} \leq e^{3 \cdot |k| - \frac{2\pi}{16} \cdot \text{Im}(\tau_{\text{or}}) \cdot |k|} \leq e^{|k| \cdot (3 - \frac{\pi}{8} \cdot \text{Im}(\tau_{\text{or}}))}
\end{aligned}$$

Now *let us assume that*

$$C \stackrel{\text{def}}{=} \frac{\pi}{8} \cdot \text{Im}(\tau_{\text{or}}) - 3 \geq 62 + \frac{2}{d} \cdot \log(n)$$

Then it follows that

$$\begin{aligned}
||\zeta_r^{\text{CG},\boldsymbol{\mu}} - \mathfrak{Z}_r^{\text{CG}}|| &\leq \sum_{k \in \mathbf{Z} \backslash (K_{\text{Crit}} \bigcup K_{\text{Semi}})} \left| \binom{k+\lambda_r}{r} \cdot q_{\text{sc}}^{\Psi(k)-\Psi(k[r])} \right| \\
&\leq \sum_{k \in \mathbf{Z} \backslash (K_{\text{Crit}} \bigcup K_{\text{Semi}})} e^{-C \cdot |k|} \\
&\leq 2 \cdot \sum_{k \geq \frac{1}{2}d} e^{-C \cdot k} \\
&\leq 2 \cdot e^{-\frac{1}{2}C \cdot d} \cdot (1 - e^{-C})^{-1} \leq 4 \cdot e^{-\frac{1}{2}C \cdot d}
\end{aligned}$$

(since $C \geq 1$ implies $e^C \geq e \geq 2$, so $(1 - e^{-C})^{-1} \leq 2$). Thus, since $\frac{1}{2}C \geq 31 + \frac{\log(n)}{d}$, we have:

$$
\begin{aligned}
||\zeta^{\boldsymbol{\mu}} - \mathfrak{z}|| &\leq \sum_{r=0}^{d-1} |\gamma_r| \cdot ||\zeta_r^{\mathrm{CG},\boldsymbol{\mu}} - \mathfrak{z}_r^{\mathrm{CG}}|| \\
&\leq d \cdot (n \cdot e^{30 \cdot d}) \cdot (4 \cdot e^{-\frac{1}{2} C \cdot d}) \\
&\leq \frac{1}{2} \cdot n \cdot e^{d \cdot (31 - \frac{1}{2} C)} \\
&\leq \frac{1}{2} \cdot n \cdot e^{-\log(n)} = \frac{1}{2}
\end{aligned}
$$

(where we use that $d \geq 4$ implies $8d \leq e^d$). Thus, since $||\mathfrak{z}|| = 1$, we obtain

$$
\frac{1}{2} \cdot ||\mathfrak{z}|| = \frac{1}{2} \leq ||\mathfrak{z}|| - ||\mathfrak{z} - \zeta^{\boldsymbol{\mu}}|| \leq ||\zeta^{\boldsymbol{\mu}}|| \leq ||\zeta^{\boldsymbol{\mu}} - \mathfrak{z}|| + ||\mathfrak{z}|| \leq \frac{3}{2} \leq 2 = 2 \cdot ||\mathfrak{z}||
$$

This is the *desired bound* on $||\zeta^{\boldsymbol{\mu}}||$. Thus, we have proven the following result, which is the *main result* of the present §:

Theorem 5.8. *Suppose that $d \geq 12$, and*

$$
\mathrm{Im}(\tau_{\mathrm{or}}) \geq \frac{32}{\pi} \cdot \log^2(d) + \frac{80}{\pi} \cdot n \cdot \log(d) + \frac{16}{\pi} \cdot n \cdot \log(n) + \frac{8}{\pi} \cdot 65
$$

and let $\zeta \stackrel{\mathrm{def}}{=} \sum_{r=0}^{d-1} \gamma_r \cdot q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot \zeta_r^{\mathrm{CG}}$ be a $\mathbf{C}$-linear combination (i.e., the $\gamma_r \in \mathbf{C}$) of the $q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot \zeta_r^{\mathrm{CG}}$. Let us write

$$
||\zeta||_{\mathrm{et}}
$$

for the $L^2(\boldsymbol{\mu}_d)$ norm (i.e., the norm for which $U_{\mathrm{sc}}^0, U_{\mathrm{sc}}^1, \ldots, U_{\mathrm{sc}}^{d-1}$ are orthonormal) of the function $(U_{\mathrm{cv}}^{-i_\chi} \cdot \zeta)|_{\boldsymbol{\mu}_d \subseteq (\mathbf{S}^1)_{\mathrm{sc}}}$. Then

(1) *If the $|\gamma_r| \leq 1$ for $r = 0, \ldots, d-1$, then we have: $||\zeta||_{\mathrm{et}} \leq 2 \cdot d^2 \cdot e^{3d} \leq e^{4d}$.*

(2) *If $||\zeta||_{\mathrm{et}} \leq 1$, then the $|\gamma_r| \leq 2n \cdot e^{30 \cdot d}$, for all $r = 0, \ldots, d-1$, and $|\gamma_r| \leq 2 \cdot e^{26 \cdot d}$ if $r \notin R_{\mathrm{Crit}}^{\mathrm{Semi}}$.*

In particular, if we orthonormalize the $q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot \zeta_r^{\mathrm{CG}}$ to form

$$
\zeta_0^{\mathrm{OC}}, \ldots, \zeta_{d-1}^{\mathrm{OC}}
$$

– which are unique if we stipulate that the leading coefficient of *ζ_r^{OC} (i.e., the coefficient of ζ_r^{CG} in the linear combination of $\zeta_0^{\mathrm{CG}}, \ldots, \zeta_r^{\mathrm{CG}}$ that forms ζ_r^{OC}) be* positive *– then the*

coefficients of the ζ_r^{CG} *in each* $\zeta_{r'}^{\mathrm{OC}}$ *have absolute value* $\leq 2n \cdot e^{30d}$*, and the absolute values of the leading coefficients are* $\geq e^{-4d}$.

Proof. One checks easily that the hypothesis of Theorem 5.8 implies both the hypothesis of Theorem 5.7, as well as the assumption

$$\frac{\pi}{8} \cdot \mathrm{Im}(\tau_{\mathrm{or}}) - 3 \geq 62 + \frac{2}{d} \cdot \log(n)$$

made above. Then (1), (2) follow from the corresponding assertions of Theorem 5.7, together with the inequalities

$$\frac{1}{2} \cdot ||\mathfrak{Z}|| \leq ||\zeta^{\boldsymbol{\mu}}|| \leq 2 \cdot ||\mathfrak{Z}||$$

derived above (which hold even if $||\mathfrak{Z}|| \neq 1$). The assertions concerning the ζ_r^{OC} follow formally from (1), (2) (cf. the proof of Theorem 5.7). ○

Remark. Thus, stated in words, Theorem 5.8 asserts that:

> *If* $\mathrm{Im}(\tau_{\mathrm{or}})$ *is sufficiently large (roughly* $\geq$ *the order of* $\log^2(d)$*, when* n *is held fixed), then up to a factor of order* $n \cdot C^d$ *(for some constant* C*), the* L^2*-norm on the complex vector space generated by the* $q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot \zeta_r^{\mathrm{CG}}$ *(for* $r = 0, \ldots, d-1$*) is the same as the norm for which these functions* $q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot \zeta_r^{\mathrm{CG}}$ *(for* $r = 0, \ldots, d-1$*) are orthonormal.*

Note that the factor of n in "$n \cdot C^d$" is precisely *the archimedean analogue of the scheme-theoretic zero locus of the determinant* given in Chapter VI, Theorem 4.1, (2) (cf. also the Remark following Lemma 4.4). In some sense, of the three models (Hermite, Legendre, and Binomial) discussed, *the (full) binomial model is the closest archimedean analogue* to what is done in Chapter VI at finite primes. Unfortunately, however, *the estimates for the binomial model only hold when the elliptic curve in question is fairly close to infinity.* This degree of proximity, i.e., $\log^2(d)$ when n is held fixed, is not so outrageously large in the sense that, roughly speaking, $\mathrm{Im}(\tau_{\mathrm{or}})$ *should be regarded as being roughly on a par with* d *in terms of size* (cf. the discussion of "natural variables over $\mathbf{F}_1$" in Remark 2 at the end of Chapter VII, §6). Thus, in some sense, to require that $\mathrm{Im}(\tau_{\mathrm{or}})$ be at least of the order of $\log^2(d)$ is a "logarithmically weak" requirement on the proximity to infinity.

§6. Relations Among Various Norms and Zeta Functions

In this §, we prepare for the discussion of the *Hodge-Arakelov Comparison Isomorphism* in §7 below by relating the material of §1 through 5 of the present Chapter to the theory of §4 through 6 of Chapter VII. That is to say, throughout Chapter VII and the present Chapter, various canonical norms on function spaces, as well as various canonical zeta functions were introduced. The reason that so many such objects were introduced was because each one relates to a certain piece of the fundamental problem of *relating the natural metrics on the function spaces of "de Rham functions" and "étale functions"* to one another (cf. the Introduction to Chapter VII). Thus, the purpose of the present § is to explain how to put these pieces together.

We continue with the notation of §5. In addition, we suppose for simplicity that

$$d \geq 25$$

Let us review the situation that we are in. First of all, we have our *"original" elliptic curve* $E_{\mathrm{or}} \stackrel{\mathrm{def}}{=} (\mathbf{G}_{\mathrm{m}})_{\mathrm{or}}/q_{\mathrm{or}}^{\mathbf{Z}}$, together with our *"scaled" elliptic curve* $E_{\mathrm{sc}} \stackrel{\mathrm{def}}{=} (\mathbf{G}_{\mathrm{m}})_{\mathrm{sc}}/q_{\mathrm{sc}}^{\mathbf{Z}}$, where $q_{\mathrm{or}} = e^{2\pi i \tau_{\mathrm{or}}}$, $q_{\mathrm{sc}} = e^{2\pi i \tau_{\mathrm{sc}}}$, $\tau_{\mathrm{or}} = d \cdot \tau_{\mathrm{sc}}$, and $(\mathbf{G}_{\mathrm{m}})_{\mathrm{or}} = (\mathbf{G}_{\mathrm{m}})_{\mathrm{sc}}$. Thus, we may think of E_{sc} as a *quotient*

$$E_{\mathrm{or}} \stackrel{\mathrm{def}}{=} (\mathbf{G}_{\mathrm{m}})_{\mathrm{or}}/q_{\mathrm{or}}^{\mathbf{Z}} \to E_{\mathrm{sc}} \stackrel{\mathrm{def}}{=} (\mathbf{G}_{\mathrm{m}})_{\mathrm{sc}}/q_{\mathrm{sc}}^{\mathbf{Z}}$$

of E_{or}. The pull-back morphism on differentials thus induces a natural *isomorphism*

$$\iota_{\mathrm{diff}} : \omega_{E_{\mathrm{or}}} \cong \omega_{E_{\mathrm{sc}}}$$

Note that this quotient map extends to a push-forward morphism on the *universal extensions* of E_{or}, E_{sc}:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_{E_{\mathrm{or}}} & \longrightarrow & E^{\dagger}_{\mathrm{or}} & \longrightarrow & E_{\mathrm{or}} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle d\cdot\iota_{\mathrm{diff}}} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & \omega_{E_{\mathrm{sc}}} & \longrightarrow & E^{\dagger}_{\mathrm{sc}} & \longrightarrow & E_{\mathrm{sc}} & \longrightarrow & 0
\end{array}$$

One checks easily that the push-forward morphism induces the map $d \cdot \iota_{\mathrm{diff}}$ on $\omega_{E_{\mathrm{or}}} = \omega_{E_{\mathrm{sc}}}$. In the following discussion, we would like to think of $\tau_{E_{\mathrm{or}}} = \omega^{\vee}_{E_{\mathrm{or}}}$ (respectively, $\tau_{E_{\mathrm{sc}}} = \omega^{\vee}_{E_{\mathrm{sc}}}$) as the space of (linear) *functions on the $\omega_{E_{\mathrm{or}}}$ (respectively, $\omega_{E_{\mathrm{sc}}}$) portion of $E^{\dagger}_{\mathrm{or}}$ (respectively, $E^{\dagger}_{\mathrm{sc}}$).* From this "function-theoretic" point of view, we obtain an *isomorphism*

$$\iota_{\mathrm{func}} : \tau_{E_{\mathrm{sc}}} \to \tau_{E_{\mathrm{or}}}$$

where $\iota_{\mathrm{func}} \stackrel{\mathrm{def}}{=} (d \cdot \iota_{\mathrm{diff}})^{\vee}$.

Next, we would like to consider various *trivializations* of $\tau_{E_{\mathrm{or}}}$, $\tau_{E_{\mathrm{sc}}}$. If we think of E_{sc} as the elliptic curve "E" of Chapter VII, §4,5, then we have the trivialization "$\theta^{\vee}$" (cf. Chapter VII, §4,5) of $\tau_{E_{\mathrm{sc}}}$, which is determined up to multiplication by an element $\in \mathbf{S^1} \subseteq \mathbf{C}^{\times}$ by the property that its norm with respect to the metric "$|| \sim ||_{\tau}$" of Chapter VII, §4, is 1. Let us denote this trivialization by $\Theta_{\mathrm{DR,sc}}$. Put another way, if we regard E_{sc} as the "E" of Chapter VII, §4,5, then $\Theta_{\mathrm{DR,sc}}$ may be identified with the *exterior derivative of the function* "T_{DR}" on Chapter VII, §4,5. Similarly, we have a metric "$|| \sim ||_{\tau}$" (as in Chapter VII, §4) on $\tau_{E_{\mathrm{or}}}$ which gives rise to a trivialization $\Theta_{\mathrm{DR,or}}$ of $\tau_{E_{\mathrm{or}}}$ whose norm is 1 and which may be normalized by requiring that it be a positive multiple of $\Theta_{\mathrm{DR,sc}} \in \tau_{E_{\mathrm{sc}}} \cong \tau_{E_{\mathrm{or}}}$ (where "$\cong$" is either $\iota_{\mathrm{diff}}^{\vee}$ or ι_{func} – both give the same result since they differ by a positive multiple). Since the metric "$|| \sim ||_{\tau}$" of Chapter VII, §4, is defined by integration, and integrating the pull-back to E_{or} of a $(1,1)$-form on E_{sc} differs by a factor of d from integrating the form on E_{sc}, we thus obtain that $d^{\frac{1}{2}} \cdot (\Theta_{\mathrm{DR,or}})^{\vee} = \iota_{\mathrm{diff}}^{-1}((\Theta_{\mathrm{DR,sc}})^{\vee})$, which implies (by taking the dual) $d^{-\frac{1}{2}} \cdot \Theta_{\mathrm{DR,or}} = \iota_{\mathrm{diff}}^{\vee}(\Theta_{\mathrm{DR,sc}})$, hence (by multiplying by $d^{\frac{1}{2}}$):

$$\Theta_{\mathrm{DR,or}} = d^{-\frac{1}{2}} \cdot \iota_{\mathrm{func}}(\Theta_{\mathrm{DR,sc}})$$

On the other hand, in Chapter VII, §5, we also considered the trivialization "$\frac{\partial}{\partial \log(U)}$" (where U is the standard coordinate on $\mathbf{G}_{\mathrm{m}}$), which gave rise to the function "T_{SW}" of Chapter VII, §5. Let us write $\Theta_{\mathrm{SW,sc}}$ (respectively, $\Theta_{\mathrm{SW,or}}$) for the trivialization of $\omega_{E_{\mathrm{sc}}}$ (respectively, ω_{or}) defined by $\frac{\partial}{\partial \log(U_{\mathrm{sc}})}$ (respectively, $\frac{\partial}{\partial \log(U_{\mathrm{or}})}$). If we think of the elliptic curve "E" of Chapter VII, §5, as being E_{sc}, then one may also think of $\Theta_{\mathrm{SW,sc}}$ as the *exterior derivative of the function* "T_{SW}" of Chapter VII, §5. In particular, it follows from Chapter VII, Lemma 5.2, that

$$\Theta_{\mathrm{DR,sc}} = \{8\pi^2 \ \mathrm{Im}(\tau_{\mathrm{sc}})\}^{\frac{1}{2}} \cdot \Theta_{\mathrm{SW,sc}} = d^{-\frac{1}{2}} \cdot C_{\mathrm{or}} \cdot \Theta_{\mathrm{SW,sc}}$$

where $C_{\mathrm{or}} \stackrel{\mathrm{def}}{=} \{8\pi^2 \cdot \mathrm{Im}(\tau_{\mathrm{or}})\}^{\frac{1}{2}}$. Similarly,

$$\Theta_{\mathrm{DR,or}} = C_{\mathrm{or}} \cdot \Theta_{\mathrm{SW,or}}$$

Remark. One convenient way to keep track of the various trivializations introduced above is to think of them as *functions* on the "$\omega_{E_{\mathrm{or}}}$ (respectively, $\omega_{E_{\mathrm{sc}}}$) portion" of $E_{\mathrm{or}}^{\dagger}$ (respectively, $E_{\mathrm{sc}}^{\dagger}$), hence as linear functions (well-defined up to some constant term, since $E_{\mathrm{or}}^{\dagger}$, $E_{\mathrm{sc}}^{\dagger}$ are *torsors*) on the *d-torsion points of $E_{\mathrm{or}}^{\dagger}$* (which map to the d-torsion points of $E_{\mathrm{sc}}^{\dagger}$). From this point of view, we see that (if we neglect the constant term, then) $\Theta_{\mathrm{SW,or}}$ (respectively, $\Theta_{\mathrm{SW,sc}}$) takes values $\in \frac{1}{d} \cdot \mathbf{Z}$ (respectively, $\in \mathbf{Z}$) on the d-torsion points (cf. Chapter III, Corollary 5.9) of $E_{\mathrm{or}}^{\dagger}$. Thus, in summary, *the values taken (if we neglect the constant term) by these functions (trivializations) on the d-torsion of $E_{\mathrm{or}}^{\dagger}$ are given by*:

$$\Theta_{\mathrm{SW,sc}} : \in \mathbf{Z} \qquad \Theta_{\mathrm{DR,sc}} : \in \frac{C_{\mathrm{or}}}{d^{\frac{1}{2}}} \cdot \mathbf{Z}$$

$$\Theta_{\mathrm{SW,or}} : \in \frac{1}{d} \cdot \mathbf{Z} \qquad \Theta_{\mathrm{DR,or}} : \in \frac{C_{\mathrm{or}}}{d} \cdot \mathbf{Z}$$

As we will see in the following discussion, these factors of d, $d^{\frac{1}{2}}$ that occur are related to various *scaling issues*, of the sort discussed at the end of Chapter VII, §3.

Next, we would like to *apply the theory of Chapter VII, §5, relating the de Rham and Schottky-Weierstrass canonical zeta functions* in the case where the elliptic curve "E" (of Chapter VII, §5) is E_{sc}. First, let us write $(\mathbf{S^1})_{\mathrm{sc}} \subseteq (\mathbf{G}_{\mathrm{m}})_{\mathrm{sc}}$ for the copy of $(\mathbf{S^1})_{\mathrm{sc}}$ in $(\mathbf{G}_{\mathrm{m}})_{\mathrm{sc}}$. Note that $(\mathbf{S^1})_{\mathrm{sc}}$ also injects into E_{sc} via the projection $(\mathbf{G}_{\mathrm{m}})_{\mathrm{sc}} \to E_{\mathrm{sc}}$. Moreover, $(\mathbf{S^1})_{\mathrm{sc}} \subseteq E_{\mathrm{sc}}$ lifts naturally (and uniquely!) to a closed subgroup $(\mathbf{S^1})_{\mathrm{sc}} \subseteq E^{\dagger}_{\mathrm{sc}}$. Thus, we may restrict the relations of Chapter VII, Theorem 5.3, to $(\mathbf{S^1})_{\mathrm{sc}} \subseteq E^{\dagger}_{\mathrm{sc}}$. Note that the functions "T_{SW}" and "T_{DR}" of Chapter VII, §5, are *zero* on $(\mathbf{S^1})_{\mathrm{sc}} \subseteq E^{\dagger}_{\mathrm{sc}}$. (This follows since the canonical sections of $E^{\dagger}_{\mathrm{sc}} \to E_{\mathrm{sc}}$ used to define "T_{SW}" and "T_{DR}" are *continuous homomorphisms*, hence map $(\mathbf{S^1})_{\mathrm{sc}} \subseteq E_{\mathrm{sc}}$ into $(\mathbf{S^1})_{\mathrm{sc}} \subseteq E^{\dagger}_{\mathrm{sc}}$.) Also, note that although the theory of Chapter VII, §5, was only given in the case of a trivial character χ, it generalizes immediately to the case of an arbitrary character χ (cf. Chapter IV, §3, especially Theorems 3.2, 3.3), for instance, by "transport of structure" relative to the automorphism of "E" given by translating by an appropriate torsion point (cf. the discussion immediately preceding Chapter VII, Theorem 6.7). Thus, by Chapter VII, Theorem 5.3, we obtain (for $r \in \mathbf{Z}_{\geq 0}$):

$$\zeta^{\mathrm{SS}}_r|_{(\mathbf{S^1})_{\mathrm{sc}}} = \frac{r!}{d^r} \cdot \zeta^{\mathrm{PD}}_r|_{(\mathbf{S^1})_{\mathrm{sc}}} = \frac{r!}{C^r_{\mathrm{or}} \cdot d^{\frac{r}{2}}} \cdot \sum_{m=0}^{[r/2]} \frac{(\pi \cdot u)^m}{m!} \cdot \zeta^{\mathrm{DR}}_{r-2m}|_{(\mathbf{S^1})_{\mathrm{sc}}}$$

(where ζ^{SS}_r is as in Chapter VII, §6). Thus, in particular, if we think of ζ^{SS}_r as a section of the line bundle $\mathcal{L}^{\chi}_{\mathrm{sc}}$ on E_{sc} corresponding to χ (cf. Chapter VII, §6) over $E^{\dagger}_{\mathrm{sc}}$, then *the leading term of* ζ^{SS}_r, i.e., the image of ζ^{SS}_r in $(F^{r+1}/F^r)(\mathcal{R}_{E^{\dagger}_{\mathrm{sc}}}) \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} \mathcal{L}^{\chi}_{\mathrm{sc}}$, is equal to a unimodular multiple (i.e., $\mathbf{S^1} \subseteq \mathbf{C}^{\times}$-multiple) of

$$\frac{1}{C^r_{\mathrm{or}} \cdot d^{\frac{r}{2}}} \cdot \Theta^{\otimes r}_{\mathrm{DR,sc}} = \frac{1}{C^r_{\mathrm{or}}} \cdot \Theta^{\otimes r}_{\mathrm{DR,or}} = \Theta^{\otimes r}_{\mathrm{SW,or}}$$

while the *norm of* ζ^{SS}_r with respect to the metric "$|| \sim ||_{L^2_{\mathrm{DR}}}$" of Chapter VII, §4,5, which we denote here by $|| \sim ||_{\mathrm{DR,sc}}$, satisfies (by Chapter VII, Corollary 5.4, and the fact that $\frac{1}{d^r} \leq \frac{1}{r!}$):

$$||\zeta^{\mathrm{SS}}_r||_{\mathrm{DR,sc}} \leq ||\zeta^{\mathrm{SS}}_0||_{\mathrm{DR,sc}} \cdot \frac{e^{\pi}}{r!} \cdot \left(\frac{8\pi \cdot r \cdot d}{\mathrm{Im}(\tau_{\mathrm{or}})}\right)^{\frac{r}{2}} = ||\zeta^{\mathrm{SS}}_0||_{\mathrm{DR,sc}} \cdot e^{\pi} \cdot \left(\frac{8\pi^{\frac{3}{2}}}{C_{\mathrm{or}}}\right)^r \cdot \frac{(r \cdot d)^{\frac{r}{2}}}{r!}$$

Finally, by Chapter VII, Lemma 6.5 (where we take "$P(-)$" to be 1, and change the variable of integration from "k" to $x \stackrel{\mathrm{def}}{=} \{\frac{2\pi}{d} \cdot \mathrm{Im}(\tau_{\mathrm{or}})\}^{\frac{1}{2}} \cdot k$), we have:

$$\begin{aligned}||\zeta_0^{\mathrm{SS}}||^2_{\mathrm{DR,sc}} &= e^{\frac{2\pi}{d} \cdot \mathrm{Im}(\tau_{\mathrm{or}}) \cdot (i_\chi/n)^2} \cdot \left(\frac{4\pi \cdot d}{8\pi^2 \cdot \mathrm{Im}(\tau_{\mathrm{or}})}\right)^{\frac{1}{2}} \int_{\mathbf{R}} e^{-x^2} dx \\ &= e^{\frac{8\pi^2}{4\pi d} \cdot \mathrm{Im}(\tau_{\mathrm{or}}) \cdot (i_\chi/n)^2} \cdot (4\pi \cdot d)^{\frac{1}{2}} \cdot C_{\mathrm{or}}^{-1} \cdot \pi^{\frac{1}{2}} = 2\pi \cdot (d/C_{\mathrm{or}}^2)^{\frac{1}{2}} \cdot e^{\frac{(C_{\mathrm{or}} \cdot i_\chi/n)^2}{4\pi d}}\end{aligned}$$

Thus:

$$||\zeta_r^{\mathrm{SS}}||_{\mathrm{DR,sc}} \leq 100 \cdot (d/C_{\mathrm{or}}^2)^{\frac{1}{4}} \cdot e^{\frac{(C_{\mathrm{or}} \cdot i_\chi/n)^2}{8\pi d}} \cdot (50/C_{\mathrm{or}})^r \cdot \frac{(r \cdot d)^{\frac{r}{2}}}{r!}$$

In the following discussion, we would like to think of the metric $|| \sim ||_{\mathrm{DR,sc}}$, as well as the metrics $|| \sim ||_{\mathrm{Tch}}$; $|| \sim ||_{w,\boldsymbol{\mu}_a}$ of §2, as being *metrics on the space* $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^\chi \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} \mathcal{R}_{E_{\mathrm{sc}}^\dagger})$, which we think of as the subspace

$$\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^\chi \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} \mathcal{R}_{E_{\mathrm{sc}}^\dagger}) \hookrightarrow L^2((\mathbf{S^1})_{\mathrm{sc}})$$

(where the map "$\hookrightarrow$" is obtained by dividing by the trivialization "$U_{\mathrm{cv}}^{i_\chi} \cdot \theta^m$") *generated by* $\zeta_0^{\mathrm{SS}}, \ldots, \zeta_r^{\mathrm{SS}}, \ldots$ (i.e., ζ_0^{SS} and its derivatives).

On the other hand, let us write $|| \sim ||_{\mathrm{SS}}$ for the metric on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^\chi \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} \mathcal{R}_{E_{\mathrm{sc}}^\dagger})$ for which the ζ_r^{SS} are *orthonormal*. Then by Chapter VII, Theorem 5.3, we have:

$$\left(\frac{d^{\frac{1}{2}}}{C_{\mathrm{or}}}\right)^r \cdot \zeta_r^{\mathrm{DR}}|_{(\mathbf{S^1})_{\mathrm{sc}}} = \sum_{m=0}^{[r/2]} \frac{d^{r-m} \cdot (-\pi \cdot u)^m}{C_{\mathrm{or}}^{2m} \cdot m! \cdot (r-2m)!} \cdot \zeta_{r-2m}^{\mathrm{SS}}|_{(\mathbf{S^1})_{\mathrm{sc}}}$$

which shows that (for $r < d$)

$$\left(\frac{d^{\frac{1}{2}}}{C_{\mathrm{or}}}\right)^r \cdot ||\zeta_r^{\mathrm{DR}}||_{\mathrm{SS}} \leq e^{3d} \cdot \sum_{m=0}^{[r/2]} \frac{\pi^m}{C_{\mathrm{or}}^{2m}} \leq e^{3d} \cdot r \cdot \{1 + (\pi/C_{\mathrm{or}}^2)\}^r$$

(where we use that

$$\begin{aligned}\frac{d^{r-m}}{m!(r-2m)!} &\leq \frac{d^{r-m}}{(r-m)!} \cdot \binom{r-m}{m} \leq 2^r \cdot \frac{d^{r-m}}{(r-m)!} \\ &\leq e^{2r} \cdot \frac{d^{r-m}}{(r-m)^{r-m}} \leq e^{2r+d} \leq e^{3d}\end{aligned}$$

by Chapter VII, Lemmas 3.5, 3.6). Since, by Chapter VII, Corollary 4.6, Lemma 6.5, we have:

$$||\zeta_r^{\mathrm{DR}}||_{\mathrm{DR,sc}} \geq \left(\frac{(2\pi)^r}{r!}\right)^{\frac{1}{2}} \cdot ||\zeta_0^{\mathrm{DR}}||_{\mathrm{DR,sc}} = \left(\frac{(2\pi)^r}{r!}\right)^{\frac{1}{2}} \cdot e^{\frac{(C_{\mathrm{or}} \cdot i_\chi/n)^2}{8\pi d}} \cdot (2\pi)^{\frac{1}{2}} \cdot (d/C_{\mathrm{or}}^2)^{\frac{1}{4}}$$

we obtain:

$$\begin{aligned} ||\zeta_r^{\mathrm{DR}}||_{\mathrm{DR,sc}}^{-1} \cdot ||\zeta_r^{\mathrm{DR}}||_{\mathrm{SS}} &\leq e^{-\frac{(C_{\mathrm{or}} \cdot i_\chi/n)^2}{8\pi d}} \cdot \frac{(r!)^{\frac{1}{2}}}{(2\pi)^{\frac{r}{2}}} \cdot (C_{\mathrm{or}}/d^{\frac{1}{2}})^{r+\frac{1}{2}} \cdot r \cdot \{1 + (\pi/C_{\mathrm{or}}^2)\}^r \cdot e^{3d} \\ &\leq e^{-\frac{(C_{\mathrm{or}} \cdot i_\chi/n)^2}{8\pi d}} \cdot (r! \cdot d^{-r})^{\frac{1}{2}} \left(\frac{C_{\mathrm{or}} + \pi \cdot C_{\mathrm{or}}^{-1}}{(2\pi)^{\frac{1}{2}}}\right)^r \cdot \frac{r \cdot C_{\mathrm{or}}^{1/2}}{d^{\frac{1}{4}}} \cdot e^{3d} \end{aligned}$$

Thus, in summary:

Lemma 6.1. *For $r \leq d$, the metrics $|| \sim ||_{\mathrm{DR,sc}}$, $|| \sim ||_{\mathrm{SS}}$ on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^\chi \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E_{\mathrm{sc}}^\dagger}))$ satisfy:*

$$\begin{aligned} e^{\frac{(C_{\mathrm{or}} \cdot i_\chi/n)^2}{8\pi d} - 3d} \cdot (d^r/r!)^{\frac{1}{2}} \cdot \left(\frac{(2\pi)^{\frac{1}{2}}}{C_{\mathrm{or}} + \pi \cdot C_{\mathrm{or}}^{-1}}\right)^r \cdot (d^{\frac{3}{4}} \cdot r \cdot C_{\mathrm{or}}^{1/2})^{-1} \cdot || \sim ||_{\mathrm{SS}} &\leq || \sim ||_{\mathrm{DR,sc}} \\ &\leq e^{\frac{(C_{\mathrm{or}} \cdot i_\chi/n)^2}{8\pi d}} \cdot \frac{(r \cdot d)^{\frac{r}{2}}}{r!} \cdot (50/C_{\mathrm{or}})^r \cdot 100 \cdot (d^5/C_{\mathrm{or}}^2)^{\frac{1}{4}} \cdot || \sim ||_{\mathrm{SS}} \end{aligned}$$

Proof. This follows immediately from the above inequalities: Indeed, if $||\phi||_{\mathrm{SS}} \leq 1$ (where $\phi \in \Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^\chi \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E_{\mathrm{sc}}^\dagger}))$), then ϕ may be written as a linear combination of the ζ_r^{SS} with complex coefficients ≤ 1. Thus, $||\phi||_{\mathrm{DR,sc}}$ may be bounded using the bound for $||\zeta_r^{\mathrm{SS}}||_{\mathrm{DR,sc}}$ obtain above. Similarly, if $||\phi||_{\mathrm{DR,sc}} \leq 1$, then ϕ may be written as a linear combination of the $||\zeta_r^{\mathrm{DR}}||_{\mathrm{DR,sc}}^{-1} \cdot \zeta_r^{\mathrm{DR}}$ with complex coefficients ≤ 1. Thus, $||\phi||_{\mathrm{SS}}$ may be bounded using the bound for $||\zeta_r^{\mathrm{DR}}||_{\mathrm{DR,sc}}^{-1} \cdot ||\zeta_r^{\mathrm{SS}}||_{\mathrm{SS}}$ obtain above. ○

Remark. Note that (for $r \leq d$), we have:

$$(d^r/r!)^{\frac{1}{2}} \cdot d^{-2} \cdot e^{-3d} \geq e^{-4d}$$

$$\frac{(r \cdot d)^{\frac{r}{2}}}{r!} \cdot d^2 \leq \frac{e^r \cdot (r \cdot d)^{\frac{r}{2}}}{r^r} \cdot d^2 \leq e^r \cdot (d/r)^{\frac{r}{2}} \cdot d^2 \leq e^r \cdot d^2 \cdot e^{\frac{d}{2}} \leq e^{2d}$$

(cf. Chapter VII, Lemmas 3.5, 3.6), where we use that $d^2 \leq e^{\frac{d}{2}}$ since $d \geq 10$. It thus follows that:

> *If τ_{or} is restricted to vary in a compact subset of the upper half-plane, then $|| \sim ||_{\mathrm{SS}}$ and $|| \sim ||_{\mathrm{DR,sc}}$ differ from each other by a factor $\leq$* $\mathrm{constant} \cdot e^{4d}$.

In fact, even if τ_{or} is not restricted to vary inside a compact set, the "constant" in this statement is bounded by (some absolute constant)^d times $e^{\frac{(C_{\mathrm{or}} \cdot i_\chi/n)^2}{8\pi d}} \cdot (C_{\mathrm{or}} + C_{\mathrm{or}}^{-1})^d$. Note, in particular, that:

> *Despite the fact that the "leading term" of ζ_r^{SS} (cf. the above discussion) goes like $\frac{1}{C_{\mathrm{or}}^r} \cdot \Theta_{\mathrm{DR,or}}^{\otimes r} = \Theta_{\mathrm{SW,or}}^{\otimes r}$, the absolute value of ζ_r^{SS} with respect to $|| \sim ||_{\mathrm{DR,sc}}$ (a norm which is based on a $\Theta_{\mathrm{DR,sc}}$-type scaling of $\mathcal{R}_{E_{\mathrm{sc}}^\dagger}$, not a $\Theta_{\mathrm{DR,or}}$-type scaling!) goes (at least for compactly varying* $\mathrm{Im}(\tau_{\mathrm{or}})$, *and up to a factor of* $\mathrm{constant} \cdot e^{4d}$*) like* ≈ 1.

This discrepancy of roughly a factor $d^{\frac{r}{2}}$ (when $\mathrm{Im}(\tau_{\mathrm{or}})$ varies compactly) in the portion of torsorial degree r, i.e., a *scaling factor* of $d^{\frac{1}{2}}$, is the phenomenon of *analytic torsion* – cf. Remark 1 following Chapter VII, Corollary 4.6, which concerns a factor of $(r!)^{\frac{1}{2}} \approx d^{\frac{r}{2}}$ (as $r \to d$) in the portion of torsorial degree r. This analytic torsion may be regarded as the *analogue at the infinite prime* of the factors of $\frac{1}{r!}$ that must be added to the integral structure of the portion of torsorial degree r in the finite prime case (cf., e.g., Chapter V, Theorem 3.1).

Next, we would like to recall from §2 the *relationship between $|| \sim ||_{\mathrm{SS}}$ and $|| \sim ||_{\mathrm{Tch}}$*. First, let us recall from the first part of the proof of Lemma 2.1 that if $||\phi||_{\mathrm{Tch}} = 1$ (for $\phi \in \Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^\chi \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E_{\mathrm{sc}}^\dagger}))$, $r \leq d$), then ϕ may be written as a linear combination of the $\zeta_{r'}^{\mathrm{SS}}$ (where $r' < r$) with complex coefficients of absolute value

$$\leq r^{\frac{7}{2}} \cdot e^{4r}$$

Thus, we conclude that on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^\chi \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E_{\mathrm{sc}}^\dagger}))$ (where $r \leq d$), we have:

$$|| \sim ||_{\mathrm{SS}} \leq r^{\frac{9}{2}} \cdot e^{4r} \cdot || \sim ||_{\mathrm{Tch}}$$

On the other hand, from the definition of $|| \sim ||_{\mathrm{Tch}}$, we have

$$\begin{aligned} ||\zeta_r^{\mathrm{SS}}||_{\mathrm{Tch}}^2 &= \frac{1}{d} \cdot \sum_{j=0}^{d-1} \left|(j/d) - l_d\right|^{2r} \\ &\leq \frac{1}{d} \cdot \sum_{j=0}^{d-1} 1 = 1 \end{aligned}$$

where l_d is as in §2, hence satisfies $0 \leq l_d \leq \frac{1}{2}$. Thus, on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}^{\chi}_{\mathrm{sc}} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E^{\dagger}_{\mathrm{sc}}}))$ (where $r \leq d$), we have:

$$|| \sim ||_{\mathrm{Tch}} \leq r \cdot || \sim ||_{\mathrm{SS}}$$

We summarize this as follows:

Lemma 6.2. *For $r \leq d$, the metrics $|| \sim ||_{\mathrm{Tch}}$, $|| \sim ||_{\mathrm{SS}}$ on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}^{\chi}_{\mathrm{sc}} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E^{\dagger}_{\mathrm{sc}}}))$ satisfy:*

$$r^{-1} \cdot || \sim ||_{\mathrm{Tch}} \leq || \sim ||_{\mathrm{SS}} \leq r^{\frac{9}{2}} \cdot e^{4r} \cdot || \sim ||_{\mathrm{Tch}}$$

while the metrics $|| \sim ||_{\mathrm{Tch}}$, $|| \sim ||_{\mathrm{DR,sc}}$ on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}^{\chi}_{\mathrm{sc}} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E^{\dagger}_{\mathrm{sc}}}))$ satisfy:

$$\begin{aligned} e^{\frac{(C_{\mathrm{or}} \cdot i_{\chi}/n)^2}{8\pi d} - 3d} \cdot (d^r/r!)^{\frac{1}{2}} \cdot \Big(\frac{(2\pi)^{\frac{1}{2}}}{C_{\mathrm{or}} + \pi \cdot C_{\mathrm{or}}^{-1}}\Big)^r \cdot (d^{\frac{3}{4}} \cdot r^2 \cdot C_{\mathrm{or}}^{1/2})^{-1} \cdot || \sim ||_{\mathrm{Tch}} \\ \leq || \sim ||_{\mathrm{DR,sc}} \\ \leq e^{\frac{(C_{\mathrm{or}} \cdot i_{\chi}/n)^2}{8\pi d}} \cdot \frac{(r \cdot d)^{\frac{r}{2}}}{r!} \cdot (e^8/C_{\mathrm{or}})^r \cdot 100 \cdot (d^{12}/C_{\mathrm{or}})^{\frac{1}{2}} \cdot || \sim ||_{\mathrm{Tch}} \end{aligned}$$

Finally, the metrics $|| \sim ||_{w,\boldsymbol{\mu}_a}$ (cf. §2), $|| \sim ||_{\mathrm{DR,sc}}$ on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}^{\chi}_{\mathrm{sc}} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E^{\dagger}_{\mathrm{sc}}}))$ satisfy:

$$\begin{aligned} e^{\frac{(C_{\mathrm{or}} \cdot i_{\chi}/n)^2}{8\pi d} - 3d} \cdot (d^r/r!)^{\frac{1}{2}} \cdot \Big(\frac{(2\pi)^{\frac{1}{2}}}{e^4 \cdot a^2 \cdot (C_{\mathrm{or}} + \pi \cdot C_{\mathrm{or}}^{-1})}\Big)^r \cdot (d^8 \cdot C_{\mathrm{or}}^{1/2})^{-1} \cdot || \sim ||_{w,\boldsymbol{\mu}_a} \\ \leq || \sim ||_{\mathrm{DR,sc}} \\ \leq e^{\frac{(C_{\mathrm{or}} \cdot i_{\chi}/n)^2}{8\pi d}} \cdot \frac{(r \cdot d)^{\frac{r}{2}}}{r!} \cdot (e^8/C_{\mathrm{or}})^r \cdot 200 \cdot (d^{12}/C_{\mathrm{or}})^{\frac{1}{2}} \cdot || \sim ||_{w,\boldsymbol{\mu}_a} \end{aligned}$$

where a is a positive integer satisfying $a \geq 8 + (\frac{\pi}{4} \cdot \mathrm{Im}(\tau_{\mathrm{or}}))^{-1}$.

Remark. Thus, stated in words, the above Lemma asserts that:

> *When $\mathrm{Im}(\tau_{\mathrm{or}})$ varies compactly, $|| \sim ||_{\mathrm{DR,sc}}$ and $|| \sim ||_{w,\boldsymbol{\mu}_a}$ differ by at most a factor of* (constant)^d.

Thus, $|| \sim ||_{\mathrm{DR,sc}}$; $|| \sim ||_{\mathrm{SS}}$; $|| \sim ||_{\mathrm{Tch}}$; $|| \sim ||_{w,\boldsymbol{\mu}_a}$ *belong to "the same equivalence class"* – i.e., in the sense that when $\mathrm{Im}(\tau_{\mathrm{or}})$ varies compactly, they differ by at most a factor of (constant)^d.

Similarly, we may recall the *relationship between* $|| \sim ||_{\mathrm{SS}}$ *and the* ζ_r^{CG}, as follows. Let us write

$$|| \sim ||_{\mathrm{CG}} \quad (\text{respectively, } || \sim ||_{\mathrm{qCG}})$$

for the metric on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^{\chi} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^d(\mathcal{R}_{E_{\mathrm{sc}}^{\dagger}}))$ for which the ζ_r^{CG} (respectively, $q_{\mathrm{sc}}^{-\Psi(k[r])} \cdot \zeta_r^{\mathrm{CG}}$ – notation of §3,4,5), where $r = 0, \ldots, d-1$, are *orthonormal.* Recall from the proof of Chapter VII, Lemma 6.2 (i.e., in essence, Chapter VII, Proposition 3.3), that ζ_r^{CG} is a linear combination of $\zeta_0^{\mathrm{SS}}, \ldots, \zeta_r^{\mathrm{SS}}$ with complex coefficients of absolute value $\leq e^{2r+d}$. Thus,

$$||\zeta_r^{\mathrm{CG}}||_{\mathrm{SS}} \leq (r+1) \cdot e^{3d}$$

In particular, on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^{\chi} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E_{\mathrm{sc}}^{\dagger}}))$ (where $r \leq d$), we have:

$$|| \sim ||_{\mathrm{SS}} \leq r^2 \cdot e^{3d} \cdot || \sim ||_{\mathrm{CG}}$$

On the other hand, if we consider a linear combination of $\zeta_0^{\mathrm{CG}}, \ldots, \zeta_r^{\mathrm{CG}}$ whose $|| \sim ||_{\mathrm{Tch}} = d^{-1}$, then we see that the issue of bounding the coefficients of such a linear combination is exactly the same as the situation considered in §3, except with $|q_{\mathrm{sc}}| = 1$. (Indeed, note that if we set $|q_{\mathrm{sc}}| = 1$, then in the context of §3, $|| \sim ||_{\mathrm{Tch}} = d^{-1}$ means that the usual $L^2((\mathbf{S^1})_{\mathrm{sc}})$-norm of the corresponding linear combination of $Z_0^{\mathrm{CG}}, \ldots, Z_r^{\mathrm{CG}}$ is $= 1$.) Thus, by specializing what we did in §3 to the case where $|q_{\mathrm{sc}}| = 1$, we see that the coefficients of this linear combination are bounded by a *sum of* 4^r *terms*, each of which has absolute value $\leq r^{2r}$ (cf. the bounding of the "binomial coefficient portion of the $C[r, r_1] \cdot C[r_1, r_2] \cdot \ldots \cdot C[r_{j-1}, r_j]$" in §3). Thus, the coefficients of the linear combination under consideration have absolute value $\leq 4^r \cdot r^{2r}$. In particular, we obtain that on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^{\chi} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E_{\mathrm{sc}}^{\dagger}}))$ (where $r \leq d$), we have:

$$|| \sim ||_{\mathrm{CG}} \leq r \cdot 4^r \cdot r^{2r} \cdot d \cdot || \sim ||_{\mathrm{Tch}}$$

Finally, the leading term of ζ_r^{CG} is equal to the leading term of ζ_r^{PD}. That is, in summary:

Lemma 6.3. *For $r \leq d$, the metrics $|| \sim ||_{\mathrm{CG}}$, $|| \sim ||_{\mathrm{SS}}$ on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^{\chi} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E_{\mathrm{sc}}^{\dagger}}))$ satisfy:*

$$d^{-1} \cdot 4^{-r} \cdot r^{-2r-2} \cdot || \sim ||_{\mathrm{CG}} \leq || \sim ||_{\mathrm{SS}} \leq r^2 \cdot e^{3d} \cdot || \sim ||_{\mathrm{CG}}$$

and (for $0 \leq r \leq d-1$*)* ζ_r^{CG} *has the same leading term (i.e., image in* $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^{\chi} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} (F^{r+1}/F^r)(\mathcal{R}_{E_{\mathrm{sc}}^{\dagger}})))$ *as* $\frac{d^r}{r!} \cdot \zeta_r^{\mathrm{SS}}$*. On the other hand, (for* $r \leq d$*) the metrics* $|| \sim ||_{\mathrm{CG}}$*,* $|| \sim ||_{\mathrm{DR,sc}}$ *on* $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^{\chi} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E_{\mathrm{sc}}^{\dagger}}))$ *satisfy:*

$$\begin{aligned}
e^{\frac{(C_{\mathrm{or}} \cdot i_\chi/n)^2}{8\pi d} - 3d} \cdot (d^r/r!)^{\frac{1}{2}} \cdot r^{-2r-2} \cdot \left(\frac{(2\pi)^{\frac{1}{2}}}{4(C_{\mathrm{or}} + \pi \cdot C_{\mathrm{or}}^{-1})} \right)^r \cdot (d^2 \cdot r \cdot C_{\mathrm{or}}^{1/2})^{-1} \cdot || \sim ||_{\mathrm{CG}} \\
\leq || \sim ||_{\mathrm{DR,sc}} \\
\leq e^{\frac{(C_{\mathrm{or}} \cdot i_\chi/n)^2}{8\pi d}} \cdot \frac{(r \cdot d)^{\frac{r}{2}}}{r!} \cdot (e^7/C_{\mathrm{or}})^d \cdot 100 \cdot (d^7/C_{\mathrm{or}})^{\frac{1}{2}} \cdot || \sim ||_{\mathrm{CG}}
\end{aligned}$$

and (for $0 \leq r \leq d-1$*)* ζ_r^{CG} *has the same leading term (i.e., image in* $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^{\chi} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} (F^{r+1}/F^r)(\mathcal{R}_{E_{\mathrm{sc}}^{\dagger}})))$ *as* $\frac{d^{\frac{r}{2}}}{C_{\mathrm{or}}^r \cdot r!} \cdot \zeta_r^{\mathrm{DR}}$.

Remark. Thus, stated in words, the above Lemma asserts that:

> *When* $\mathrm{Im}(\tau_{\mathrm{or}})$ *varies compactly, the* $||\zeta_r^{\mathrm{CG}}||_{\mathrm{DR,sc}}$ *are bounded by a factor of* $(\mathrm{constant})^d$*, but if one expands* ζ_r^{CG} *in terms of an orthonormal basis for* $|| \sim ||_{\mathrm{DR,sc}}$*, the resulting coefficients have rather large absolute values, i.e., of the order* $d^{\mathrm{constant} \cdot d}$.

Put another way, whereas $|| \sim ||_{\mathrm{DR,sc}}$; $|| \sim ||_{\mathrm{SS}}$; $|| \sim ||_{\mathrm{Tch}}$; $|| \sim ||_{w,\boldsymbol{\mu}_a}$ *belong to "the same equivalence class"* – i.e., in the sense that when $\mathrm{Im}(\tau_{\mathrm{or}})$ varies compactly, they differ by at most a factor of $(\mathrm{constant})^d$ – the metric $|| \sim ||_{\mathrm{CG}}$ *lies outside this equivalence class.* That is to say, it is bounded below by metrics in this equivalence class, but not above. Moreover, this obstruction to bounding it above by metrics in the said equivalence class arises not from letting the elliptic curve in question degenerate, but from the *combinatorial* portion of the coefficients involved.

Finally, we consider the *Hermite model.* Let us write

$$|| \sim ||_{\mathrm{HM}_d}$$

for the metric on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^{\chi} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^d(\mathcal{R}_{E_{\mathrm{sc}}^{\dagger}}))$ for which

$$||P(\delta^*) \cdot \zeta_0||_{\mathrm{HM}_d}^2 \stackrel{\mathrm{def}}{=} \int_{\mathbf{R}} \left| P(\mathbf{s}) \right|^2 \cdot e^{-\frac{1}{2}(\mathbf{s}/\gamma_d)^2} \cdot d\mathbf{s}$$

where $\gamma_d \stackrel{\mathrm{def}}{=} \{d/4\pi \cdot \mathrm{Im}(\tau_{\mathrm{or}})\}^{\frac{1}{2}}$ (as in Chapter VII, Theorem 6.7). Thus, the $\zeta_0^{\mathrm{HM}_d}, \ldots, \zeta_{d-1}^{\mathrm{HM}_d}$ (of Chapter VII, Theorem 6.7) are *orthogonal* for $|| \sim ||_{\mathrm{HM}_d}$, and $\gamma_d^{-1} \cdot ||\zeta_r^{\mathrm{HM}_d}||_{\mathrm{HM}_d}^2 =$

$(2\pi)^{\frac{1}{2}} \cdot r!$. Recall that the $\zeta_r^{\mathrm{HM}_d}$ are the canonical zeta functions associated to the (scaled) *Hermite polynomials*

$$H_r(\mathbf{s}/\gamma_d)$$

where $H_r(-)$ is as in Chapter VII, Proposition 2.2. Let us write

$$|| \sim ||_{\mathbf{R},\mathrm{sc}}$$

for the metric on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^{\chi} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^d(\mathcal{R}_{E_{\mathrm{sc}}^{\dagger}}))$ denoted by $|| \sim ||_{L^2(E_{\mathrm{sc}})}$ in Chapter VII, §6, and by $|| \sim ||_{L^2_{\mathbf{R}}}$ in Chapter VII, §4. Note that

$$\begin{aligned} ||\zeta_0^{\mathrm{SS}}||^2_{\mathbf{R},\mathrm{sc}} = ||\zeta_0^{\mathrm{SS}}||^2_{\mathrm{DR},\mathrm{sc}} &= e^{\frac{2\pi}{d} \cdot \mathrm{Im}(\tau_{\mathrm{or}}) \cdot (i_\chi/n)^2} \cdot \left(\frac{d}{4\pi \cdot \mathrm{Im}(\tau_{\mathrm{or}})}\right)^{\frac{1}{2}} (2\pi)^{\frac{1}{2}} \\ &= (2\pi)^{\frac{1}{2}} \cdot \gamma_d \cdot e^{\frac{2\pi}{d} \cdot \mathrm{Im}(\tau_{\mathrm{or}}) \cdot (i_\chi/n)^2} \end{aligned}$$

(cf. the discussion preceding Lemma 6.1), and

$$\left|\left| \left(\frac{r!}{(2\pi)^{\frac{r}{2}}}\right) \cdot \zeta_r^{\mathrm{DR}} \right|\right|^2_{\mathbf{R},\mathrm{sc}} = r! \cdot ||\zeta_0^{\mathrm{DR}}||^2_{\mathbf{R},\mathrm{sc}} = (2\pi)^{\frac{1}{2}} \cdot r! \cdot \gamma_d \cdot e^{\frac{2\pi}{d} \cdot \mathrm{Im}(\tau_{\mathrm{or}}) \cdot (i_\chi/n)^2}$$

(cf. Chapter VII, Theorem 4.5). Now, if we write

$$||\zeta||_{\mathrm{et}}$$

(as in Theorem 5.8) for the $L^2(\boldsymbol{\mu}_d)$ norm (i.e., the norm for which $U_{\mathrm{sc}}^0, U_{\mathrm{sc}}^1, \ldots, U_{\mathrm{sc}}^{d-1}$ are orthonormal) of the function $(U_{\mathrm{cv}}^{-i_\chi} \cdot \zeta)|_{\boldsymbol{\mu}_d \subseteq (\mathbf{S}^1)_{\mathrm{sc}}}$, then we obtain from Chapter VII, Theorem 6.7, that:

Lemma 6.4. *If we fix a nonnegative integer $r \leq d$ and let $d \to \infty$, then on $\Gamma(E_{\mathrm{sc}}, \mathcal{L}_{\mathrm{sc}}^{\chi} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E_{\mathrm{sc}}^{\dagger}}))$, we have:*

$$\lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot || \sim ||_{\mathrm{et}} \;=\; \lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot || \sim ||_{\mathrm{HM}_d} \;=\; \lim_{d\to\infty} \gamma_d^{-\frac{1}{2}} \cdot || \sim ||_{\mathbf{R},\mathrm{sc}}$$

(where all three sides are finite*) and $|| \sim ||_{\mathbf{R},\mathrm{sc}} \leq || \sim ||_{\mathrm{DR},\mathrm{sc}} \leq e^{\pi + r} \cdot || \sim ||_{\mathbf{R},\mathrm{sc}}$.*

Proof. Note that the inequalities relating $|| \sim ||_{\mathbf{R},\mathrm{sc}}$ and $|| \sim ||_{\mathrm{DR},\mathrm{sc}}$ result from Chapter VII, Corollary 4.6 (for $\rho = 1$). ○

§7. The Comparison Isomorphism at the Infinite Prime

In this §, we complete the proof of a theorem, which is the main theorem of this paper, concerning the behavior of the comparison isomorphism of Chapter VI, Theorem 3.1, at all the primes of a number field. The behavior of this comparison isomorphism at finite primes is discussed in Chapter VI, Theorem 4.1; the behavior of this comparison isomorphism at archimedean primes is the general topic of Chapter VII and the present Chapter. We refer to the result which encompasses integrality issues at *all primes of a number field* as the *Hodge-Arakelov Comparison Isomorphism* (cf. the Introduction to this Chapter). In the present §, we complete the archimedean portion of this result, which consists of showing that the natural metrics on the function spaces of "de Rham functions" and "étale functions" are close to one another in three senses. These three senses correspond, respectively, to the *Hermite, Legendre, and Binomial Models* treated in Chapter VII and the present Chapter.

We continue with the notation of §5. Write $\mathcal{L}^{\chi}_{\mathrm{or}} \stackrel{\mathrm{def}}{=} \mathcal{L}^{\chi}_{\mathrm{sc}}|_{E_{\mathrm{or}}}$. Thus, $\mathcal{L}^{\chi}_{\mathrm{or}}$ is a line bundle of degree d on E_{or}. Let

$$\mathcal{G} \stackrel{\mathrm{def}}{=} \mathcal{G}_{\mathcal{L}^{\chi}_{\mathrm{or}}}$$

be the *theta group* associated to this line bundle (cf. Chapter IV, §1). Let us write

$$\mathcal{G}_{\mathbf{S}^1} \subseteq \mathcal{G}$$

for the *maximal compact subgroup*. Thus, we have an exact sequence

$$0 \to \mathbf{S^1} \to \mathcal{G}_{\mathbf{S}^1} \to K_{\mathcal{L}^{\chi}_{\mathrm{or}}} \to 0$$

Note that the kernel of $E_{\mathrm{or}} \to E_{\mathrm{sc}}$, together with the line bundle $\mathcal{L}^{\chi}_{\mathrm{sc}}$, define a *Lagrangian subgroup* (cf. Chapter IV, the Remark following Theorem 1.4) $H \subseteq \mathcal{G}$. Suppose that V is an *irreducible* $\mathbf{C}[\mathcal{G}]$*-module* (such that $\mathbf{G}_{\mathrm{m}} \subseteq \mathcal{G}$ acts on V in the usual fashion). Thus, V is a d-dimensional complex vector space, and the subspace of H-invariants $V^H \subseteq V$ is a 1-dimensional complex subspace. Let $v \subseteq V^H$ be nonzero.

Lemma 7.1. *There exists a unique $\mathcal{G}_{\mathbf{S}^1}$-invariant Hermitian metric $(-,-)$ on V such that $(v, v) = 1$.*

Proof. Write $K_H \subseteq K_{\mathcal{L}_{\rm or}^{\times}}$ for the image of H in $K_{\mathcal{L}_{\rm or}^{\times}}$. Let $H' \subseteq \mathcal{G}$ be a finite cyclic subgroup of order d with the property that the image $K_{H'}$ of H' in $K_{\mathcal{L}_{\rm or}^{\times}}$ is such that $K_{\mathcal{L}_{\rm or}^{\times}} = K_H \times K_{H'}$. (It is not difficult to see that such an H' always exists – cf., e.g., Chapter IV, Example 1.2). Since V is irreducible, it follows (cf. Chapter IV, Example 1.2) that

$$V = \bigoplus_{g \in H'} \mathbf{C} \cdot g(v)$$

For $g, g' \in H'$ such that $g \neq g'$, define $(g(v), g(v)) \stackrel{\rm def}{=} 1$, $(g(v), g'(v)) \stackrel{\rm def}{=} 0$. This gives us a well-defined metric $(-,-)$ on V. On checks easily that this metric is H'-invariant. Because of the well-known structure of $\mathcal{G}$, (cf. Chapter IV, §1), it follows that if $g \in H$, $g' \in H'$, then $g(g'(v)) = \lambda \cdot g'(g(v)) = \lambda \cdot g'(v)$, where $\lambda \in \mathbf{S^1} \subseteq \mathbf{C}^{\times}$. Thus, it is clear that H also preserves this metric. Thus, it follows that this metric is $\mathcal{G}_{\mathbf{S}^1}$-invariant. Uniqueness follows from the fact that V is *irreducible* as an $\mathcal{G}_{\mathbf{S}^1}$-module. ◯

Lemma 7.2. *Let $\{M; (-,-)_M\}$ be a pair consisting of a $\mathcal{G}_{\mathbf{S}^1}$-module M and a $\mathcal{G}_{\mathbf{S}^1}$-invariant metric $(-,-)_M$ on M. Then the correspondence*

$$\{M; (-,-)_M\} \mapsto \{M^H; (-,-)_{M^H} \stackrel{\rm def}{=} (-,-)_M|_{M^H}\}$$

given by taking H-invariants is an equivalence of categories (cf. Chapter IV, Theorem 1.4) between the category of finite dimensional (i.e., over $\mathbf{C}$) $\mathbf{C}[\mathcal{G}_{\mathbf{S}^1}]$-modules equipped with $\mathcal{G}_{\mathbf{S}^1}$-invariant Hermitian metrics and the category of finite dimensional complex vector spaces equipped with Hermitian metrics.

Proof. Indeed, the functor "$\mapsto$" is clearly well-defined. If $\{N; (-,-)_N\}$ is a finite dimensional complex vector space equipped with a Hermitian metric, and $\{V; (-,-)_V\}$ is a $\mathbf{C}[\mathcal{G}_{\mathbf{S}^1}]$-module as in Lemma 7.1 equipped with the metric of Lemma 7.1 (relative to some $v \in V^H$), then $N \otimes_{\mathbf{C}} V$ admits a natural structure of $\mathcal{G}_{\mathbf{S}^1}$-module (where $g \in \mathcal{G}_{\mathbf{S}^1}$ acts by $g(\alpha \otimes \beta) = \alpha \otimes g(\beta)$, for $\alpha \in N$, $\beta \in V$), as well as a natural metric (obtained by tensoring $(-,-)_N$ with $(-,-)_V$), which is clearly $\mathcal{G}_{\mathbf{S}^1}$-invariant. Moreover, it is clear that "$\mapsto$" associates $\{N \otimes V; (-,-)_{N \otimes V}\}$ to the pair $\{N; (-,-)_N\}$. Since maps between "$\{N; (-,-)_N\}$'s" clearly induce maps between "$\{N \otimes V; (-,-)_{N \otimes V}\}$'s," we thus see that the functor "$\mapsto$" is full and essentially surjective. Faithfulness follows from Chapter IV, Theorem 1.4. ◯

Thus, applying Lemma 7.2 shows that the various metrics:

$$|| \sim ||_{\rm DR,sc}; \quad || \sim ||_{\mathbf{R},\rm sc}; \quad || \sim ||_{\rm SS}; \quad || \sim ||_{\rm Tch}$$

$$|| \sim ||_{w,\boldsymbol{\mu}_a}; \quad || \sim ||_{\mathrm{et}}; \quad || \sim ||_{\mathrm{CG}}; \quad || \sim ||_{\mathrm{qCG}}$$

on the space $\Gamma(E_{\mathrm{sc}}, \mathcal{L}^{\chi}_{\mathrm{sc}} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^d(\mathcal{R}_{E^{\dagger}_{\mathrm{sc}}}))$ that we discussed in §5, 6 naturally define *metrics on the space* $\Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^d(\mathcal{R}_{E^{\dagger}_{\mathrm{sc}}}))$. (By abuse of notation) we shall denote the resulting metrics on $\Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^d(\mathcal{R}_{E^{\dagger}_{\mathrm{sc}}}))$ by the same names.

Note that the morphism $E^{\dagger}_{\mathrm{or}} \to E^{\dagger}_{\mathrm{sc}}$ considered in §6 defines an *isomorphism*

$$E^{\dagger}_{\mathrm{or},[d]} \cong E^{\dagger}_{\mathrm{sc}} \times_{E_{\mathrm{sc}}} E_{\mathrm{or}}$$

(where $E^{\dagger}_{\mathrm{or},[d]}$ is as in Chapter V, §2) which, if we identify $\omega_{E_{\mathrm{or}}}$ with $\omega_{E_{\mathrm{sc}}}$ via ι_{diff} (cf. the discussion at the beginning of §6), induces an *isometry* between $\omega_{E_{\mathrm{or}}} \subseteq E^{\dagger}_{\mathrm{or},[d]}$ and $\omega_{E_{\mathrm{or}}} = \omega_{E_{\mathrm{sc}}} \subseteq E^{\dagger}_{\mathrm{sc}} \times_{E_{\mathrm{sc}}} E_{\mathrm{or}}$ – i.e., both may be identified naturally with "d times the $\omega_{E_{\mathrm{or}}} \subseteq E^{\dagger}_{\mathrm{or}}$." Thus, we may naturally identify $\Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}} \otimes_{\mathcal{O}_{E_{\mathrm{sc}}}} F^r(\mathcal{R}_{E^{\dagger}_{\mathrm{sc}}}))$ with $\Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}} \otimes_{\mathcal{O}_{E_{\mathrm{or}}}} F^r(\mathcal{R}_{E^{\dagger}_{\mathrm{or},[d]}}))$. Note, moreover, that these spaces $\Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}} \otimes_{\mathcal{O}_{E_{\mathrm{or}}}} F^r(\mathcal{R}_{E^{\dagger}_{\mathrm{or},[d]}}))$ are exactly the spaces the appear in the domain of the *evaluation map* of Chapter VI, Theorem 4.1, (2) (when applied to E_{or}).

Next, let us consider the action of $\mathbf{Z}$ on $\mathfrak{H} \stackrel{\mathrm{def}}{=} \{\tau \in \mathbf{C} \mid \mathrm{Im}(\tau) > 0\}$ defined by letting $1 \in \mathbf{Z}$ act as $\tau \mapsto \tau + 1$. Write

$$\mathcal{T}_{\infty} \stackrel{\mathrm{def}}{=} \mathfrak{H}/\mathbf{Z}$$

for the quotient of $\mathfrak{H}$ by this action. Thus, the function $q \stackrel{\mathrm{def}}{=} \exp(2\pi i \tau)$ defines a biholomorphic isomorphism of $\mathcal{T}_{\infty}$ with the punctured unit disk $\{z \in \mathbf{C} \mid 0 < |z| < 1\}$. In the present context, we would like to think of $\mathcal{T}_{\infty}$ as a *covering space of the analytic stack* $\overline{\mathcal{M}}^{\mathrm{an}}_{1,0} = \mathfrak{H}/SL_2(\mathbf{Z})$ (where the quotient $\mathfrak{H}/SL_2(\mathbf{Z})$ is the quotient of $\mathfrak{H}$ by the standard action of $SL_2(\mathbf{Z})$ *in the sense of analytic stacks*).

Then let us observe that *as the moduli of* E_{or} *vary (i.e., where take "q" in the preceding paragraph to be* q_{or}*), the metrics defined above all descend naturally from* $\mathfrak{H}$ *to* $\mathcal{T}_{\infty}$. (Indeed, this follows immediately from the definitions.) Moreover, I *claim* that:

The metrics $|| \sim ||_{\mathrm{DR,sc}}$*;* $|| \sim ||_{\mathbf{R},\mathrm{sc}}$*;* $|| \sim ||_{\mathrm{et}}$ *descend naturally to* $\overline{\mathcal{M}}^{\mathrm{an}}_{1,0}$.

Indeed, $|| \sim ||_{\mathbf{R},\mathrm{sc}}$ is defined by using the *canonical real analytic section* $(\kappa_{\mathrm{sc}})_{\mathbf{R}} : (E_{\mathrm{sc}})_{\mathbf{R}} \to (E^{\dagger}_{\mathrm{sc}})_{\mathbf{R}}$ (which defines a real analytic splitting of $\mathcal{R}_{E^{\dagger}_{\mathrm{sc}}}$) to restrict sections of $\mathcal{L}^{\chi}_{\mathrm{sc}}$ over $E^{\dagger}_{\mathrm{sc}}$

to sections over $\mathcal{L}^{\chi}_{\mathrm{sc}}$. On the other hand, one checks easily (for instance, by considering torsion points) that the canonical real analytic splitting $(\kappa_{\mathrm{or}})_{\mathbf{R}} : (E_{\mathrm{or}})_{\mathbf{R}} \to (E^{\dagger}_{\mathrm{or}})_{\mathbf{R}}$ is *compatible* with $(\kappa_{\mathrm{sc}})_{\mathbf{R}}$ with respect to the natural morphism $E^{\dagger}_{\mathrm{or}} \to E^{\dagger}_{\mathrm{sc}}$. Moreover, $(\kappa_{\mathrm{or}})_{\mathbf{R}}$ manifestly descends from $\mathfrak{H}$ to $\overline{\mathcal{M}}^{\mathrm{an}}_{1,0}$. On the other hand, the metric with translation-invariant curvature on $\mathcal{L}^{\chi}_{\mathrm{sc}}$ (that was used in Chapter VII, §4) pulls back to a metric with translation-invariant curvature on $\mathcal{L}^{\chi}_{\mathrm{or}}$. Since metrics with translation-invariant curvature are unique up to multiplication by a positive real number, we thus see that such a metric descends from $\mathfrak{H}$ to $\overline{\mathcal{M}}^{\mathrm{an}}_{1,0}$. Thus, in summary, all the ingredients used to define $|| \sim ||_{\mathbf{R},\mathrm{sc}}$ descend naturally to $\overline{\mathcal{M}}^{\mathrm{an}}_{1,0}$, so we conclude that $|| \sim ||_{\mathbf{R},\mathrm{sc}}$ itself descends naturally to $\overline{\mathcal{M}}^{\mathrm{an}}_{1,0}$.

Similarly, the only additional object used to define $|| \sim ||_{\mathrm{DR},\mathrm{sc}}$ is the metric on $\omega_{E_{\mathrm{sc}}}$ defined in Chapter VII, §4, but this metric is equal to $d^{-\frac{1}{2}}$ times the analogously defined metric on $\omega_{E_{\mathrm{or}}}$ (which manifestly descends to $\overline{\mathcal{M}}^{\mathrm{an}}_{1,0}$) – cf. the discussion at the beginning of §6 involving $\Theta_{\mathrm{DR},\mathrm{sc}} = d^{\frac{1}{2}} \cdot \Theta_{\mathrm{DR},\mathrm{or}}$. Thus, $|| \sim ||_{\mathrm{DR},\mathrm{sc}}$ also descends naturally to $\overline{\mathcal{M}}^{\mathrm{an}}_{1,0}$.

Finally, the metric on $|| \sim ||_{\mathrm{et}}$ on $\Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}} \otimes_{\mathcal{O}_{E_{\mathrm{or}}}} F^r(\mathcal{R}_{E^{\dagger}_{\mathrm{or},[d]}}))$ is clearly equal to the metric obtained as follows: Since we have a natural inclusion ${}_dE_{\mathrm{or}} \subseteq E^{\dagger}_{\mathrm{or},[d]}$ of the d-torsion points ${}_dE_{\mathrm{or}}$ of E_{or} in $E^{\dagger}_{\mathrm{or},[d]}$, we may *restrict* sections $s \in \Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}} \otimes_{\mathcal{O}_{E_{\mathrm{or}}}} F^r(\mathcal{R}_{E^{\dagger}_{\mathrm{or},[d]}}))$ to ${}_dE_{\mathrm{or}}$ to obtain sections $s|_{{}_dE_{\mathrm{or}}} \in \mathcal{L}^{\chi}_{\mathrm{or}}|_{{}_dE_{\mathrm{or}}}$. Then

$$||s||^2_{\mathrm{et}} = \frac{1}{d^2} \cdot \sum_{\alpha \in {}_dE_{\mathrm{or}}} |s(\alpha)|^2$$

(where $| \sim |$ is the chosen metric on $\mathcal{L}^{\chi}_{\mathrm{or}}$ with translation-invariant curvature), which manifestly descends to $\overline{\mathcal{M}}^{\mathrm{an}}_{1,0}$. This completes the proof of the *claim*.

Next, we pause to take a closer look at the metric $|| \sim ||_{\mathbf{R},\mathrm{sc}}$ on $\Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}} \otimes_{\mathcal{O}_{E_{\mathrm{or}}}} F^r(\mathcal{R}_{E^{\dagger}_{\mathrm{or},[d]}}))$. Note that we have a natural surjection:

$$\begin{aligned}\Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}} \otimes_{\mathcal{O}_{E_{\mathrm{or}}}} F^{r+1}(\mathcal{R}_{E^{\dagger}_{\mathrm{or},[d]}})) &\to \Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}} \otimes_{\mathcal{O}_{E_{\mathrm{or}}}} (F^{r+1}/F^r)(\mathcal{R}_{E^{\dagger}_{\mathrm{or},[d]}}))\\ &= (d \cdot \tau_{E_{\mathrm{or}}})^r \otimes \Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}})\end{aligned}$$

Moreover, since all the metrics dealt with here are $\mathcal{G}_{\mathbf{S}^1}$*-invariant*, it follows that, if we equip $d \cdot \tau_{E_{\mathrm{or}}}$ with its natural metric (i.e., the metric for which $\Theta_{\mathrm{DR},\mathrm{or}}$ has norm d^{-1}), then the metric induced by $|| \sim ||_{\mathbf{R},\mathrm{sc}}$ on $(d \cdot \tau_{E_{\mathrm{or}}})^r \otimes \Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}})$ is equal to *some constant* times the metric induced by $|| \sim ||_{\mathbf{R},\mathrm{sc}}$ on $\Gamma(E_{\mathrm{or}}, \mathcal{L}^{\chi}_{\mathrm{or}})$. By Chapter VII, Theorem 4.5, it follows that this constant is $(d^{-1} \cdot \Theta_{\mathrm{DR},\mathrm{or}}/\Theta_{\mathrm{DR},\mathrm{sc}})^r \cdot r! \cdot \left(\frac{(2\pi)^r}{r!}\right)^{\frac{1}{2}}$. Since $(d^{-1} \cdot \Theta_{\mathrm{DR},\mathrm{or}}/\Theta_{\mathrm{DR},\mathrm{sc}})^r = d^{-\frac{3r}{2}}$, we thus obtain:

Lemma 7.3. *We have:*

$$\text{Metric induced by } || \sim ||_{\mathbf{R},\mathrm{sc}} \text{ on } (d \cdot \tau_{E_{\mathrm{or}}})^r \otimes \Gamma(E_{\mathrm{or}}, \mathcal{L}_{\mathrm{or}}^{\chi})$$
$$= \text{Metric induced by } \left(\frac{(2\pi)^r \cdot r!}{d^{3r}}\right)^{\frac{1}{2}} \cdot || \sim ||_{\mathbf{R},\mathrm{sc}} \text{ on } \Gamma(E_{\mathrm{or}}, \mathcal{L}_{\mathrm{or}}^{\chi})$$

Remark. The meaning of Lemma 7.3 is the following: We saw in Chapter VI, Theorem 4.1, that *at finite primes*, the necessary modification to the integral structure of the space "$(d \cdot \tau_{E_{\mathrm{or}}})^r \otimes \Gamma(E_{\mathrm{or}}, \mathcal{L}_{\mathrm{or}}^{\chi})$" that allows one to obtain an exact comparison isomorphism is a factor of $(r!)^{-1}$ (cf. Chapter V, Theorem 3.1). Thus, by the *product formula*, assuming that the modifications to the integral structure are distributed evenly with respect to r, *one expects that the modification to the integral structure to "$(d \cdot \tau_{E_{\mathrm{or}}})^r \otimes \Gamma(E_{\mathrm{or}}, \mathcal{L}_{\mathrm{or}}^{\chi})$" at the infinite primes should be of the order of $(r!)^{-1} \approx (d!)^{-1}$ (as $r \to d$)* (cf. the discussion of the "Fundamental Combinatorial Model" in Chapter VII, §3, especially the proof of Proposition 3.4). On the other hand, Lemma 7.3 states that in fact, for the metric $|| \sim ||_{\mathbf{R},\mathrm{sc}}$ (hence also for $|| \sim ||_{\mathrm{DR},\mathrm{sc}}$ – by Lemma 6.4), the corresponding modification to the integral structure is roughly

$$\left(\frac{(2\pi)^r \cdot r!}{d^{3r}}\right)^{\frac{1}{2}} \approx (d^{-3r} \cdot r!)^{\frac{1}{2}} \approx (d!)^{-1}$$

(where we ignore factors of the order $(\text{constant})^d$) as $r \to d$. Moreover, let us recall that these "modifications to the integral structure" are also known as *analytic torsion*. Thus, in summary:

> *Lemma 7.3 asserts that for the metrics $|| \sim ||_{\mathrm{DR},\mathrm{sc}}$, $|| \sim ||_{\mathbf{R},\mathrm{sc}}$, the resulting analytic torsion at the infinite prime is roughly the same as the analytic torsion that one would expect from applying the product formula (of elementary number theory) to the analytic torsion at the finite primes.*

Put another way, Lemma 7.3 is the *analogue for general smooth elliptic curves of Chapter VII, Proposition 3.4* (which, in effect, concerns "degenerate elliptic curves"). In particular, Lemma 7.3 leads one to suspect that:

> The metrics $|| \sim ||_{\mathrm{DR},\mathrm{sc}}$, $|| \sim ||_{\mathbf{R},\mathrm{sc}}$ are *natural (rough) candidates for* (at least the "combinatorial" or "arithmetic" portion – i.e., the portion that does not arise from allowing the elliptic curve in question to degenerate – of) *the integral structure on*
>
> $$\Gamma(E_{\mathrm{or}}, \mathcal{L}_{\mathrm{or}}^{\chi} \otimes_{\mathcal{O}_{E_{\mathrm{or}}}} F^d(\mathcal{R}_{E^{\dagger}_{\mathrm{or},[d]}}))$$

arising from the "étale side" of the comparison isomorphism, i.e., the metric $|| \sim ||_{\mathrm{et}}$.

The extensive calculations of Chapter VII and the present Chapter imply that *this intuitive argument based on the product formula does, in fact, give the right answer*:

Theorem 7.4. **(The Comparison Isomorphism at the Infinite Prime)** *Let $E \stackrel{\mathrm{def}}{=} \mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$ (where $q = \exp(2\pi i\tau) \in \mathcal{T}_\infty$) be an elliptic curve over $\mathbf{C}$. Let $d, m \geq 1$ be integers such that m does* **not** *divide d; write $n \stackrel{\mathrm{def}}{=} 2m$. Let*

$$\mathcal{L} \stackrel{\mathrm{def}}{=} \mathcal{O}_E(d \cdot [\eta])$$

where $\eta \in E$ is a point of order equal to m. Write $E^\dagger_{[d]}$ for the result of pushing forward the universal extension $E^\dagger \to E$ of E by the map $\omega_E \to \omega_E$ given by multiplication by d. Write ${}_dE \subseteq E$ for the subscheme of d-torsion points; thus, we have a natural inclusion ${}_dE \subseteq E^\dagger_{[d]}$. Consider the **Comparison Isomorphism**

$$\Xi : \Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^d(\mathcal{R}_{E^\dagger_{[d]}})) \cong \mathcal{L}|_{{}_dE}$$

of Chapter VI, Theorem 4.1 (given by restriction sections of $\mathcal{L}$ over $E^\dagger_{[d]}$ to ${}_dE \subseteq E^\dagger_{[d]}$). Choose a metric $| \sim |_{\mathcal{L}}$ on $\mathcal{L}$ whose curvature is translation-invariant. Let us regard $\mathcal{L}|_{{}_dE}$ as equipped with the L^2-metric defined by $| \sim |_{\mathcal{L}}$, i.e.:

$$||s||^2 \stackrel{\mathrm{def}}{=} \frac{1}{d^2} \cdot \sum_{\alpha \in {}_dE} |s(\alpha)|^2$$

*for $s \in \mathcal{L}|_{{}_dE}$. Write $|| \sim ||_{\mathrm{et}}$ for the "***étale metric***," i.e., the metric on $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^d(\mathcal{R}_{E^\dagger_{[d]}}))$ induced by this L^2-metric. On the other hand, the* **canonical real analytic splitting** *$\kappa_{\mathbf{R}} : E_{\mathbf{R}} \to (E^\dagger_{[d]})_{\mathbf{R}}$ (i.e., the unique continuous section which is a group homomorphism) defines a metric $|| \sim ||_{\mathbf{R}}$ on $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^d(\mathcal{R}_{E^\dagger_{[d]}}))$ by restricting sections of holomorphic sections of $\mathcal{L}$ over $E^\dagger_{[d]}$ to real analytic sections of $\mathcal{L}$ over E, where one has the* **$\mathbf{L^2}$-metric***:*

$$||s||^2_{\mathbf{R}} \stackrel{\mathrm{def}}{=} \int_E |\kappa^*_{\mathbf{R}}(s)|^2_{\mathcal{L}} \cdot d\mu$$

(where $s \in \Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^d(\mathcal{R}_{E^\dagger_{[d]}}))$, and $d\mu$ is the unique translation-invariant $(1,1)$-form whose integral is 1). Finally, the section $\kappa_{\mathbf{R}} : E_{\mathbf{R}} \to (E^\dagger_{[d]})_{\mathbf{R}}$ determines a real analytic direct sum decomposition

$$(\mathcal{R}_{E^\dagger_{[d]}})_{\mathbf{R}} \cong \bigoplus_{r \geq 0} \tau_E^{\otimes r} \otimes \mathcal{O}_{E_{\mathbf{R}}}$$

(where the subscript "$\mathbf{R}$'s" denote that the decomposition is in the real analytic category). If we use this decomposition to write $s \in \Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^d(\mathcal{R}_{E^\dagger_{[d]}}))$ as $s = s[0] + s[1] + \ldots + s[r] + \ldots$, and, moreover, equip ω_E with the metric given by:

$$||\alpha||^2 \stackrel{\text{def}}{=} d \cdot \int_E \alpha \wedge \overline{\alpha}$$

*then we obtain another metric, the "**de Rham metric**" $|| \sim ||_{\text{DR}}$, on $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^d(\mathcal{R}_{E^\dagger_{[d]}}))$:*

$$||s||_{\text{DR}} \stackrel{\text{def}}{=} \sum_{r \geq 0} \int_E |s[r]|^2_{\mathcal{L}} \cdot d\mu$$

(cf. Chapter VII, §4, for more details).

These metrics satisfy:

$$|| \sim ||_{\mathbf{R}} \leq || \sim ||_{\text{DR}} \leq e^{\pi + r} \cdot || \sim ||_{\mathbf{R}}$$

Moreover,

(1) (Hermite Model) *If we fix a nonnegative integer $r \leq d$ and let $d \to \infty$, then on $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^r(\mathcal{R}_{E^\dagger_{[d]}}))$, we have:*

$$\lim_{d \to \infty} \gamma_d^{-\frac{1}{2}} \cdot || \sim ||_{\text{et}} \;=\; \lim_{d \to \infty} \gamma_d^{-\frac{1}{2}} \cdot || \sim ||_{\text{HM}_d} \;=\; \lim_{d \to \infty} \gamma_d^{-\frac{1}{2}} \cdot || \sim ||_{\mathbf{R}}$$

Here $\gamma_d \stackrel{\text{def}}{=} \left\{\frac{d}{4\pi \cdot \text{Im}(\tau)}\right\}^{\frac{1}{2}}$, and $|| \sim ||_{\text{HM}_d}$ is the metric on the space $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^r(\mathcal{R}_{E^\dagger_{[d]}}))$ defined by considering **Hermite polynomials scaled by** *γ_d in the derivatives of the theta functions $\in \Gamma(E, \mathcal{L})$ (cf. §6 for more details on $|| \sim ||_{\text{HM}_d}$).*

(2) (Legendre Model) *There is a natural metric $|| \sim ||_{\text{Tch}}$ on $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^d(\mathcal{R}_{E^\dagger_{[d]}}))$ which admits a natural orthonormal basis obtained by considering* **discrete Tchebycheff polynomials** *(cf. Chapter VII, Proposition 3.1)* **scaled by** *d in the derivatives of the theta functions $\in \Gamma(E, \mathcal{L})$ (cf. §2 for more details on $|| \sim ||_{\text{Tch}}$). In the limit, as $d \to \infty$, with this scaling, these polynomials converge to the* **Legendre polynomials***. Moreover, if $d \geq 25$, then for $r \leq d$, the metrics $|| \sim ||_{\text{Tch}}$; $|| \sim ||_{\text{DR}}$ on $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^r(\mathcal{R}_{E^\dagger_{[d]}}))$ satisfy:*

$$\frac{|q|^{-\frac{1}{2d} \cdot (i_\chi / n)^2}}{d^3 \cdot C^{1/2} \cdot e^{5d} \cdot (C + C^{-1})^r} \cdot || \sim ||_{\text{Tch}} \leq || \sim ||_{\text{DR}}$$
$$\leq \left(\frac{e^5 \cdot d^6}{C^{\frac{1}{2}}}\right) \cdot \left(\frac{e^{10}}{C}\right)^d \cdot |q|^{-\frac{1}{2d} \cdot (i_\chi / n)^2} \cdot || \sim ||_{\text{Tch}}$$

where $C \overset{\mathrm{def}}{=} \{8\pi^2 \cdot \mathrm{Im}(\tau)\}^{\frac{1}{2}}$. *Moreover, if* a *is a natural number, then there is a natural metric* $|| \sim ||_{w,\boldsymbol{\mu}_a}$ *on* $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^d(\mathcal{R}_{E^\dagger_{[d]}}))$ *obtained by* **averaging** $|| \sim ||_{\mathrm{et}}$ *with respect to translates by the* $a \cdot d$*-torsion points of* $E = \mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$ *arising from the torsion points of* $\mathbf{G}_{\mathrm{m}}$ *(cf. §1,2, for more details). The metrics* $|| \sim ||_{w,\boldsymbol{\mu}_a}$*;* $|| \sim ||_{\mathrm{DR}}$ *on* $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^r(\mathcal{R}_{E^\dagger_{[d]}}))$ *satisfy:*

$$\frac{|q|^{-\frac{1}{2d}\cdot(i_\chi/n)^2}}{d^8 \cdot C^{1/2} \cdot e^{9d} \cdot \{a^2 \cdot (C + C^{-1})\}^r} \cdot || \sim ||_{w,\boldsymbol{\mu}_a} \leq || \sim ||_{\mathrm{DR}}$$
$$\leq \left(\frac{e^6 \cdot d^6}{C^{\frac{1}{2}}}\right) \cdot \left(\frac{e^{10}}{C}\right)^d \cdot |q|^{-\frac{1}{2d}\cdot(i_\chi/n)^2} \cdot || \sim ||_{w,\boldsymbol{\mu}_a}$$

whenever $a \geq 8 + (\frac{\pi}{4} \cdot \mathrm{Im}(\tau))^{-1}$.

(3) (Binomial Model) *There is a natural metric* $|| \sim ||_{\mathrm{CG}}$ *on* $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^d(\mathcal{R}_{E^\dagger_{[d]}}))$ *which admits a natural orthonormal basis obtained by considering certain* **binomial coefficient polynomials** *(cf. Chapter V, Theorem 4.8)* **(scaled by** $1 = d^0$**)** *in the derivatives of the theta functions* $\in \Gamma(E, \mathcal{L})$ *(cf. §2 for more details on* $|| \sim ||_{\mathrm{Tch}}$*). If* $d \geq 25$*, for* $r \leq d$*, then the metrics* $|| \sim ||_{\mathrm{CG}}$*,* $|| \sim ||_{\mathrm{DR}}$ *on* $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^d(\mathcal{R}_{E^\dagger_{[d]}}))$ *satisfy:*

$$\frac{|q|^{-\frac{1}{2d}\cdot(i_\chi/n)^2}}{e^{3d} \cdot r^{2r} \cdot d^5 \cdot C^{1/2} \cdot \{4\pi(C + C^{-1})\}^r} \cdot || \sim ||_{\mathrm{CG}} \leq || \sim ||_{\mathrm{DR}}$$
$$\leq \left(\frac{e^5 \cdot d^4}{C^{\frac{1}{2}}}\right) \cdot \left(\frac{e^9}{C}\right)^d \cdot |q|^{-\frac{1}{2d}\cdot(i_\chi/n)^2} \cdot || \sim ||_{\mathrm{CG}}$$

Moreover, there is a natural metric $|| \sim ||_{\mathrm{qCG}}$ *on* $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^d(\mathcal{R}_{E^\dagger_{[d]}}))$ *which admits a natural orthonormal basis obtained by* **dividing the binomial coefficient polynomials** *in the derivatives of the theta functions considered above* **by certain powers of** q*. If* $d \geq 12$*, and* $\mathrm{Im}(\tau) \geq 200\{\log^2(d) + n \cdot \log(d) + n \cdot \log(n)\}$*, then this metric satisfies:*

$$n^{-1} \cdot e^{-32d} \cdot || \sim ||_{\mathrm{qCG}} \leq || \sim ||_{\mathrm{et}} \leq e^{4d} \cdot || \sim ||_{\mathrm{qCG}}$$

In particular, for each of these models, the **combinatorial/arithmetic portion of the analytic torsion** *(i.e., the portion not arising from letting the elliptic curve* E *degenerate) induced on* $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} (F^{r+1}/F^r)(\mathcal{R}_{E^\dagger_{[d]}}))$ *by the metrics* $|| \sim ||_{\mathbf{R}}$*;* $|| \sim ||_{\mathrm{DR}}$*;* $|| \sim ||_{\mathrm{HM}_d}$*;* $|| \sim ||_{\mathrm{Tch}}$*;* $|| \sim ||_{w,\boldsymbol{\mu}_a}$*;* $|| \sim ||_{\mathrm{qCG}}$ *(in their respective domains of applicability) as* $r \to d$*, goes (modulo factors of the order* $(\mathrm{constant})^d$*) as*

$$\approx (d!)^{-1}$$

which is precisely what you would expect by applying the **product formula** *to the computation of the "analytic torsion" in the* **finite prime case** *(cf. Chapter V, Theorem 3.1; Chapter VI, Theorem 4.1; Chapter VII, Proposition 3.4). Finally, all of the metrics appearing above on the space* $\Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} F^r(\mathcal{R}_{E^\dagger_{[d]}}))$ *are invariant with respect to the maximal compact subgroup* $\mathcal{G}_{\mathbf{S}^1} \subseteq \mathcal{G}_{\mathcal{L}}$ *of the theta group* $\mathcal{G}_{\mathcal{L}}$ *associated to* $\mathcal{L}$*.*

Proof. Note that relative to the discussion preceding Theorem 7.4, we replace E_{or} (respectively, q_{or}; DR, sc; $\mathbf{R}, \mathrm{sc}$) by the simpler notation E (respectively, q; DR; $\mathbf{R}$) in the statement of Theorem 7.4. The inequality relating $|| \sim ||_{\mathbf{R}}$; $|| \sim ||_{\mathrm{DR}}$ follows from Lemma 6.4. The estimates of (1) (respectively, (2); (3)) follow by weakening/simplifying the estimates of Lemma 6.4 (respectively, Lemma 6.2; Lemma 6.3, Theorem 5.8). For stronger estimates, we refer to the various estimates of §5, 6. (Note that to simplify these estimates, we make use of Chapter VII, Lemmas 3.5, 3.6.) The statement concerning the relationship between the analytic torsion and the comparison theorem at finite primes follows from Lemma 7.3 and the Remark following Lemma 7.3. The statement concerning invariance with respect to $\mathcal{G}_{\mathbf{S}^1}$ follows from the definitions of the metrics using Lemma 7.2. ○

Remark. Note that the three models correspond roughly to three "domains of validity/applicability":

$$\underline{\text{Hermite Model}} \text{ (scaling factor } = d^{\frac{1}{2}}) : \text{ nondegenerating } E, \text{ fixed } r < d$$

$$\underline{\text{Legendre Model}} \text{ (scaling factor } = d) : \text{ nondegenerating } E, \text{ varying } r < d$$

$$\underline{\text{Binomial Model}} \text{ (scaling factor } = 1) : \text{ degenerating } E$$

Relative to $\mathrm{Im}(\tau)$ (where $E = \mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$, $q = \exp(2\pi i\tau)$), the Hermite Model (respectively, Legendre Model, Binomial Model) corresponds to the case where/is most useful when $\mathrm{Im}(\tau)$ is fixed (respectively, $\to 0$; $\to \infty$). Finally, as remarked at the end of §5, the factor of n^{-1} appearing at the beginning of the last line of inequalities of Theorem 7.4, (3), is the *archimedean analogue* of the description of the scheme-theoretic *zero locus of the determinant* given in Chapter VI, Theorem 4.1, (2).

Chapter IX:
The Arithmetic Kodaira-Spencer Morphism

§0. Introduction

In this Chapter, we apply *the Hodge-Arakelov Comparison Theorem* (Chapter VIII, Theorem A) to construct an *arithmetic version of the well-known Kodaira-Spencer morphism* of a family of elliptic curves $E \to S$ (satisfying a mild technical assumption concerning its 2-torsion):

$$\kappa_E^{\mathrm{arith}} : \Pi_S \to \mathfrak{Filt}(\mathcal{H}_{\mathrm{DR}})(S)$$

Roughly speaking, this arithmetic Kodaira-Spencer morphism is a canonical map from the *algebraic fundamental group* of $S_{\mathbf{Q}} \stackrel{\mathrm{def}}{=} S \otimes_{\mathbf{Z}} \mathbf{Q}$ (i.e., if $S = \mathrm{Spec}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of a number field K, then one may think of Π_S as the *absolute Galois group of* K) to a flag variety of *filtrations* of a module which is a certain analogue of the *de Rham cohomology* of the elliptic curve. Moreover, this morphism has certain *remarkable integrality properties (in the Arakelov sense)* at all the primes (both finite and infinite) of a number field. This arithmetic Kodaira-Spencer morphism is constructed in §3 of this Chapter. In §1,2, we first give a construction of the Kodaira-Spencer morphism in the complex and p-adic cases which is entirely analogous to the construction to be given in the arithmetic case in §3, but which shows quite explicitly how this construction is related (in the complex and p-adic cases) to the "classical Kodaira-Spencer morphism" that appears in the theory of moduli of algebraic varieties. Conceptually speaking, the *main point* in all of these constructions consists, as depicted in the following diagram:

<u>Kodaira-Spencer morphism</u>:

motion in base-space $\mapsto$ *induced deformation of Hodge filtration*

of the idea that the Kodaira-Spencer morphism is the map which associates to a "motion" in the base-space of a family of elliptic curves, the deformation in the Hodge filtration of the de Rham cohomology of the elliptic curve induced by the motion. More concretely, the main idea consists of a certain *"recipe" for constructing "Kodaira-Spencer-type morphisms" out of "comparison isomorphisms between de Rham and étale/singular cohomology."* In §3, we carry out this recipe in the case when the comparison isomorphism is the *the Hodge-Arakelov Comparison Isomorphism* (Chapter VIII, Theorem A); in §1,2, we discuss certain novel approaches to the well-known comparison isomorphisms for elliptic curves in the complex and p-adic cases, which, on the one hand, make the connection with the classical Kodaira-Spencer morphism explicit, and, on the other hand, show how the Hodge-Arakelov Comparison Isomorphism is entirely analogous to the well-known complex and p-adic comparison isomorphisms.

§1. The Complex Case: The Classical "Modular Theory" of the Upper Half-Plane

In this §, we review the de Rham isomorphism of a complex elliptic curve, showing how this isomorphism may be regarded as being analogous in a fairly precise sense to the *Comparison Isomorphism* of Chapter VIII, Theorem A. We then discuss the theory of the Kodaira-Spencer morphism of a family of complex elliptic curves in the universal case, but we formulate this theory in a somewhat novel fashion, showing how the Kodaira-Spencer morphism may be derived directly from the de Rham isomorphism in a rather geometric way. This formulation will allow us to make the connection with the global arithmetic theory of §3.

We begin our discussion by considering a single *elliptic curve* E over $\mathbf{C}$. Frequently in the following discussion, we shall also write "E" for the complex manifold defined by the original algebraic curve. Recall (cf. Chapter III, §3) that we have a commutative diagram

$$\begin{array}{ccccccc}
H^1_{\mathrm{DR}}(E, \mathcal{O}_E) = \widetilde{E}^\dagger & \cong & H^1_{\mathrm{sing}}(E, \mathbf{C}) & \supseteq & H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbf{R}) & \supseteq & H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbf{Z}) \\
\downarrow \exp & & \downarrow \exp & & \downarrow \exp & & \downarrow \exp \\
H^1_{\mathrm{DR}}(E, \mathcal{O}^\times_E) = E^\dagger & \cong & H^1_{\mathrm{sing}}(E, \mathbf{C}^\times) & \supseteq & H^1_{\mathrm{sing}}(E, \mathbf{S^1}) = E_{\mathbf{R}} & \supseteq & \text{identity elt.}
\end{array}$$

Here, the horizontal isomorphisms are the *de Rham isomorphisms* relating de Rham cohomology to singular cohomology. Note that in characteristic zero, line bundles with connection are necessarily of degree zero, so $E^\dagger$ may be naturally identified with $H^1_{\mathrm{DR}}(E, \mathcal{O}^\times_E)$, the group of line bundles equipped with a connection on E. (Similarly, the (topological) universal cover $\widetilde{E}^\dagger$ of $E^\dagger$ may be identified with $H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$.) The vertical maps are the morphisms induced on cohomology by the exponential map; $\mathbf{S^1} \subseteq \mathbf{C}^\times$ is the unit circle (equipped with its usual group structure). Finally, $E_{\mathbf{R}} \subseteq E^\dagger$ is the real analytic submanifold (discussed in Chapter III, §3) which is equal to the closure of the torsion points of $E^\dagger$ and maps bijectively onto E via the natural projection $E^\dagger \to E$.

Here, we would like to consider the issue of *precisely how the de Rham isomorphisms of the above diagram are defined.* Of course, there are many possible definitions for these isomorphisms, but the point that we would like to make here is the following:

> *If one thinks of* $H^1_{\mathrm{sing}}(E, \mathbf{S^1}) = E_{\mathbf{R}}$ *(respectively,* $T_v(E) \stackrel{\mathrm{def}}{=} H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbf{R})$*) as the "$v$-divisible group of torsion points of E" (respectively, the*

"v-adic Tate module") — where $v = \infty$ is the archimedean prime of $\mathbf{Q}$ *— then, roughly speaking, one may think of the de Rham isomorphisms as being given by the diagram*

$$\begin{array}{ccc}
\text{Hol. fns. on } H^1_{\text{DR}}(E, \mathcal{O}_E) = \widetilde{E}^{\dagger} & \text{“}\cong\text{”} & \text{Real an. fns. on } H^1_{\text{sing}}(E, 2\pi i \cdot \mathbf{R}) \\
\bigcup & & \bigcup \\
\text{Hol. fns. on } H^1_{\text{DR}}(E, \mathcal{O}^{\times}_E) = E^{\dagger} & \text{“}\cong\text{”} & \text{Real an. fns. on } H^1_{\text{sing}}(E, \mathbf{S^1}) = E_{\mathbf{R}}
\end{array}$$

where the horizontal isomorphisms "$\cong$" are given by restricting holomorphic functions on $E^{\dagger}$, $\widetilde{E}^{\dagger}$ to real analytic functions on the "∞-adic torsion points/Tate module" $E_{\mathbf{R}}$, $T_{\infty}(E) \stackrel{\text{def}}{=} H^1_{\text{sing}}(E, 2\pi i \cdot \mathbf{R})$.

Here, we say "roughly speaking" (and write "$\cong$") for the following reason: Although this restriction morphism is *injective*, the correspondence between holomorphic functions on the de Rham objects $E^{\dagger}$, $\widetilde{E}^{\dagger}$ and real analytic functions on the v-adic torsion point objects $E_{\mathbf{R}}$, $T_{\infty}(E)$ is, strictly speaking, only true on an "infinitesimal neighborhood" of $E_{\mathbf{R}} \subseteq E^{\dagger}$, $T_{\infty}(E) \subseteq \widetilde{E}^{\dagger}$. (That is to say, although a real analytic function on $E_{\mathbf{R}}$ always corresponds to a holomorphic function on some open neighborhood of $E_{\mathbf{R}}$ in $E^{\dagger}$, whether or not this holomorphic function extends to a holomorphic function defined over all of $E^{\dagger}$ involves subtle convergence issues and, in fact, is not always the case.) Thus, here, in order to get a precise statement, we shall work with *polynomial functions* on $\widetilde{E}^{\dagger}$ and $T_{\infty}(E)$. Then one sees immediately that *the de Rham isomorphism of the first commutative diagram of this § may be formulated as the isomorphism*

$$\text{Holom}^{\text{Poly}}(\widetilde{E}^{\dagger}) \quad \cong \quad \text{Real An}^{\text{Poly}}(T_{\infty}(E))$$

given by restricting holomorphic polynomials on $\widetilde{E}^{\dagger}$ to the ∞-adic torsion points so as to obtain real analytic polynomials on $T_{\infty}(E)$. When formulated from this point of view, one sees that *the Comparison Isomorphism of Chapter VIII, Theorem A, is analogous in a very direct sense to the classical de Rham isomorphism in the complex case*, i.e.:

Both may be thought of as being bijections between algebraic/ holomorphic functions on de Rham-type objects and arbitrary/ real analytic functions on torsion points – bijections given by restricting algebraic/ holomorphic functions on de Rham-type objects to the torsion points lying inside those de Rham-type objects.

This observation may be thought of as the philosophical starting point of the theory of this paper.

Remark. Note that the collection of "holomorphic functions on $H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$" includes, in particular, the *theta functions* (cf., e.g., [Mumf], Chapter I, §3) associated to the elliptic curve E. Moreover, these functions are "fairly representative of" (roughly speaking, "generate") the set of *all* holomorphic functions on $H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$ that arise by pull-back via the projection $H^1_{\mathrm{DR}}(E, \mathcal{O}_E) \to H^1(E, \mathcal{O}_E)$ (defined by the *Hodge filtration*) from holomorphic functions on $H^1(E, \mathcal{O}_E)$. *This observation played a fundamental motivating role in the development of the theory of the present paper.*

Next, we shift gears and discuss various versions of the *Kodaira-Spencer morphism for the universal family of complex elliptic curves.* First, let us write $\mathfrak{H} \stackrel{\mathrm{def}}{=} \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$ for the upper half-plane, and

$$E_{\mathfrak{H}} \to \mathfrak{H}$$

for the *universal family of complex elliptic curves* over $\mathfrak{H}$. That is to say, over a point $z \in \mathfrak{H}$, the fiber E_z of this family is given by $E_z = \mathbf{C}/ < 1, z >$ (where $< 1, z >$ denotes the $\mathbf{Z}$-submodule generated by $1, z$).

Let us fix a "base-point" $z_0 \in \mathfrak{H}$. Write

$$H^1_{\mathrm{DR}}(E_{z_0}) \stackrel{\mathrm{def}}{=} H^1_{\mathrm{DR}}(E_{z_0}, \mathcal{O}_{E_{z_0}})$$

Thus, $H^1_{\mathrm{DR}}(E_{z_0})$ is a two-dimensional complex vector space. Recall that, in fact, the correspondence $z \mapsto H^1_{\mathrm{DR}}(E_z)$ defines a rank two vector bundle $\mathcal{E}$ on $\mathfrak{H}$ equipped with a natural (integrable) connection (the "Gauss-Manin connection"). Since the underlying topological space of $\mathfrak{H}$ is *contractible*, parallel transport via this connection thus gives rise to a natural trivialization of this rank two vector bundle $\mathcal{E}$, i.e., a natural isomorphism

$$\mathcal{E} \cong \mathfrak{H} \times H^1_{\mathrm{DR}}(E_{z_0})$$

Recall that the *Hodge filtration* of de Rham cohomology defines a subbundle $F^1(\mathcal{E}) \subseteq \mathcal{E}$ of rank one. This subbundle induces a natural holomorphic morphism

$$\kappa_{\mathfrak{H}} : \mathfrak{H} \to P \stackrel{\mathrm{def}}{=} \mathbf{P}^1(H^1_{\mathrm{DR}}(E_{z_0}))$$

that maps a point $z \in \mathfrak{H}$ to the subspace of $H^1_{\mathrm{DR}}(E_{z_0}) = H^1_{\mathrm{DR}}(E_z)$ defined by $F^1(\mathcal{E}) \subseteq \mathcal{E}$ at z.

Now let us recall that we have a *natural action* of $SL_2(\mathbf{R})$ on $\mathfrak{H}$ given by linear fractional transformations. This action allows us to define a morphism

$$\kappa_{SL_2} : SL_2(\mathbf{R}) \to P$$

by letting $\kappa_{SL_2}(\gamma) \stackrel{\text{def}}{=} \kappa_{\mathfrak{H}}(\gamma \cdot z_0)$ (for $\gamma \in SL_2(\mathbf{R})$). If we then differentiate κ_{SL_2} at z_0, we obtain a *morphism on tangent spaces* that fits into a commutative diagram:

$$\begin{array}{ccc} sl_2(\mathbf{R}) & \stackrel{\kappa_{sl_2}}{\longrightarrow} & \tau_{P,p_0} \\ \downarrow & & \uparrow \kappa_\tau \\ sl_2(\mathbf{R})/so_2 & \cong & \tau_{\mathfrak{H},z_0} \end{array}$$

Here, *the vertical morphism on top is the derivative of κ_{SL_2} at the origin of $SL_2(\mathbf{R})$*; τ_{P,p_0} (respectively, $\tau_{\mathfrak{H},z_0}$) is the tangent space to P (respectively, $\mathfrak{H}$) at $p_0 \stackrel{\text{def}}{=} \kappa_{\mathfrak{H}}(z_0)$ (respectively, z_0). This vertical morphism clearly factors through the quotient $sl_2(\mathbf{R}) \to sl_2(\mathbf{R})/so_2$, where $sl_2(\mathbf{R})$ (respectively, so_2) is the Lie algebra associated to $SL_2(\mathbf{R})$ (respectively, the subgroup of $SL_2(\mathbf{R})$ that fixes z_0). Moreover, all tangent vectors to $z_0 \in \mathfrak{H}$ are obtained by acting by various elements of $sl_2(\mathbf{R})$ on $\mathfrak{H}$ at z_0; thus, one may identify $sl_2(\mathbf{R})/so_2$ with $\tau_{\mathfrak{H},z_0}$ (the vertical isomorphism on the bottom).

Definition 1.1. We shall refer to κ_{SL_2} (respectively, κ_{sl_2}; κ_τ) as the *group-theoretic* (respectively, *Lie-theoretic*; *classical*) *Kodaira-Spencer morphism (of the family $E_{\mathfrak{H}} \to \mathfrak{H}$ at z_0).*

Thus, the "classical Kodaira-Spencer morphism" κ_τ is obtained (cf. the above commutative diagram) simply by using the fact that κ_{sl_2} factors through the quotient $sl_2(\mathbf{R}) \to sl_2(\mathbf{R})/so_2$. One checks easily that this morphism is indeed the usual Kodaira-Spencer morphism associated to the family $E_{\mathfrak{H}} \to \mathfrak{H}$. In particular, κ_τ is an isomorphism.

The reason that we feel that it is natural also to regard κ_{SL_2} and κ_{sl_2} as "Kodaira-Spencer morphisms" is the following: The essence of the notion of a "Kodaira-Spencer morphism" is that of a *correspondence* that associates to a *motion* in the base-space the *induced deformation of the Hodge filtration of the de Rham cohomology*, i.e., symbolically,

<u>Kodaira-Spencer morphism</u>:

motion in base-space $\mapsto$ *induced deformation of Hodge filtration*

In the case of the "group-theoretic Kodaira-Spencer morphism" (respectively, "Lie-theoretic Kodaira-Spencer morphism"; "classical Kodaira-Spencer morphism"), this motion is a motion given by the "Lie group $SL_2(\mathbf{R})$ of motions of $\mathfrak{H}$" (respectively, the Lie algebra associated to this Lie group of motions; a tangent vector in $\mathfrak{H}$). That is to say, all three types of Kodaira-Spencer morphism discussed here fit into the general pattern just described.

It turns out that the *group-theoretic Kodaira-Spencer morphism is the version which is most suited to generalization to the arithmetic case* (cf. the discussions of §2,3 below).

Finally, *we make the connection between the theory of the Kodaira-Spencer morphism just discussed and the function-theoretic approach to the de Rham isomorphism discussed* *at the beginning of this* §. First of all, let us observe that the action of $SL_2(\mathbf{R})$ on $\mathfrak{H}$ lifts naturally to an action on $\mathcal{E}$. Moreover, if one thinks of $SL_2(\mathbf{R})$ as the group of unimodular (i.e., with determinant $= 1$) $\mathbf{R}$-linear automorphisms of the two-dimensional $\mathbf{R}$-vector space $T_\infty(E_{z_0})$, that is, if one makes the identification

$$SL_2(\mathbf{R}) = SL(T_\infty(E_{z_0}))$$

then the action of $SL_2(\mathbf{R})$ on $\mathcal{E} \cong \mathfrak{H} \times H^1_{\mathrm{DR}}(E_{z_0})$ corresponds to the natural action of $SL(T_\infty(E_{z_0}))$ on $H^1_{\mathrm{DR}}(E_{z_0}) \cong T_\infty(E_{z_0}) \otimes_{\mathbf{R}} \mathbf{C}$ (where the isomorphism here is the de Rham isomorphism). It thus follows that *the group-theoretic Kodaira-Spencer morphism κ_{SL_2} may also be defined as the morphism*

$$SL_2(\mathbf{R}) = SL(T_\infty(E_{z_0})) \to P = \mathbf{P}(H^1_{\mathrm{DR}}(E_{z_0}))$$

given by $\gamma \mapsto \gamma \cdot p_0$, where the expression "$\gamma \cdot p_0$" is relative to the natural action of $SL(T_\infty(E_{z_0}))$ on $P = \mathbf{P}(H^1_{\mathrm{DR}}(E_{z_0})) \cong \mathbf{P}(T_\infty(E_{z_0}) \otimes_{\mathbf{R}} \mathbf{C})$ (where the isomorphism here is that derived from the de Rham isomorphism). This approach to defining κ_{SL_2} shows that:

> *The group-theoretic Kodaira-Spencer morphism κ_{SL_2} may essentially be defined directly from the de Rham isomorphism.*

This observation brings us one step closer to the discussion of the arithmetic case in §3. In particular, in light of the above "function-theoretic approach to the de Rham isomorphism," it motivates the following *point of view*:

Note that (in the notation of the discussion at the beginning of this §) the space $\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger)$ of holomorphic polynomials on $\widetilde{E}^\dagger$ has a *Hodge filtration*

$$\ldots F^d(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger)) \subseteq \ldots \subseteq \mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger)$$

given by letting $F^d(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger)) \subseteq \mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger)$ denote the subspace of polynomials whose "torsorial degree" (cf. Chapter III, Definition 2.2), i.e., degree as a polynomial in the relative variable of the torsor $\widetilde{E}^\dagger \to \widetilde{E}$ (where $\widetilde{E}$ is the universal covering space of E), is $< d$. Note that relative to the "function-theoretic de Rham isomorphism" $\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger) \cong \text{Real An}^{\mathrm{Poly}}(T_\infty(E))$, $F^d(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger))$ corresponds to the subspace of $\text{Real An}^{\mathrm{Poly}}(T_\infty(E))$ annihilated by $\overline{\partial}^d$. (Here, $\overline{\partial}$ is the usual "del-bar" operator of complex analysis on $T_\infty(E) = \widetilde{E}_{\mathbf{R}}$, relative to the complex structure on $\widetilde{E}_{\mathbf{R}}$ defined by $\widetilde{E}$.) Let us write

$$\mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^\dagger))$$

for the (infinite-dimensional) flag-manifold of $\mathbf{C}$-linear filtrations $\{F^d\}_{d \in \mathbf{Z}_{\geq 0}}$ of $\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^{\dagger})$ such that $F^0 = 0$. Then the "Hodge filtration" just defined determines a point

$$p_E^{\mathrm{func}} \in \mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^{\dagger}))$$

Similarly, any one-dimensional complex quotient $\widetilde{E}^{\dagger} \to Q$ defines a filtration of $\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^{\dagger})$ (given by looking at the degree with respect to the variable corresponding to the kernel of $\widetilde{E}^{\dagger} \to Q$). In particular, we get an immersion

$$\mathbf{P}(\widetilde{E}^{\dagger}) \hookrightarrow \mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^{\dagger}))$$

Thus, returning to the discussion of the *group-theoretic Kodaira-Spencer morphism*, we see that we may think of the composite

$$\kappa_{SL_2}^{\mathrm{func}} : SL_2(\mathbf{R}) = SL(T_{\infty}(E_{z_0})) \to \mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^{\dagger}))$$

of κ_{SL_2} with the inclusion $\mathbf{P}(\widetilde{E}^{\dagger}) \hookrightarrow \mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^{\dagger}))$ as being defined as follows:

> *The natural action of* $SL(T_{\infty}(E_{z_0}))$ *on* $\mathrm{Real\ An}^{\mathrm{Poly}}(T_{\infty}(E_{z_0}))$ *induces, via the "function-theoretic de Rham isomorphism"*
>
> $$\mathrm{Real\ An}^{\mathrm{Poly}}(T_{\infty}(E_{z_0})) \cong \mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^{\dagger}_{z_0})$$
>
> *an action of* $SL(T_{\infty}(E_{z_0}))$ *on* $\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^{\dagger}_{z_0})$*; then the* "function-theoretic version of the group-theoretic Kodaira-Spencer morphism"
>
> $$\kappa_{SL_2}^{\mathrm{func}} : SL(T_{\infty}(E_{z_0})) \to \mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^{\dagger}_{z_0}))$$
>
> *is defined by* $\gamma \mapsto \gamma \cdot p_{E_{z_0}}^{\mathrm{func}}$*, where* $p_{E_{z_0}}^{\mathrm{func}} \in \mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^{\dagger}_{z_0}))$ *is the natural point defined by the Hodge filtration on* $\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^{\dagger}_{z_0})$.

It is this point of view that forms the basis of our approach to the arithmetic case in §3.

Remark. The theory discussed in §1,2, of this Chapter generalizes immediately to the case of *higher-dimensional abelian varieties*. Since, however, the Hodge-Arakelov Comparison Theorem (Chapter VIII, Theorem A) is only available (at the time of writing) for elliptic curves, we restrict ourselves both in the present and the following §'s to the case of elliptic curves.

§2. The p-adic Case: The Hodge-Tate Decomposition as an Evaluation Map

The purpose of this § is to exhibit the morphism which defines the Hodge-Tate decomposition of the p-adic Tate module of an elliptic curve as an *evaluation map*, given by *restricting certain functions on the universal extension of the elliptic curve to the p-power torsion points of the elliptic curve.* This renders explicit the analogy between the Hodge-Arakelov Comparison Isomorphism of Chapter VIII, Theorem A, and the p-adic Hodge theory of an elliptic curve. We then explain how, if one applies *the general recipe* – which is the theme of the present Chapter – *for obtaining Kodaira-Spencer-type morphisms from evaluation map-based comparison isomorphisms* (cf. §1 in the complex case and in §3 in the global Arakelov case) to the present p-adic situation, one obtains a sort of *"p-adic arithmetic Kodaira-Spencer morphism,"* which, by Faltings' computation of certain Galois cohomology groups, may be identified with the classical Kodaira-Spencer morphism of a family of elliptic curves.

Remark. The material of the present § is in principle "well-known," and, in fact, inspired by the techniques and points of view of [Falt1,2], [Font], but I do not know of an adequate explicit reference for this material. Roughly speaking, the idea here is what is usually referred to as the *p-adic period map for abelian varieties* – cf. [Coln], [Colz], [Wint]. Indeed, our point of view here essentially coincides with what is done in [Coln], [Colz], [Wint], and [Font], *when restricted to the subspace of the de Rham cohomology of an elliptic curve arising from differentials of the first kind* (i.e., globally holomorphic differentials). Our treatment of the "other half" of the de Rham cohomology, however, appears to differ from these other sources. Indeed, Coleman ([Coln]) and Fontaine ([Font]) do not construct a morphism on the entire de Rham cohomology module. On the other hand, Colmez ([Colz], Théorème 5.2) constructs such a morphism by thinking of the remainder of the de Rham cohomology as being defined by differentials of the second kind (i.e., meromorphic differentials which are "locally integrable"), which he integrates to construct his period map; in particular, Colmez does not use the universal extension as we do here. Finally, Wintenberger ([Wint], §4) uses the universal extension, but in a somewhat different fashion from what is done here.

Let k be a *perfect field of characteristic* $p > 2$. (In fact, the case $p = 2$ may be handled without much more difficulty, but for simplicity, we assume here that $p > 2$.) Write $A \stackrel{\mathrm{def}}{=} W(k)$; $K \stackrel{\mathrm{def}}{=} A \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Let S be a *formal p-adic A-scheme*, which is formally smooth and formally of finite type over A. Suppose that S is equipped with an A-flat divisor with normal crossings $D \subseteq S$. Write $S^{\log}$ for the resulting log scheme. Suppose that

$$C^{\log} \to S^{\log}$$

is a *log elliptic curve* over $S^{\log}$ (cf. Chapter III, Definition 1.1).

Next, let us suppose that S is *affine*, i.e., of the form

$$\mathrm{Spf}(R)$$

where R is a p-adic ring, and *small* (cf. [Falt2], II., (a.)), i.e., R is étale over the p-adic completion $A[t_1, \ldots, t_r]^\wedge$ of $A[t_1, \ldots, t_r]$, where $t_1, \ldots, t_r$ are indeterminates such that the schematic zero locus of $t \stackrel{\mathrm{def}}{=} t_1 \cdot \ldots \cdot t_r$ in S is equal to D. Write

$$\widehat{\overline{R}}$$

for the p-adic completion of the normalization $\overline{R}$ of R in the maximal étale extension of $R_{\mathbf{Q}_p}[t^{-1}]$ (where $R_{\mathbf{Q}_p} \stackrel{\mathrm{def}}{=} R \otimes \mathbf{Q}_p$). (Here, "maximal" means "among those extensions whose Spf is connected.") Also, let us write

$$\Gamma_R \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{R}_{\mathbf{Q}_p}/R_{\mathbf{Q}_p})$$

Then as in [Falt2], II., (b.), we may form the *ring of p-adic periods*

$$B^+(R)$$

This ring admits a natural Γ_R-action, together with a Γ_R-equivariant projection $B^+(R) \to \widehat{\overline{R}}$ whose kernel I is a "*principal divided power ideal.*" In the discussion of the present §, we will work with the following *truncated version* of $B^+(R)$:

$$B \stackrel{\mathrm{def}}{=} B^+(R)/I^{[2]}$$

(where $I^{[2]}$ is the second divided power of I). Write $J \stackrel{\mathrm{def}}{=} I/I^{[2]} \subseteq B$ for the image of I in B. Thus, we have an exact sequence of Γ_R-modules:

$$0 \to J \to B \to \widehat{\overline{R}} \to 0$$

Moreover, J is a *free* $\widehat{\overline{R}}$*-module of rank* 1. If one further takes the Γ_R-action on J into account, then one has a natural identification (cf. [Falt2], II., (b.))

$$\mathbf{Z}_p(1) \cdot \widehat{\overline{R}} = p^{\frac{1}{(p-1)}} \cdot J$$

(where the "(1)" is a Tate twist). Put another way, there is a Γ_R-equivariant homomorphism

$$\beta_{\mathbf{Q}_p} : \mathbf{Q}_p(1) \to B^\times$$

whose projection to $\widehat{\widehat{R}}^{\times}$ is the natural inclusion $\mathbf{Q}_p(1)/\mathbf{Z}_p(1) \hookrightarrow \widehat{\widehat{R}}^{\times}$ and whose restriction to $\mathbf{Z}_p(1)$ is equal to 1 plus the homomorphism $\beta : \mathbf{Z}_p(1) \to J$.

Note that since $\widehat{\widehat{R}}$ contains all roots of the q-parameter at infinity, we may form

$$E_{\infty} \to \mathrm{Spf}(\widehat{\widehat{R}}); \quad C_{\infty} \to \mathrm{Spf}(\widehat{\widehat{R}})$$

as in Chapter IV, §4. Note that we have a projection $C_{\infty} \to C$, and an open immersion $E_{\infty} \hookrightarrow C_{\infty}$. Thus, in particular, by pulling back via this projection and this open immersion the extension to C (cf. Chapter III, Corollary 4.3)

$$E_C^{\dagger} \to C$$

of the universal extension $E^{\dagger} \to E$ of E, we obtain

$$E_{\infty}^{\dagger} \to \mathrm{Spf}(\widehat{\widehat{R}}); \quad E_{C_{\infty}}^{\dagger} \to \mathrm{Spf}(\widehat{\widehat{R}})$$

Next, let us recall that the universal extension $E_{\infty}^{\dagger}$ defines a *crystal* $E_{\infty,\mathrm{crys}}^{\dagger}$ (cf., e.g., [Mess], for the non-logarithmic case) over $\mathrm{Spf}(\widehat{\widehat{R}})^{\log}$ (where "log" refers to the natural log structure defined by roots of the q-parameter at infinity). Since we may regard $\mathrm{Spf}(B)$ as a *PD-thickening* of $\mathrm{Spf}(\widehat{\widehat{R}})^{\log}$, we thus see that it makes sense to speak of $E_{\infty,\mathrm{crys}}^{\dagger}(B)$, or even

$$E_{\infty,\mathrm{crys}}^{\dagger}(B_{\mathbf{Q}_p})$$

where $B_{\mathbf{Q}_p} \stackrel{\mathrm{def}}{=} B \otimes \mathbf{Q}_p$. In particular, we see that, for $n \geq 0$,

$${}_{p^n}E_{\infty,\mathrm{crys}}^{\dagger}(B_{\mathbf{Q}_p}) \subseteq E_{\infty,\mathrm{crys}}^{\dagger}(B_{\mathbf{Q}_p})$$

forms a group (noncanonically isomorphic to) $(\mathbf{Z}/p^n \cdot \mathbf{Z})^2$. In fact, the sections of the crystal $E_{\infty,\mathrm{crys}}^{\dagger}$ defined by ${}_{p^n}E_{\infty,\mathrm{crys}}^{\dagger}(B_{\mathbf{Q}_p})$ are all *horizontal*, hence preserved by base-changes of $E_{\infty,\mathrm{crys}}^{\dagger}$ among various thickenings of $B_{\mathbf{Q}_p}$. Observe also that as $n \to \infty$, the ${}_{p^n}E_{\infty,\mathrm{crys}}^{\dagger}(B_{\mathbf{Q}_p})$ form a natural inverse system (all of whose transition morphisms are surjective); the inverse limit of this inverse system is the *Tate module*

$$T_E$$

of E. Note that as a topological group, $T_E \cong {\mathbf{Z}_p}^2$. Moreover, T_E has a natural structure of continuous Γ_R-module, and $T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}$ may be naturally identified with ${}_{p^n}E_{\infty,\mathrm{crys}}^{\dagger}(B_{\mathbf{Q}_p})$.

Next, we would like to consider *functions*. Let us write

$$\mathcal{R}_{E^{\dagger}_{\infty}}$$

for the push-forward to E_{∞} of the structure sheaf $\mathcal{O}_{E^{\dagger}_{\infty}}$ on $E^{\dagger}_{\infty}$. Recall that this structure sheaf has a *filtration* $F^j(\mathcal{R}_{E^{\dagger}_{\infty}}) \subseteq \mathcal{R}_{E^{\dagger}_{\infty}}$ (for $j \in \mathbf{Z}$), where we take $F^j(\mathcal{R}_{E^{\dagger}_{\infty}})$ to be the subsheaf of sections of torsorial degree (cf. Chapter III, Definition 2.2) $\leq -j$. Note that the filtration index that we use here differs from the filtration index of Chapters III-VIII in that "the index used here = 1− the index used there." In particular, $F^j(\mathcal{R}_{E^{\dagger}_{\infty}})$ as defined here is a vector bundle on E_{∞} of rank $1-j$, when $j \leq 0$. Thus, we have an exact sequence:

$$0 \longrightarrow \mathcal{O}_{E_{\infty}} \longrightarrow F^{-1}(\mathcal{R}_{E^{\dagger}_{\infty \widehat{R}}}) \longrightarrow \tau_E \otimes_{\mathcal{O}_S} \mathcal{O}_{E^{\dagger}_{\infty}} \longrightarrow 0$$

In particular, if we apply this filtration to the *crystal* $E^{\dagger}_{\infty,\mathrm{crys}}$ evaluated on the thickening B, where we regard B itself as equipped with the filtration given by:

$$F^{\geq 2}(B) = 0; \quad F^1(B) = J; \quad F^{\leq 0}(B) = B$$

then we see that the sheaf $\mathcal{R}_{E^{\dagger}_{\infty,\mathrm{crys}}(B)}$ of functions on $E^{\dagger}_{\infty,\mathrm{crys}}(B)$ (pushed forward topologically to E_{∞}) gets a filtration whose $F^0(-)$ we denote by

$$\mathcal{R}^0$$

If one restricts functions on $E^{\dagger}_{\infty,\mathrm{crys}}(B)$ to $E^{\dagger}_{\infty,\mathrm{crys}}(\widehat{\overline{R}}) = E^{\dagger}_{\infty}(\widehat{\overline{R}})$, then one obtains a surjection $\mathcal{R}^0 \to \mathcal{O}_{E_{\infty}}$ which fits into an exact sequence

$$0 \longrightarrow J \otimes_{\widehat{\overline{R}}} F^{-1}(\mathcal{R}_{E^{\dagger}_{\infty}}) \longrightarrow \mathcal{R}^0 \longrightarrow \mathcal{O}_{E_{\infty}} \longrightarrow 0$$

Let us write

$$\mathcal{R}^J \stackrel{\mathrm{def}}{=} \mathcal{R}^0 / J \cdot \mathcal{O}_{E_{\infty}}$$

Thus, we have an exact sequence

$$0 \longrightarrow (J \otimes_R \tau_E) \otimes_{\widehat{\overline{R}}} \mathcal{O}_{E_{\infty}} \longrightarrow \mathcal{R}^J \longrightarrow \mathcal{O}_{E_{\infty}} \longrightarrow 0$$

The *evaluation map* that we will consider in this § (which gives rise to the comparison isomorphism of Hodge-Tate theory) will be defined by *restricting functions in $\mathcal{R}^0$ to the p-power torsion points* ${}_{p^n}E^{\dagger}_{\infty,\mathrm{crys}}(B_{\mathbf{Q}_p})$ (and then noting that this restriction map factors through $\mathcal{R}^J$).

More precisely, since sections of $\mathcal{R}^0$ are functions on $E^{\dagger}_{\infty,\mathrm{crys}}(B)$, hence define functions on $E^{\dagger}_{\infty,\mathrm{crys}}(B_{\mathbf{Q}_p})$, we may restrict them to the points of ${}_{p^n}E^{\dagger}_{\infty,\mathrm{crys}}(B_{\mathbf{Q}_p})$ to obtain a morphism

$$\Xi_{\mathbf{Q}_p}[p^n] : \mathcal{R}^0 \to \mathbf{Func}({}_{p^n}E^{\dagger}_{\infty,\mathrm{crys}}(B_{\mathbf{Q}_p}), B_{\mathbf{Q}_p}) \stackrel{\mathrm{def}}{=} \bigoplus_{\gamma \in {}_{p^n}E^{\dagger}_{\infty,\mathrm{crys}}(B_{\mathbf{Q}_p})} B_{\mathbf{Q}_p}$$

Now we have the following:

Lemma 2.1. *The morphism $\Xi_{\mathbf{Q}_p}[p^n]$ reduced modulo $J_{\mathbf{Q}_p} \subseteq B_{\mathbf{Q}_p}$ maps into $\widehat{\widehat{R}} \subseteq \widehat{\widehat{R}}_{\mathbf{Q}_p}$.*

Proof. Indeed, this follows from the definition of $\mathcal{R}^0$, together with the fact that the image of ${}_{p^n}E^{\dagger}_{\infty,\mathrm{crys}}(\widehat{\widehat{R}}_{\mathbf{Q}_p})$ in $E_\infty(\widehat{\widehat{R}}_{\mathbf{Q}_p})$ maps into $E_\infty(\widehat{\widehat{R}}) \subseteq E_\infty(\widehat{\widehat{R}}_{\mathbf{Q}_p})$ (i.e., all the p-torsion points of E_∞ over $\widehat{\widehat{R}}_{\mathbf{Q}_p}$ are already defined over $\widehat{\widehat{R}}$). ○

Thus, it follows that $\Xi_{\mathbf{Q}_p}[p^n]$ maps $J \cdot \mathcal{O}_{(E_\infty)_{\widehat{\widehat{R}}}} \subseteq \mathcal{R}^0$ into $\mathbf{Func}({}_{p^n}E^{\dagger}_{\infty,\mathrm{crys}}(B_{\mathbf{Q}_p}), J) \subseteq \mathbf{Func}({}_{p^n}E^{\dagger}_{\infty,\mathrm{crys}}(B_{\mathbf{Q}_p}), B_{\mathbf{Q}_p})$. In particular, if we compose $\Xi_{\mathbf{Q}_p}[p^n]$ with the projection

$$\mathbf{Func}({}_{p^n}E^{\dagger}_{\infty,\mathrm{crys}}(B_{\mathbf{Q}_p}), B_{\mathbf{Q}_p}) \to \mathbf{Func}({}_{p^n}E^{\dagger}_{\infty,\mathrm{crys}}(B_{\mathbf{Q}_p}), B_{\mathbf{Q}_p/\mathbf{Z}_p})$$

(where $B_{\mathbf{Q}_p/\mathbf{Z}_p} \stackrel{\mathrm{def}}{=} B \otimes \mathbf{Q}_p/\mathbf{Z}_p$) we get a morphism $\mathcal{R}^0 \to \mathbf{Func}({}_{p^n}E^{\dagger}_{\infty,\mathrm{crys}}(B_{\mathbf{Q}_p}), B_{\mathbf{Q}_p/\mathbf{Z}_p})$ which vanishes on the kernel of $\mathcal{R}^0 \to \mathcal{R}^J$ and maps into $\mathbf{Func}({}_{p^n}E^{\dagger}_{\infty,\mathrm{crys}}(B_{\mathbf{Q}_p}), J_{\mathbf{Q}_p/\mathbf{Z}_p})$, i.e., a morphism

$$\Xi_{\mathcal{R}^J}[p^n] : \mathcal{R}^J \to \mathbf{Func}(T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}, J_{\mathbf{Q}_p/\mathbf{Z}_p})$$

(since $T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z} = {}_{p^n}E^{\dagger}_{\infty,\mathrm{crys}}(B_{\mathbf{Q}_p})$).

Let us analyze this morphism $\Xi_{\mathcal{R}^J}[p^n]$ in more detail. First, let us observe from the definitions that $\mathcal{R}^0$ has a natural *ring structure* (inherited from the ring structure of $\mathcal{O}_{E^{\dagger}_{\infty,\mathrm{crys}}}$). Moreover, it follows immediately from the definitions that the evaluation map $\Xi_{\mathbf{Q}_p}[p^n]$ is a *ring homomorphism.* Note also that $\mathbf{Func}(T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}, J_{\mathbf{Q}_p/\mathbf{Z}_p})$ has a natural $\mathcal{R}^0$-module structure defined by $\Xi_{\mathbf{Q}_p}[p^n]$.

Lemma 2.2. *The restriction $\mathcal{R}^0 \to \mathbf{Func}(T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}, J_{\mathbf{Q}_p/\mathbf{Z}_p})$ of $\Xi_{\mathcal{R}^J}[p^n]$ to $\mathcal{R}^0$ is a* derivation*.*

Proof. Suppose that f, g are local sections of $\mathcal{R}^0$. Write:

$$(\Xi_{\mathbf{Q}_p}[p^n])(f) = \alpha_0 + \alpha_J; \quad (\Xi_{\mathbf{Q}_p}[p^n])(g) = \beta_0 + \beta_J$$

where $\alpha_0, \beta_0 \in B$, and $\alpha_J, \beta_J \in J_{\mathbf{Q}_p}$. (Note that the fact that such $\alpha_0, \beta_0, \alpha_J, \beta_J$ exist follows from Lemma 2.1.) Then since $\Xi_{\mathbf{Q}_p}[p^n]$ is a ring homomorphism, we obtain that:

$$\begin{aligned}(\Xi_{\mathbf{Q}_p}[p^n])(fg) &= (\Xi_{\mathbf{Q}_p}[p^n])(f) \cdot (\Xi_{\mathbf{Q}_p}[p^n])(g) \\ &= (\alpha_0 + \alpha_J) \cdot (\beta_0 + \beta_J) \\ &= \alpha_0 \cdot \beta_J + \beta_0 \cdot \alpha_J + \alpha_0 \cdot \beta_0\end{aligned}$$

But $\alpha_0 \cdot \beta_0 \in B$, so modulo B, we obtain:

$$\begin{aligned}(\Xi_{\mathbf{Q}_p}[p^n])(fg) &\equiv \alpha_0 \cdot \beta_J + \beta_0 \cdot \alpha_J \\ &\equiv f \cdot (\Xi_{\mathbf{Q}_p}[p^n])(g) + g \cdot (\Xi_{\mathbf{Q}_p}[p^n])(f)\end{aligned}$$

This completes the proof. ○

Next, let us observe that the crystal $E^{\dagger}_{\infty,\mathrm{crys}}$ gives rise to a *crystal* $\Omega_{E^{\dagger}_{\infty,\mathrm{crys}}}$ (the sheaf of differentials of $E^{\dagger}_{\infty,\mathrm{crys}}$ over the base, i.e., thickening of $\widehat{\overline{R}}$, in question) of modules over $\mathcal{R}_{E^{\dagger}_{\infty,\mathrm{crys}}}$ (where $\mathcal{R}_{E^{\dagger}_{\infty,\mathrm{crys}}}$ is the topological push-forward to E_∞ of the structure sheaf of $E^{\dagger}_{\infty,\mathrm{crys}}$). Let us denote by

$$\Omega^{\mathrm{inv}}_{E^{\dagger}_{\infty,\mathrm{crys}}} \subseteq \Omega_{E^{\dagger}_{\infty,\mathrm{crys}}}$$

the subsheaf of *invariant differentials* (cf. the discussion at the beginning of the Appendix). Thus, $\Omega^{\mathrm{inv}}_{E^{\dagger}_{\infty,\mathrm{crys}}}$ forms a crystal in (rank two) vector bundles over $\mathrm{Spf}(\widehat{\overline{R}})$, and we have an isomorphism of crystals:

$$\Omega^{\mathrm{inv}}_{E^{\dagger}_{\infty,\mathrm{crys}}} \otimes \mathcal{R}_{E^{\dagger}_{\infty,\mathrm{crys}}} \cong \Omega_{E^{\dagger}_{\infty,\mathrm{crys}}}$$

(where the tensor product is over the base, i.e., thickening of $\widehat{\overline{R}}$, in question). Note that $\Omega^{\mathrm{inv}}_{E^{\dagger}_{\infty}}$ (i.e., $\Omega^{\mathrm{inv}}_{E^{\dagger}_{\infty,\mathrm{crys}}}$ evaluated on $\mathrm{Spf}(\widehat{\overline{R}})$) admits a natural two-step *Hodge filtration*, whose

$F^0 = \Omega^{\mathrm{inv}}_{E^\dagger_\infty}$; $F^2 = 0$; and F^1 consists of the invariant differentials ω_E on E_∞. (Thus, F^0/F^1 may be identified with τ_E.) Tensoring this filtration with the filtration discussed above on $\mathcal{R}_{E^\dagger_\infty}$ gives rise to a filtration on $\Omega_{E^\dagger_\infty}$. Moreover, these filtrations define filtrations on the evaluations (on various thickenings of $\mathrm{Spf}(\widehat{\overline{R}})$) of the corresponding crystals. Thus, if we evaluate these crystals on $\mathrm{Spf}(B)$, we obtain:

$$F^1(\Omega^{\mathrm{inv}}_{E^\dagger_{\infty,\mathrm{crys}}})|_{\mathrm{Spf}(B)} \subseteq F^1(\Omega_{E^\dagger_{\infty,\mathrm{crys}}})|_{\mathrm{Spf}(B)} \subseteq \Omega_{E^\dagger_{\infty,\mathrm{crys}}}|_{\mathrm{Spf}(B)}$$

Note that we have a natural inclusion

$$F^1(\Omega^{\mathrm{inv}}_{E^\dagger_\infty}) \otimes_{\widehat{\overline{R}}} \mathrm{Ker}(\mathcal{R}^0 \to \mathcal{O}_{E_\infty})) \hookrightarrow F^1(\Omega_{E^\dagger_{\infty,\mathrm{crys}}}|_{\mathrm{Spf}(B)})$$

Let us write

$$\Omega^J_{E^\dagger_\infty} \stackrel{\mathrm{def}}{=} F^1(\Omega_{E^\dagger_\infty}|_{\mathrm{Spf}(B)})/(F^1(\Omega^{\mathrm{inv}}_{E^\dagger_\infty}) \otimes_{\widehat{\overline{R}}} \mathrm{Ker}(\mathcal{R}^0 \to \mathcal{O}_{E_\infty}))$$

$$\Omega^{\mathrm{Inv},J}_{E^\dagger_\infty} \stackrel{\mathrm{def}}{=} (F^1/F^2)(\Omega^{\mathrm{inv}}_{E^\dagger_\infty})|_{\mathrm{Spf}(B)}$$

Thus, we have a commutative diagram of $\widehat{\overline{R}}$-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J \otimes_R \tau_E & \longrightarrow & \Omega^{\mathrm{Inv},J}_{E^\dagger_\infty} & \longrightarrow & \omega_E \otimes_R \widehat{\overline{R}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (J \otimes_R \tau_E) \otimes_{\widehat{\overline{R}}} \mathcal{O}_{E_\infty} & \longrightarrow & \Omega^J_{E^\dagger_\infty} & \longrightarrow & \Omega_{E_\infty/\widehat{\overline{R}}} = \omega_E \otimes_R \mathcal{O}_{E_\infty} & \longrightarrow & 0
\end{array}$$

In particular, we see that we obtain a natural isomorphism

$$\Omega^{\mathrm{Inv},J}_{E^\dagger_\infty} \otimes_{\widehat{\overline{R}}} \mathcal{O}_{E_\infty} \cong \Omega^J_{E^\dagger_\infty}$$

On the other hand, it follows immediately from the definitions of the various filtrations that the exterior derivative d (over B) on $\mathcal{R}_{E^\dagger_{\infty,\mathrm{crys}}}|_{\mathrm{Spf}(B)}$ induces a morphism $\mathcal{R}^0 \to F^1(\Omega_{E^\dagger_\infty}|_{\mathrm{Spf}(B)})$, hence (by projecting) a morphism

$$\mathcal{R}^0 \to \Omega^J_{E^\dagger_\infty}$$

which is easily seen to vanish on $J \cdot \mathcal{O}_{(E_\infty)_{\widehat{R}}}$. Thus, we obtain that the exterior derivative induces a *derivation*

$$\mathcal{R}^J \to \Omega^J_{E^\dagger_\infty}$$

from the ring $\mathcal{R}^J$ to the module $\Omega^J_{E^\dagger_\infty}$, hence a natural morphism

$$\Omega_{\mathcal{R}^J/\widehat{R}} \to \Omega^J_{E^\dagger_\infty}$$

But one computes easily (using the exact sequence considered above in which $\mathcal{R}^J$ appears as the term in the middle) that this morphism is an *isomorphism* (where we use that $p \neq 2$!), i.e., we have proven the following

Lemma 2.3. *We have:* $\Omega_{\mathcal{R}^J/\widehat{R}} \cong \Omega^J_{E^\dagger_\infty}$.

Thus, the well-known relation between derivations and differentials implies (by Lemmas 2.2, 2.3) that $\Xi_{\mathcal{R}^J}[p^n]$ factors as the composite of the tautological derivation

$$d_{\mathcal{R}^J} : \mathcal{R}^J \to \Omega_{\mathcal{R}^J/\widehat{R}} \cong \Omega^J_{E^\dagger_\infty}$$

and a morphism

$$\Omega^J_{E^\dagger_\infty} \to \mathbf{Func}(T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}, J_{\mathbf{Q}_p/\mathbf{Z}_p})$$

If we then restrict to the *invariant differentials* $\Omega^{\mathrm{Inv},J}_{E^\dagger_\infty} \subseteq \Omega^J_{E^\dagger_\infty}$, we get a morphism

$$\Psi[p^n] : \Omega^{\mathrm{Inv},J}_{E^\dagger_\infty} \to \mathbf{Func}(T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}, J_{\mathbf{Q}_p/\mathbf{Z}_p})$$

Lemma 2.4. *The image of* $\Psi[p^n]$ *lies in the subset*

$$\mathrm{Hom}(T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}, J_{\mathbf{Q}_p/\mathbf{Z}_p}) \subseteq \mathbf{Func}(T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}, J_{\mathbf{Q}_p/\mathbf{Z}_p})$$

of "functions that are compatible with the additive group operations on $T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}$ *and* $J_{\mathbf{Q}_p/\mathbf{Z}_p}$*."*

Proof. Indeed, this follows by observing that since the definition of $\Psi[p^n]$ is "natural," it is *functorial* with respect to the homomorphism $\mu : E_\infty \times_{\widehat{R}} E_\infty \to E_\infty$ (defined by the group law on the group scheme E_∞). On the other hand, (by the functoriality of invariant differentials with respect to homomorphisms) the pull-back to $E_\infty \times_{\widehat{R}} E_\infty$ of any section α of $\Omega^{\mathrm{Inv},J}_{E^\dagger_\infty}$ is simply (α, α). Put another way, this implies that to evaluate $(\Psi[p^n])(\alpha)$ on the sum of two elements of $T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}$ is the same as the sum of the values of $(\Psi[p^n])(\alpha)$ at each of these two elements, i.e., the function $(\Psi[p^n])(\alpha) \in \mathbf{Func}(T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}, J_{\mathbf{Q}_p/\mathbf{Z}_p})$ is *additive*, as desired. ◯

Note that the additivity property of Lemma 2.4 implies that: (i) the image of $\Psi[p^n]$ is annihilated by p^n; (ii) for $\alpha \in \Omega^{\mathrm{Inv},J}_{E^\dagger_\infty}$, the function $(\Psi[p^{n-1}])(\alpha)$ is the function induced by $(p \cdot \Psi[p^n])(\alpha)$ on the quotient $T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z} \to T_E \otimes \mathbf{Z}/p^{n-1} \cdot \mathbf{Z}$. In particular, by taking the inverse limit, we obtain a morphism:

$$\Psi_{\mathbf{Z}_p} : \Omega^{\mathrm{Inv},J}_{E^\dagger_\infty} \to \mathrm{Hom}(T_E, J)$$

This morphism is the *morphism that defines the Hodge-Tate decomposition* (cf., e.g., the final Theorem – i.e., the "relative version" – in [Hyodo], §0.3). The following result may be regarded as the *main theorem of the Hodge-Tate theory of an elliptic curve*:

Theorem 2.5. *The morphism*

$$\Psi_{\mathbf{Z}_p} : \Omega^{\mathrm{Inv},J}_{E^\dagger_\infty} \to \mathrm{Hom}(T_E, J)$$

is invertible over $\mathbf{Q}_p$, and its inverse has poles annihilated by $p^{\frac{1}{(p-1)}}$.

Proof. Note that $\Psi_{\mathbf{Z}_p}$ is an $\widehat{\widehat{R}}$-linear morphism between free $\widehat{\widehat{R}}$-modules of rank 2. Moreover, its determinant may be regarded as a morphism

$$\det(\Psi_{\mathbf{Z}_p}) : J \to \det(T_E)^\vee \otimes_{\mathbf{Z}_p} J^{\otimes 2} = J^{\otimes 2}(-1)$$

i.e., this determinant may be thought of as an element

$$D \in \mathrm{Hom}_{\widehat{R}}(J, J^{\otimes 2}(-1)) = J(-1) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{Z}_p(1), J)$$

I *claim* that (up to a sign) this element is the morphism $\beta : \mathbf{Z}_p(1) \to J$ discussed at the beginning of this §. (Note that this is sufficient to complete the proof of Theorem 2.5.)

In order to prove this claim, let us first observe that it suffices to prove it when the present S is *replaced by the (formal scheme given by the) completion of S along D*. Indeed, this is clear when this completion along the divisor at infinity D is "schematically dense" in S (i.e., the map from functions on S to functions on this completion is injective). But this holds in the "universal case," i.e., when the original S is étale over (the p-adic completion of) the moduli stack $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}_p}$. Thus, it suffices to work in a formal neighborhood of infinity. In particular, for the rest of the proof, we assume that S is such a formal neighborhood.

Then, by Chapter III, Theorem 2.1, we have an exact sequence of group objects

$$0 \to (\mathbf{G}_{\mathrm{m}})_S \to E_\infty^\dagger \to W_E \to 0$$

over S. (Here, W_E is the affine group object associated to ω_E.) Let us observe that, relative to the structure of $E_\infty^\dagger$ as a *crystal* over $\mathrm{Spf}(\widehat{\overline{R}})$, this exact sequence is *horizontal*. Indeed, the subobject $(\mathbf{G}_{\mathrm{m}})_S \subseteq E_\infty^\dagger$ may be recovered as the (formal) schematic closure of the prime-to-p torsion points of $E_\infty^\dagger$ that lie inside $(\mathbf{G}_{\mathrm{m}})_S$. But it is clear that the subschemes defined by these torsion points are horizontal. Thus, we see that this exact sequence is horizontal, as desired.

Next, observe that, since we are working in a formal neighborhood of infinity, the Γ_R-module fits into a natural exact sequence of Γ_R-modules:

$$0 \to \mathbf{Z}/p^n \cdot \mathbf{Z}(1) \to T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z} \to \mathbf{Z}/p^n \cdot \mathbf{Z} \to 0$$

Moreover, if we consider the affine coordinate "T" on W_E corresponding to the trivialization of ω_E given by "$d\log(U)$" (notation of Chapter III, §5), then it follows from Chapter III, Corollary 5.9, that the image under $\Xi_{\mathcal{R}^J}[p^n]$ of the section of $\mathcal{R}^J$ defined by "$T \cdot j$" (where $j \in J$) is the function on $T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}$ that arises by pulling back (to $T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}$) the J-valued function on $\mathbf{Z}/p^n \cdot \mathbf{Z}$ that maps $b \in \mathbf{Z}/p^n \cdot \mathbf{Z}$ to $b \cdot j$. Thus, in particular, applying $\mathrm{Hom}(-, J)$ to the inverse limit as $n \to \infty$ of the above exact sequence — which yields an exact sequence

$$0 \to J \to \mathrm{Hom}(T_E, J) \to J(-1) \to 0$$

— we see that $\Psi_{\mathbf{Z}_p}$ maps $T \cdot J \subseteq \Omega^{\mathrm{Inv},J}_{E_\infty^\dagger}$ to $J \subseteq \mathrm{Hom}(T_E, J)$ *isomorphically* via the morphism given by $T \mapsto 1$.

Thus, it remains to consider what happens when functions on $(\mathbf{G}_{\mathrm{m}})_S$ are restricted to $\mathbf{Z}/p^n \cdot \mathbf{Z}(1) \subseteq T_E \otimes \mathbf{Z}/p^n \cdot \mathbf{Z}$. In particular, we would like to investigate the image of the invariant differential $d\log(U)$ under $\Psi[p^n]$. But since $\Psi[p^n]$ was defined by factoring the derivation $\Xi_{\mathcal{R}^J}[p^n]$ (cf. Lemma 2.2) through the tautological derivation, it follows that this amounts to investigating the behavior of the function "$\log(U)$" on $\mathbf{Z}/p^n \cdot \mathbf{Z}(1) \subseteq (\mathbf{G}_{\mathrm{m}})_S$. But then it follows from the *definition* of the morphism $\beta : \mathbf{Z}_p(1) \to J$ (which is essentially given by taking the logarithm of p-power roots of unity! — cf. [Falt2], II., (b.)) that (up to a sign), if we take the inverse limit as $n \to \infty$, the image of $d\log(U) \in \omega_E \subseteq \Omega^{\mathrm{inv}}_{E_\infty^\dagger}/(J \otimes_R \tau_E)$

under the composite of $\Psi_{\mathbf{Z}_p}$ with the quotient $\mathrm{Hom}(T_E, J) \to J(-1)$ is precisely $\beta \in \mathrm{Hom}(\mathbf{Z}_p(1), J) = J(-1)$. More explicitly, let $\zeta \in \widehat{\overline{R}}$ be a p^n-th root of unity. Write $\widetilde{\zeta} \in p^{-n} \cdot \mathbf{Z}_p(1) \subseteq \mathbf{Q}_p(1)$ for an element such that $z \stackrel{\mathrm{def}}{=} \beta_{\mathbf{Q}_p}(\widetilde{\zeta}) \equiv \zeta$ modulo J. Note that $z \in B$ itself is not necessarily a p^n-th root of unity. If $\delta \stackrel{\mathrm{def}}{=} -p^{-n} \cdot \zeta \cdot \beta(p^n \cdot \widetilde{\zeta}) \in p^{-n} \cdot J \subseteq J_{\mathbf{Q}_p}$, then

$$(z + \delta)^{p^n} = z^{p^n} + p^n \cdot z^{p^n - 1} \cdot \delta = 1 + \beta(p^n \cdot \widetilde{\zeta}) + p^n \cdot \zeta^{-1} \cdot \delta = 1$$

i.e., $z + \delta \in \boldsymbol{\mu}_{p^n}(B)$. Thus, the evaluation map "modulo B" (cf. the definition of $\Xi_{\mathcal{R}^J}[p^n]$ above) maps the function U on $(\mathbf{G}_{\mathrm{m}})_S$ to the image of $z + \delta$ in $B_{\mathbf{Q}_p/\mathbf{Z}_p}$. In fact, this image is equal to the image $\overline{\delta} \in J_{\mathbf{Q}_p/\mathbf{Z}_p}$ of δ. Put another way, we obtain that $dU \mapsto \overline{\delta}$, so $d\ \log(U) = U^{-1} \cdot dU \mapsto \zeta^{-1} \cdot \overline{\delta} = -p^{-n} \cdot \beta(p^n \cdot \widetilde{\zeta}) \pmod{J} \ \in J_{\mathbf{Q}_p/\mathbf{Z}_p}$. Taking the inverse limit as $n \to \infty$, we thus obtain that, in this inverse limit, $d\ \log(U) \mapsto -\beta$, as desired. Thus, combining this with what was done in the preceding paragraph, we see that the determinant D in question is equal to $\pm\beta$, as claimed. ○

Remark. The isomorphism of Theorem 2.5 is the *Hodge-Tate Comparison Isomorphism* of an elliptic curve. If one restricts it to a point α of S valued in some finite extension L of K, then it follows from the facts $H^1(\Gamma_L, \widehat{\overline{L}}(1)) = 0$, $H^0(\Gamma_L, \widehat{\overline{L}}) = L$ (where $\Gamma_L \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{L}/L)$) that $\Omega^{\mathrm{inv}}_{E^\dagger_\infty}|_\alpha$ admits a unique Γ_L-invariant splitting

$$\Omega^{\mathrm{inv}}_{E^\dagger_\infty}|_\alpha = \omega_E|_\alpha \oplus \tau_E(1)|_\alpha$$

(where we use that $J_{\mathbf{Q}_p} = \widehat{\overline{R}}_{\mathbf{Q}_p}(1)$), hence that Theorem 2.5 gives us an isomorphism

$$\omega_E|_\alpha \oplus \tau_E(1)|_\alpha \cong T_E \otimes_{\mathbf{Z}_p} \widehat{\overline{L}}$$

This is the usual form in which the Hodge-Tate "comparison isomorphism"/decomposition is stated.

The important point here is that this well-known isomorphism is *essentially defined by restricting functions on the universal extension of E to p-power torsion points* (cf. the definition of $\Xi_{\mathbf{Q}_p}[p^n]$), i.e.,

> The Hodge-Tate Comparison Isomorphism is defined in a fashion which is *entirely analogous* to the definition of the Hodge-Arakelov Comparison Isomorphism (Chapter VIII, Theorem A).

Note, moreover, that after reducing $\Xi_{\mathbf{Q}_p}[p^n]$ "modulo $\mathbf{Z}_p$," and factoring through the tautological derivation $d_{\mathcal{R}^J}$, we restricted to the *invariant differentials* on the universal extension. Thus, put another way, the comparison isomorphism of Theorem 2.5 was defined essentially by *integrating invariant differentials, and then restricting the resulting functions to p-power torsion points*. (Thus, these "resulting functions" are essentially "logarithm" type functions, of the sort reviewed in the Appendix.) Regarded from this point of view, we thus also see *the explicit analogy between this p-adic comparison isomorphism and its complex counterpart in §1*.

Finally, we make *the connection between the above discussion of the Hodge-Tate Comparison Isomorphism and the arithmetic Kodaira-Spencer morphism*. In keeping with the analogy to the theory of §1,3, we start off with the *comparison isomorphism* (Theorem 2.5):

$$\Psi_{\mathbf{Q}_p} : \mathcal{E} \stackrel{\text{def}}{=} (\Omega^{\text{Inv},J}_{E^{\dagger}_{\infty}})_{\mathbf{Q}_p} \cong \text{Hom}(T_E, J_{\mathbf{Q}_p}) = V_E^{\vee}(1) \otimes_{\mathbf{Q}_p} \widehat{\overline{R}}_{\mathbf{Q}_p} \cong V_E \otimes_{\mathbf{Q}_p} \widehat{\overline{R}}_{\mathbf{Q}_p}$$

(where $\Psi_{\mathbf{Q}_p} \stackrel{\text{def}}{=} \Psi_{\mathbf{Z}_p} \otimes \mathbf{Q}_p$, $V_E \stackrel{\text{def}}{=} T_E \otimes \mathbf{Q}_p$). Since $V_E \otimes_{\mathbf{Q}_p} \widehat{\overline{R}}_{\mathbf{Q}_p}$ has a natural Γ_R-action, we thus obtain a natural Γ_R-action on $\mathcal{E}$. This technique of obtaining a *Galois action on the de Rham side of the comparison isomorphism by pulling back (via the comparison isomorphism) the natural Galois action on the étale side* is entirely analogous to what is done in §1,3.

Thus, we have a natural Γ_R-action on $\mathcal{E}$. Note, moreover, that we have an exact sequence of $\widehat{\overline{R}}_{\mathbf{Q}_p}$-modules:

$$0 \to \tau_E(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p} \to \mathcal{E} \to \omega_E \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p} \to 0$$

(where we identify $J_{\mathbf{Q}_p}$ with $\widehat{\overline{R}}_{\mathbf{Q}_p}(1)$ via β). Moreover, it is easy to see from the definition of $\Xi_{\mathcal{R}^J}[p^n]$ that the natural Galois action on $\tau_E(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p} \subseteq \mathcal{E}$ is compatible with the Galois action on $V_E \otimes_{\mathbf{Q}_p} \widehat{\overline{R}}_{\mathbf{Q}_p}$. Thus, this submodule is stabilized by Γ_R, and the above exact sequence is an exact sequence of Γ_R-modules. (Note that the fact that the Galois action on $\omega_E \otimes_R \widehat{\overline{R}}$ arising from the Galois action on $\mathcal{E}$ is the expected action may be derived using determinants, as in the proof of Theorem 2.5.) We observe that this exact sequence is essentially the same as that of [Mzk3], Proposition 2.4.

This exact sequence thus defines a *cohomology class*

$$\eta_{\text{KS}} \in H^1(\Gamma_R, Hom_{\widehat{\overline{R}}_{\mathbf{Q}_p}}(\omega_E \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p}, \tau_E(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p})) = H^1(\Gamma_R, \tau_E^{\otimes 2}(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p})$$

Such a cohomology class may be thought of as an *equivalence class of twisted homomorphisms* $\Gamma_R \to \tau_E^{\otimes 2}(1)_{\widehat{\overline{R}}_{\mathbf{Q}_p}} \stackrel{\text{def}}{=} \tau_E^{\otimes 2}(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p}$, where we consider twisted homomorphisms that differ by (the twisted homomorphism defined by) a coboundary to be equivalent. We would like to regard this equivalence class of twisted homomorphisms which, by abuse of notation, we write

$$\kappa_R^{\mathrm{arith},p} : \Gamma_R \to \tau_E^{\otimes 2}(1)_{\widehat{\overline{R}}_{\mathbf{Q}_p}}$$

as a sort of *p-adic (Galois) group-theoretic/arithmetic Kodaira-Spencer morphism*, analogous to the morphisms "$\kappa_{SL_2}^{\mathrm{func}}$" of §1, and "$\kappa_E^{\mathrm{arith}}$" in §3. Note that *the construction of* $\kappa_R^{\mathrm{arith},p}$ *starting from the comparison isomorphism is entirely analogous to what is done in* §*1,3*.

To explain *why it is natural to regard* $\kappa_R^{\mathrm{arith},p}$ *as a sort of Kodaira-Spencer morphism*, we must recall the theory of [Falt1]. In [Falt1], I., §4, (b), (f), a certain natural extension of Γ_R-modules

$$0 \to p^{-\frac{1}{(p-1)}} \cdot \widehat{\overline{R}} \to E_\rho \to \Omega_S^{\log}(-1) \otimes_R \widehat{\overline{R}} \to 0$$

(where $\Omega_S^{\log} \stackrel{\mathrm{def}}{=} \Omega_{S^{\log}/A}$) is constructed. If we write $\Theta_S^{\log} \stackrel{\mathrm{def}}{=} (\Omega_S^{\log})^\vee$ for the logarithmic tangent bundle of $S^{\log}$ over A, then we get an element

$$\eta_{\mathrm{Falt}} \in H^1(\Gamma_R, \Theta_S^{\log}(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p})$$

This element may be thought of as a twisted homomorphism

$$\xi_R^{\mathrm{Falt}} : \Gamma_R \to \Theta_S^{\log}(1)_{\widehat{\overline{R}}_{\mathbf{Q}_p}}$$

On the other hand, we have the *classical Kodaira-Spencer morphism*

$$\kappa_R^{\mathrm{class}} : \Theta_S^{\log} \to \tau_E^{\otimes 2}$$

given by observing the extent to which the Hodge filtration of the crystal $\Omega_{E^\dagger_\infty}$ varies as one moves in various tangent directions $\in \Theta_S^{\log}$. Then the relationship between the *p-adic arithmetic Kodaira-Spencer morphism* $\kappa_R^{\mathrm{arith},p}$ and the *classical Kodaira-Spencer morphism* $\kappa_R^{\mathrm{class}}$ is given by Faltings' morphism ξ_R^{Falt}, as described in Theorem 2.6 below.

Remark. If we write

$$\Delta_R \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Gamma_R \to \Gamma_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K))$$

for the *geometric portion of the Galois group* Γ_R, then according to the theory of "almost étale extensions" discussed in [Falt1], "$\Delta_R \otimes_{\mathbf{Z}_p} \widehat{\overline{R}}_{\mathbf{Q}_p}$" (notation, which, of course, is not defined rigorously, but is meant to be suggestive of what is going on) may roughly be identified with $\Theta_S^{\log}(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p}$, that is to say, over $\widehat{\overline{R}}_{\mathbf{Q}_p}$, the *geometric Galois group* Δ_R *of* $S^{\log}$ *is "almost equivalent" to the tangent bundle of* $S^{\log}$, hence may be thought of as a group of "*motions in* $S^{\log}$." From this "almost" point of view, it is thus *natural that the domain of our p-adic arithmetic Kodaira-Spencer morphism* $\kappa_R^{\mathrm{arith},p}$ *is the Galois group* Γ_R.

Theorem 2.6. *We have a commutative diagram:*

$$\begin{array}{ccc} \Gamma_R & \xrightarrow{-\xi_R^{\mathrm{Falt}}} & \Theta_S^{\log}(1)_{\widehat{\overline{R}}_{\mathbf{Q}_p}} \\ \Big\downarrow \mathrm{id} & & \Big\downarrow \kappa_R^{\mathrm{class}}(1)_{\widehat{\overline{R}}_{\mathbf{Q}_p}} \\ \Gamma_R & \xrightarrow{\kappa_R^{\mathrm{arith},p}} & \tau_E^{\otimes 2}(1)_{\widehat{\overline{R}}_{\mathbf{Q}_p}} \end{array}$$

i.e., in terms of cohomology classes, the change of coefficients morphism

$$H^1(\Gamma_R, \Theta_S^{\log}(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p}) \to H^1(\Gamma_R, \tau_E^{\otimes 2}(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p})$$

defined by $\kappa_R^{\mathrm{class}}$ *maps* $-\eta_{\mathrm{Falt}} \mapsto \eta_{\mathrm{KS}}$.

Proof. First of all, since both η_{Falt} and η_{KS} are clearly functorial in R, it suffices to prove the result in the *universal case*, i.e., the case where S is (formally) étale over the p-adic completion of the moduli stack $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}_p}$. Thus, for the rest of the proof, we assume that we are dealing with such an S.

Let us write

$$\Delta_R \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Gamma_R \to \Gamma_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K))$$

Then observe that by the theory of [Falt1] (cf. especially [Falt1], p. 270, Theorem 4.4, (i)), it follows that $H^0(\Gamma_K, H^1(\Delta_R, \tau_E^{\otimes 2}(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p}))$ is a *free* $R_{\mathbf{Q}_p}$*-module of rank* 1, while $H^1(\Gamma_K, H^0(\Delta_R, \tau_E^{\otimes 2}(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p})) = 0$ (cf. the well-known fact that $H^1(\Gamma_K, \overline{K}(1)) = 0$; [Mzk3], Lemma 2.2). Thus, by the Leray-Serre spectral sequence, we obtain that

$$H^1(\Gamma_R, \tau_E^{\otimes 2}(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p}) \hookrightarrow H^0(\Gamma_K, H^1(\Delta_R, \tau_E^{\otimes 2}(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p})) \quad (\cong R_{\mathbf{Q}_p})$$

i.e., that two classes in $H^1(\Gamma_R, \tau_E^{\otimes 2}(1) \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p})$ coincide if and only if they coincide after *restriction to a formal neighborhood of the divisor at infinity* D (cf. the proof of Theorem 2.5). Thus, it suffices to prove the result in the case where S is a such a formal neighborhood.

In this case, however, it follows from the construction of the extension "E_ρ" in [Falt1], I., §4, that η_{Falt} is essentially given by the image of the class $\in H^1(\Gamma_R, \mathbf{Z}_p(1))$ obtained by extracting p-power roots of the q-parameter under the change of coefficients morphism

$$H^1(\Gamma_R, \mathbf{Z}_p(1)) \to H^1(\Gamma_R, \Theta_S^{\log}(1) \otimes_R \widehat{\widehat{R}}_{\mathbf{Q}_p})$$

defined by the "tangent vector" $\partial/\partial\log(q)$. On the other hand, it follows from the discussion in the proof of Theorem 2.5 that the extension η_{KS} is simply the extension class of the exact sequence

$$0 \to \mathbf{Z}_p(1) \to T_E \to \mathbf{Z}_p \to 0$$

pushed forward by the change of coefficients morphism

$$H^1(\Gamma_R, \mathbf{Z}_p(1)) \to H^1(\Gamma_R, \tau_E^{\otimes 2}(1) \otimes_R \widehat{\widehat{R}}_{\mathbf{Q}_p})$$

defined by $(\partial/\partial U)^{\otimes 2}$ (notation of the proof of Theorem 2.5). On the other hand, since it is well-known that *the classical Kodaira-Spencer morphism maps* $\partial/\partial\log(q) \mapsto -(\partial/\partial U)^{\otimes 2}$ (cf., e.g., [FC], p. 84, the second paragraph preceding Lemma 9.3, beginning "As an example..."), it thus follows that $-\eta_{\mathrm{Falt}} \mapsto \eta_{\mathrm{KS}}$, as desired. ○

Remark. Put another way, the message of the proof of Theorem 2.6, and indeed of this entire §, is that:

> The relationship between the p-adic Galois-theoretic/arithmetic Kodaira-Spencer morphism and the classical Kodaira-Spencer morphism is the essential content of *Serre-Tate theory.*

For more on this point of view in the case of abelian varieties, we refer to [Katz]. For more on this point of view in a more general context (in particular, the case of hyperbolic curves), we refer to [Mzk1], Introduction; Chapter V, §1; as well as to [Mzk2], Introduction, especially §2.3.

§3. The Global Arithmetic Case: Application of the Hodge-Arakelov Comparison Isomorphism

In this §, we apply the *Hodge-Arakelov Comparison Isomorphism* (cf. Chapter VIII, Theorem A) to construct a *global arithmetic analogue* of the Kodaira-Spencer morphism of a family of elliptic curves. The technique of construction is motivated by the point of view discussed in §1,2, in the complex and p-adic cases.

We begin by discussing *the behavior of the trivializations arising from theta groups at archimedean primes*. Thus, let E be an *elliptic curve over* $\mathbf{C}$. Fix a positive integer d, and write

$$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(d \cdot [e])$$

Let $| \sim |_{\mathcal{L}}$ be a metric on $\mathcal{L}$ whose curvature is translation-invariant. Recall from Chapter IV, §1, the subscheme of *symmetric* elements

$$\mathcal{S}_{\mathcal{L}} \subseteq \mathcal{G}_{\mathcal{L}}$$

of the theta group $\mathcal{G}_{\mathcal{L}}$ of $\mathcal{L}$.

Lemma 3.1. *The automorphisms of* $(E, \mathcal{L})$ *defined by points of* $\mathcal{S}_{\mathcal{L}}$ *preserve* $| \sim |_{\mathcal{L}}$.

Proof. Indeed, clearly any translation of a translation-invariant $(1,1)$-form is again translation-invariant. Thus, it follows that for $\gamma = (\alpha, \iota) \in \mathcal{S}_{\mathcal{L}}$, the metric $\gamma(| \sim |_{\mathcal{L}})$ that one obtains on $\mathcal{L}$ by first pulling back $| \sim |_{\mathcal{L}}$ via $\iota : \mathcal{T}_{\alpha}^* \mathcal{L} \cong \mathcal{L}$, and then applying $(\mathcal{T}_{\alpha}^{-1})^*$ to produce (by transport of structure) a metric on $\mathcal{L}$ has *translation-invariant curvature*. Thus, the metrics $\gamma(| \sim |_{\mathcal{L}})$ and $| \sim |_{\mathcal{L}}$ both have translation-invariant curvature, hence differ by a positive constant λ_{γ}. Moreover, since $\gamma \in \mathcal{S}_{\mathcal{L}}$ implies $\gamma^N \in \mathcal{S}_{\mathcal{L}}$ ($\forall\, N \in \mathbf{Z}$), it follows that the correspondence $\gamma \mapsto \lambda_{\gamma}$ defines a *homomorphism* from $< \gamma >$ (the subgroup of $\mathcal{G}_{\mathcal{L}}$ generated by γ, which lies inside $\mathcal{S}_{\mathcal{L}}$) to $\mathbf{R}_{>0}$ (equipped with its multiplicative group structure). Since $< \gamma >$ is a *finite group*, it thus follows that this homomorphism is *trivial*, so $\lambda_{\gamma} = 1$, as desired. ○

Now recall the *"theta trivialization"*

$$\Theta_{\alpha} : \mathcal{L}|_{\mathcal{T}_{\alpha}^* K_{\mathcal{L}}} \cong (\mathcal{L}|_{\alpha^0}) \otimes_{\mathcal{O}_{K_{\mathcal{L}}{}^0}} \mathcal{O}_{K_{\mathcal{L}}}$$

(where $\alpha \in K_{\mathcal{L}}$) of the discussion following Chapter IV, Theorem 1.6. Note that both sides of this isomorphism may be regarded as finite-dimensional complex vector spaces. Moreover, both of these vector spaces have natural metrics, arising from $| \sim |_{\mathcal{L}}$, and the fact that since we are working over $\mathbf{C}$, the ring $\mathcal{O}_{K_{\mathcal{L}}}$ may be regarded as the ring of $\mathbf{C}$-valued functions on the d-torsion points ${}_dE$ of E, hence admits a natural L^2-metric

$$||f||^2 \stackrel{\text{def}}{=} \frac{1}{d^2} \cdot \sum_{\tau \in {}_dE} |f(\tau)|^2$$

(for $f \in \mathcal{O}_{K_{\mathcal{L}}}$). Since this trivialization is defined by the splitting $\sigma : 2 \cdot K_{\mathcal{L}} \to \mathcal{G}_{\mathcal{L}}$ of $\mathcal{G}_{\mathcal{L}} \to K_{\mathcal{L}}$ discussed in Chapter IV, Theorem 1.6, which factors through $\mathcal{S}_{\mathcal{L}}$, it follows from Lemma 3.1 that:

Lemma 3.2. *If one equips both sides of the theta trivialization*

$$\Theta_\alpha : \mathcal{L}|_{\mathcal{T}_\alpha^* K_{\mathcal{L}}} \cong (\mathcal{L}|_{\alpha^0}) \otimes_{\mathcal{O}_{K_{\mathcal{L}}{}^0}} \mathcal{O}_{K_{\mathcal{L}}}$$

determined by the splitting σ of Chapter IV, Theorem 1.6, with their natural metrics (arising from $| \sim |_{\mathcal{L}}$, and the L^2-metric on $\mathcal{O}_{K_{\mathcal{L}}}$), then this trivialization Θ is an isometry.

Remark. In particular, we may conclude the following: Recall the *Comparison Isomorphism* (cf. Chapter V, Theorem 3.1, (2); or, alternatively, Chapter VIII, Theorem A, (2))

$$\Xi : \Gamma(E^\dagger_{[d]}, \mathcal{L}_\eta)^{<d} \cong \mathcal{L}_\eta|_{{}_dE}$$

where $\mathcal{L}_\eta \stackrel{\text{def}}{=} \mathcal{T}_\eta^* \mathcal{L}$, and $\eta \in E$ is a torsion point of order m not dividing d. Thus, if we compose this isomorphism with the trivialization

$$\Theta_\eta : \mathcal{L}|_{\mathcal{T}_\eta^* K_{\mathcal{L}}} = \mathcal{L}_\eta|_{{}_dE} \cong (\mathcal{L}|_{\eta^0}) \otimes_{\mathcal{O}_{K_{\mathcal{L}}{}^0}} \mathcal{O}_{K_{\mathcal{L}}}$$

we get an isomorphism

$$\Psi : \Gamma(E^\dagger_{[d]}, \mathcal{T}_\eta^* \mathcal{L})^{<d} \cong (\mathcal{L}|_{\eta^0}) \otimes_{\mathcal{O}_{K_{\mathcal{L}}{}^0}} \mathcal{O}_{K_{\mathcal{L}}}$$

of d-dimensional vector spaces over $\mathbf{C}$. Next, suppose that we are given an *automorphism* Φ of the $\mathbf{C}$-algebra $\mathcal{O}_{K_{\mathcal{L}}}$ which *preserves and induces the identity on the* $\mathbf{C}$*-subalgebra* $\mathcal{O}_{K_{\mathcal{L}}{}^0} \subseteq \mathcal{O}_{K_{\mathcal{L}}}$ arising from the quotient $K_{\mathcal{L}} \to K_{\mathcal{L}}{}^0$. Observe (by thinking about the fact that Φ necessarily arises as the morphism induced on functions by some automorphism of the *set* of d-torsion points of E) that such an automorphism Φ is an *isometry* of $\mathcal{O}_{K_{\mathcal{L}}}$ onto itself (where we equip $\mathcal{O}_{K_{\mathcal{L}}}$ with the natural L^2-metric of the discussion above). Note, moreover, that Φ induces an (isometric) automorphism of $(\mathcal{L}|_{\eta^0}) \otimes_{\mathcal{O}_{K_{\mathcal{L}}{}^0}} \mathcal{O}_{K_{\mathcal{L}}}$, hence (by conjugating by Ψ) an automorphism of $\Gamma(E^{\dagger}_{[d]}, \mathcal{T}^*_\eta \mathcal{L})^{<d}$. By Lemma 3.2 (and the definition of the "étale metric" $|| \sim ||_{\text{et}}$ on $\Gamma(E^{\dagger}_{[d]}, \mathcal{T}^*_\eta \mathcal{L})^{<d}$), we thus obtain that:

> Φ *induces an automorphism of* $\Gamma(E^{\dagger}_{[d]}, \mathcal{T}^*_\eta \mathcal{L})^{<d}$ *which is an isometry with respect to the* $|| \sim ||_{\text{et}}$*-metric* (cf. Chapter VIII, Theorem A, (4)).

This observation will be of central importance in the following discussion.

Now let $S^{\log}$ be any *log scheme* whose underlying scheme is *connected, flat and of finite type over* $\mathbf{Z}$, and whose log structure is defined by a divisor with normal crossings. Write $\text{Int}(S^{\log}) \subseteq S$ for the *interior* of $S^{\log}$, i.e., the open subscheme where the log structure is trivial. Let

$$C^{\log} \to S^{\log}$$

be a *log elliptic curve* over $S^{\log}$ as in Chapter VIII, Theorem A, i.e.:

(1) the divisor at infinity $D \subseteq S$ (i.e., the pull-back of the divisor at infinity of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ via the classifying morphism $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$) is a Cartier divisor on S;

(2) étale locally on the completion of S along D, the pull-back of the Tate parameter q to this completion admits a d-th root.

Here, we also make the technical assumption that if d is *even*, then *the* 2*-torsion points of* $E \to S$ *over the interior of* $S_{\mathbf{Q}}$ *are all rational* (i.e., defined over the interior of $S_{\mathbf{Q}}$). Let us write (as usual)

$$_dE \to S$$

for the finite, flat group scheme of d-torsion points of $E_{\infty,S}$. Note that if we tensor with $\mathbf{Q}$, the resulting morphism $(_dE)_{\mathbf{Q}} \to S_{\mathbf{Q}}$ is *finite étale*. Let us fix a *geometric point* $\overline{s}$ of $\text{Int}(S^{\log}_{\mathbf{Q}})$ valued in an algebraically closed field of characteristic 0, and write

$$\Pi_S \stackrel{\text{def}}{=} \pi_1((S^{\log})_{\mathbf{Q}}, \overline{s})$$

for the *algebraic fundamental group* of $\text{Int}(S^{\log}_{\mathbf{Q}})$. Also, let us fix a *geometric point* $\overline{s}_d$ of $_dE$ lying over $\overline{s}$. Thus, it is clear that Π_S acts naturally on $(_dE)_{\mathbf{Q}} \to S_{\mathbf{Q}}$.

Now I *claim* that:

The natural action of Π_S on $({}_dE)_{\mathbf{Q}} \to S_{\mathbf{Q}}$ extends (uniquely) to an action on ${}_dE \to S$, except for possible poles annihilated by 4.

The *uniqueness* of the action follows immediately from the flatness of the morphisms ${}_dE \to S$ and $S \to \mathrm{Spec}(\mathbf{Z})$. To prove *existence*, let us first observe that it suffices to verify this claim for the *"universal cases"* $S = (\mathcal{M}_{1,0})_{\mathbf{Z}}$ (with the trivial log structure) and $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathbf{Z}[[q^{\frac{1}{d}}]])$, endowed with the log structure defined by the divisor $V(q^{\frac{1}{d}})$. In these cases, it follows from the theory of [KM] (cf., [KM], Chapter 5, especially Theorem 5.1.1) that the closed subscheme $F_{d'}$ of ${}_dE$ corresponding to points of a *given order* d' is *regular*, hence *normal*. Thus, it follows immediately that the action of Π_S extends to the $F_{d'}$. Next, one checks that, for instance, at the prime 2, regular functions on the disjoint union of the $F_{d'}$ *regarded as rational functions on the original* ${}_dE$ have poles annihilated by 4. Indeed, in our situation, since the base S is regular of dimension 2, it suffices to verify this assertion on the *ordinary locus*, where it essentially amounts to the corresponding assertion for $\boldsymbol{\mu}_{2^N}$ (as opposed to ${}_dE$), where we write 2^N for the maximal power of 2 dividing d. But then $\mathcal{O}_{F_{2^i} \bigcap F_{2^j}}$ (where $i < j \leq N$) is annihilated by $\zeta_{2^i} - \zeta_{2^j}$ (where $\zeta_{??}$ is a primitive ??-th root of unity), which has the same 2-adic valuation as $2^{2^{-j+1}}$. Thus, we see that the structure sheaf of the intersection of F_{2^i} with the union of the other $F_{d'}$'s is annihilated by 2 to the power

$$i \cdot 2^{-i+1} + 2^{-(i+1)+1} + 2^{-(i+2)+1} + \ldots + 2^{-N+1} \leq i \cdot 2^{-i+1} + 2^{-i+1} = (i+1) \cdot 2^{-i+1} \leq 2$$

i.e., annihilated by 4 (which implies that the "rational function on $\boldsymbol{\mu}_{2^N}$" which is 1 on F_{2^i} and 0 on the other $F_{d'}$'s has poles annihilated by 4, as desired). This completes the proof of the *claim* at the prime 2. At odd primes, the extendability of the action of Π_S follows from the fact that since the universal bases in question (i.e., $S = (\mathcal{M}_{1,0})_{\mathbf{Z}}$ and $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathbf{Z}[[q^{\frac{1}{d}}]])$) are "absolutely unramified," the fact that ${}_dE^{\sigma}$ (where $\sigma \in \Pi_S$) and ${}_dE$ are "equal" over $\mathbf{Q}$ implies that their integral structures at the prime p are equal, as well (cf. [Falt2], Theorems 2.6, 7.1).

Now let us assume that we are also given a *torsion point*

$$\eta \in E_{\infty,S}(S_{\infty})$$

of order precisely $m \in \mathbf{Z}_{\geq 1}$ (where m does not divide d) which allows us to define a (Zhang-theoretically metrized) *line bundle* $\overline{\mathcal{L}}$ as in Chapter VIII, Theorem A. Note that $K_{\overline{\mathcal{L}}} \to S$ may be identified with ${}_dE \to S$. On the other hand, by the isomorphism in the discussion at the end of Chapter V, §1 (where we take "α" to be e), we have a natural isomorphism:

$$\overline{\mathcal{L}}|_{{}_dE} = \overline{\mathcal{L}}|_{K_{\overline{\mathcal{L}}}} \cong (\overline{\mathcal{L}}|_{e^0}) \otimes_{\mathcal{O}_{K^0_{\overline{\mathcal{L}}}}} \mathcal{O}_{K_{\overline{\mathcal{L}}}}$$

Indeed, this follows immediately from the discussion at the end of Chapter V, §1, in the case when $m = 2d$ (which ensures that $\overline{\mathcal{L}}$ is *symmetric*). But this case differs (cf. the definiton of $\overline{\mathcal{L}}$ in Chapter V, §1) from the case of an arbitrary m by a translation, plus a modification of the integral structure at infinity by a d-invariant distribution. Translations are not a problem (by "transport of structure"), while modifications of the integral structure at infinity by a d-invariant distribution are not a problem since they are preserved by the action of theta groups (cf. Chapter IV, Proposition 5.1). Thus, in particular, we see (by the assumption concerning the rationality of the 2-torsion points!) that we get a *natural action (up to poles annihilated by* 4*) of* Π_S *on*

$$(\overline{\mathcal{L}}|_{e^0}) \otimes_{\mathcal{O}_{K^0_{\overline{\mathcal{L}}}}} \mathcal{O}_{K_{\overline{\mathcal{L}}}}$$

(where e is the identity section of $C \to S$).

Thus, we obtain a *natural action (up to poles annihilated by* 4*) of* Π_S *on*

$$\overline{\mathcal{L}}|_{_dE} = \overline{\mathcal{L}}|_{_dE^{\dagger}_{\infty}}$$

i.e., on the range of the *evaluation map* of Chapter VIII, Theorem A:

$$\Xi\{\infty, \mathrm{et}\} : (f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<d}\{\infty, \mathrm{et}\} \to (f_S)_*(\overline{\mathcal{L}}|_{(_dE^{\dagger}_{\infty})})$$

By Chapter VIII, Theorem A, (2), it follows that the poles of the inverse morphism to this evaluation map are contained in the divisor $[\eta \bigcap (_dE)]$. Thus, we see that we get a *natural action of* Π_S *up to poles in the divisor* $[\eta \bigcap (_dE)] + V(4)$ *on*

$$\mathcal{H}_{\mathrm{DR}} \stackrel{\mathrm{def}}{=} (f_S)_*(\overline{\mathcal{L}}|_{E^{\dagger}_{\infty,[d]}})^{<d}\{\infty, \mathrm{et}\}$$

(where $V(4)$ is the zero locus of 4). Here by the expression "with poles in (a divisor)," we mean that the action of any element of Π_S induces an endomorphism of $\mathcal{H}_{\mathrm{DR}} \otimes \mathbf{Q}$, which takes $\mathcal{H}_{\mathrm{DR}}$ into the subsheaf of $\mathcal{H}_{\mathrm{DR}} \otimes \mathbf{Q}$ given by meromorphic sections of $\mathcal{H}_{\mathrm{DR}}$ which are integral everywhere, except for possible poles contained in the divisor stated. Moreover, the Remark following Lemma 3.2 implies that if we put a metric on $\mathcal{L}_{\mathbf{C}}$ (where $\mathcal{L}_{\mathbf{C}} \stackrel{\mathrm{def}}{=} \mathcal{L} \otimes_{\mathbf{Z}} \mathbf{C}$) whose curvature on the fibers of $E_{\mathbf{C}} \to S_{\mathbf{C}}$ is *translation-invariant*, then the resulting *"étale metric"* $|| \sim ||_{\mathrm{et}}$ (cf. Chapter VIII, Theorem A, (4)) *is preserved by the action of* Π_S *on* $\mathcal{H}_{\mathrm{DR}}$. We summarize this discussion as follows:

Theorem 3.3. *Let $S^{\log}$ be any connected log scheme, whose underlying scheme is* flat and of finite type over $\mathbf{Z}$, *and such that the log structure is defined by a divisor with normal crossings. Fix positive integers m, d, where m does not divide d. Let*

$$C^{\log} \to S^{\log}$$

be a log elliptic curve *over $S^{\log}$, and $\eta \in E_{\infty,S}(S_\infty)$ be a torsion point of order precisely m satisfying the hypotheses of Chapter VIII, Theorem A. Fix geometric points $\overline{s}_d$, $\overline{s}$ of ${}_dE$ (the scheme of d-torsion points of $E_{\infty,S}$), $\mathrm{Int}(S^{\log})_{\mathbf{Q}}$ such that $\overline{s}_d$ maps to $\overline{s}$. Write*

$$\Pi_S \stackrel{\mathrm{def}}{=} \pi_1((S^{\log})_{\mathbf{Q}}, \overline{s})$$

for the algebraic fundamental group *of* $\mathrm{Int}(S^{\log}_{\mathbf{Q}})$. *Let us assume that if d is* even, *then* Π_S acts trivially on the 2-torsion points *of the given log elliptic curve. Fix a metric on $\mathcal{L}_{\mathbf{C}} \stackrel{\mathrm{def}}{=} \mathcal{L} \otimes_{\mathbf{Z}} \mathbf{C}$ whose curvature on the fibers of $E_{\mathbf{C}} \to S_{\mathbf{C}}$ is* translation-invariant. *Then there is a* natural action of Π_S on the metrized vector bundle

$$\mathcal{H}_{\mathrm{DR}} \stackrel{\mathrm{def}}{=} (f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\infty, \mathrm{et}\}$$

on S which is integral (over S) except for possible poles contained in the divisor

$$[\eta \bigcap ({}_dE)] + V(4)$$

and preserves the "étale metric" $|| \sim ||_{\mathrm{et}}$ (cf. Chapter VIII, Theorem A) determined by the chosen metric on $\mathcal{L}$.

Next, recall that the metrized vector bundle $\mathcal{H}_{\mathrm{DR}}$ is equipped with a natural filtration, which we refer to as the *Hodge filtration.* The subquotients of this filtration admit natural isomorphisms (cf. Chapter VIII, Theorem A, (3)):

$$(F^{j+1}/F^j)(\mathcal{H}_{\mathrm{DR}}) \cong \frac{1}{j!} \cdot \exp(-(\mathbf{a}_\iota)_j) \cdot (f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}}) \otimes_{\mathcal{O}_S} \tau_E^{\otimes j}$$

for $j = 0, \ldots, d-1$. Here, the "$\exp(-(\mathbf{a}_\iota)_j)$" are powers of the q-parameter, as in Chapter VIII, Theorem A, (3), i.e.,

$$\exp((\mathbf{a}_\iota)_j) = q^{\approx \frac{j^2}{8d}}$$

Let us write

$$\mathfrak{Filt}(\mathcal{H}_{\mathrm{DR}})$$

for the *flag variety* over S of filtrations of $\mathcal{H}_{\mathrm{DR}}$ which are of the same "type" (i.e., each "F^j" has the same rank as $F^j(\mathcal{H}_{\mathrm{DR}})$) as the Hodge filtration. (Thus, each fiber of $\mathfrak{Filt}(\mathcal{H}_{\mathrm{DR}}) \to S$ is noncanonically isomorphic to the flag variety of flags $F^1(V) \subseteq F^2(V) \subseteq \ldots F^{d-1}(V) \subseteq F^d(V) = V$ in a d^2-dimensional vector space V for which $\dim(F^j(V)) = j \cdot d$.) Note that if we *apply the action of* Π_S *on* $\mathcal{H}_{\mathrm{DR}}$ *to the Hodge filtration*, we thus obtain a morphism

$$\kappa_E^{\mathrm{arith}} : \Pi_S \to \mathfrak{Filt}(\mathcal{H}_{\mathrm{DR}})(S)$$

(where the "(S)" denotes the "S-valued points"). Note that

> *The integrality statements of Theorem 3.3 (at both the finite and infinite primes) imply that the image of each element $\gamma \in \Pi_S$ under $\kappa_E^{\mathrm{arith}}$ is a filtration $\{\gamma(F^j(\mathcal{H}_{\mathrm{DR}})\}$ such that (up to denominators contained in the divisor $[\eta \bigcap ({}_dE)]+V(4)$) each $\gamma(F^j(\mathcal{H}_{\mathrm{DR}})\}$ is globally isomorphic (i.e., isomorphic as a vector bundle on S, equipped with a metric over $S_{\mathbf{C}}$) to $F^j(\mathcal{H}_{\mathrm{DR}})$.*

Note, further that the construction of $\kappa_E^{\mathrm{arith}}$ is *entirely analogous* to the construction of the "group-theoretic Kodaira-Spencer morphism"

$$\kappa_{SL_2}^{\mathrm{func}} : SL_2(\mathbf{R}) \to \mathfrak{Filt}(\mathrm{Holom}^{\mathrm{Poly}}(\widetilde{E}^{\dagger}))$$

of §1, as well as to the morphism

$$\kappa_R^{\mathrm{arith},p} : \Gamma_R \to \tau_E^{\otimes 2}(1)_{\widehat{\overline{R}}_{\mathbf{Q}_p}}$$

of §2.

Definition 3.4. We shall refer to

$$\kappa_E^{\mathrm{arith}} : \Pi_S \to \mathfrak{Filt}(\mathcal{H}_{\mathrm{DR}})(S)$$

as the *arithmetic Kodaira-Spencer morphism associated to* $E \to S$.

Example 3.5. Suppose, for instance, that

$$S = \mathrm{Spec}(\mathcal{O}_K)$$

where $\mathcal{O}_K$ is the ring of integers of a *number field* K (i.e., $[K : \mathbf{Q}] < \infty$). Suppose, moreover, that the log structure of $S^{\log}$ is defined by the divisor $D_{\mathrm{red}} \subseteq S$ (where D is the "divisor at infinity" associated to $C^{\log} \to S^{\log}$). In this example, let us assume that

$$d \geq 12$$

and write $d_2 \stackrel{\mathrm{def}}{=} 2^{\mathrm{ord}_2(d)}$ (i.e., d_2 divides d and d/d_2 is odd). Here, let us take $m \stackrel{\mathrm{def}}{=} 2d_2$, $n = 4d_2$. Suppose that the elliptic curve E_K over K is *sufficiently close to infinity at all the archimedean primes*, in the following sense:

> For each embedding σ of K into $\mathbf{C}$, the resulting elliptic curve E_σ (over $\mathbf{C}$) can be written as $\mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$, where $q = \exp(2\pi i \tau)$ satisfies:
>
> $$\mathrm{Im}(\tau) \geq 200\{\log^2(d) + 4d_2 \cdot \log(d) + 4d_2 \cdot \log(4d_2)\}$$

(Thus, if $d >> d_2$ (for instance, if d is odd, in which case $d_2 = 1$), then the lower bound on $\mathrm{Im}(\tau)$ goes roughly as $\log^2(d)$.) Note that it is not difficult, for a fixed d, to construct lots of examples of E_K satisfying this condition at the archimedean primes.

Now it follows from Chapter VIII, Theorem A, (4), (C), that:

$$e^{-33d} \cdot || \sim ||_{\mathrm{qCG}} \leq \frac{1}{4d_2} \cdot e^{-32d} \cdot || \sim ||_{\mathrm{qCG}} \leq || \sim ||_{\mathrm{et}} \leq e^{4d} \cdot || \sim ||_{\mathrm{qCG}}$$

Thus, in particular, it follows that if we include both the poles at the finite primes (cf. Theorem 3.3; the first Remark following Chapter VI, Theorem 4.1) as well as the poles at the archimedean primes, then:

> *The global divisor of poles of the action on $\mathcal{H}_{\mathrm{DR}}$ of an element of Π_S has Arakelov-theoretic degree (cf. Chapter I, §1) $\leq 38d \cdot [K : \mathbf{Q}]$, and is concentrated at the archimedean primes, and the primes that divide $m = 2d_2$.*

(where the "38" is the sum of the "4" and the "33" appearing in the inequalities above, plus "1" more, to take care of the contribution from the finite primes). Put another way, the action of each element of Π_S maps $\mathcal{H}_{\mathrm{DR}}$, whose subquotients are given by

$$\frac{1}{j!} \cdot \exp(-(\mathbf{a}_\iota)_j) \cdot \tau_E^{\otimes j}$$

(tensored with a "common factor" of $(f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}})$) to a module whose subquotients are given by

$$\frac{1}{j!} \cdot \exp(-(\mathbf{a}_\iota)_j) \cdot \tau_E^{\otimes j} \cdot (\text{poles})$$

(tensored with a "common factor" of $(f_S)_*(\overline{\mathcal{L}}|_{E_{\infty,S}})$) — where the "(poles)" admit bounds as described in the italicized caption above — in a fashion which is *integral at all the primes (both archimedean and non-archimedean) of the number field* K.

Remark. It is thus tempting, relative to the analogue with the Kodaira-Spencer morphism in the geometric case, to try to apply this theory to obtain bounds on the height of an elliptic curve over a number field, *as conjectured by Szpiro and Vojta* (cf., e.g., [Lang], [Vojta]). There are, unfortunately, a number of technical difficulties here. Perhaps the most obvious is the factors of "$\exp(-(\mathbf{a}_\iota)_j)$" that appear in the subquotients – we refer to these factors as *Gaussian poles* (since they grow like a Gaussian distribution) – since *these factors distort one's ability to compute degrees as one would like.* That is to say, by analogy to the geometric case, ideally one wishes for only the factor "$\tau_E^{\otimes j} \cdot$ (poles)" to appear in these subquotients. Relative to this point of view, *poles of the order* "$C^j \approx C^d$" (as $j \to d$) *are not a substantial problem* (at least if one concentrates on the subquotients where j is close to d), since this just adds an extra constant term to the Arakelov-theoretic degree of τ_E. From this point of view, one might a priori think that the factors of $\frac{1}{j!}$ are also a cause for concern (since they are not of the order C^d). Note that this is why, in the theory of Chapter VIII – cf. especially the statement concerning analytic torsion and the product formula at the end of Chapter VIII, Theorem A – we were so concerned with making sure that the "denominators" that occurred in the theory at the archimedean primes were of the order "C^d" (i.e., *even* when one takes the factors of $\frac{1}{j!}$ mentioned above into account).

Another technical difficulty (by comparison to the geometric case) is the *general nonlinearity* that appears – e.g., of Π_S, by comparison to the tangent bundle in the geometric case; of the action of Π_S on $\mathcal{H}_{\mathrm{DR}}$, which does not appear to arise from, say, taking symmetric powers of the some action on $F^2(\mathcal{H}_{\mathrm{DR}})$, as in the geometric case. Perhaps this nonlinearity is not surprising in view of the general phenomenon that Arakelov theory tends to give rise to nonlinear objects which are analogous to linear objects in the classical geometric theory (for instance, the global integral sections of a line bundle in Arakelov theory are not, in general, closed under addition).

Yet another technical difficulty is the fact that in Example 3.5 above, we needed to assume that E_K is *rather close to infinity at the archimedean primes.*

It is the hope of the author that these technical difficulties can be overcome in a future paper.

Appendix:
Formal Uniformization of Smooth Abelian Group Schemes

In this Appendix, we review various well-known facts concerning the "exponential map" of an abelian group scheme.

Let S be a noetherian scheme. Let

$$f : G \to S$$

be a *smooth, abelian group scheme* with identity section $e : S \hookrightarrow G$. Observe that the morphism

$$G \times_S G \to G \times_S G$$

defined on T-valued points (where T is an S-scheme) $g, h \in G(T)$ by $(g, h) \mapsto (gh, g)$ induces an isomorphism of the diagonal embedding $G \hookrightarrow G \times_S G$ with the embedding $G \hookrightarrow G \times_S G$ given by $g \mapsto (g, e)$ (where e is the identity element). Since the sheaf of differentials $\Omega_{G/S}$ on G is the conormal bundle of the former embedding, we thus obtain a natural isomorphism

$$f^* e^* \Omega_{G/S} \cong \Omega_{G/S}$$

Write

$$\Omega \stackrel{\text{def}}{=} e^* \Omega_{G/S}$$

Thus, Ω is a locally free sheaf on S, and the above isomorphism induces a natural morphism $\Omega \to f_* \Omega_{G/S}$. The differentials in the image of this morphism are called *invariant differentials on G*.

Now let

$$\mathcal{A}$$

be the "pro-algebra" (i.e., inverse limit of finite locally free $\mathcal{O}_S$-algebras) on S given by taking the *completed PD-envelope in G of e* (cf., e.g., [BO] for more details). Thus, $\mathcal{A}$ is equipped with a natural augmentation $\mathcal{A} \to \mathcal{O}_S$ ("evaluation at e") whose kernel $\mathcal{I}$ is a PD-ideal of $\mathcal{A}$. If n is a positive integer, let us write $\mathcal{I}^{[n]}$ for the n^{th} divided power of $\mathcal{I}$. Thus,

$$\mathcal{A}_n \stackrel{\text{def}}{=} \mathcal{A}/\mathcal{I}^{[n]}$$

is a finite locally free $\mathcal{O}_S$-algebra, and $\mathcal{I}^{[n]}/\mathcal{I}^{[n+1]}$ is a finite locally free $\mathcal{O}_S$-module. Moreover, $\mathcal{A}$ is complete with respect to the *filtration*

$$\ldots \subseteq \mathcal{I}^{[n]} \subseteq \ldots \subseteq \mathcal{I}^{[2]} \subseteq \mathcal{I}$$

Intuitively speaking, if $x_1, \ldots, x_r$ are a complete set of local parameters on G (over S) at e, then $\mathcal{A}$ may be described as the ring of *formal power series in divided powers* of the x_i with coefficients in $\mathcal{O}_S$. Moreover, note that $\mathcal{A}$ may be regarded as also being equipped with an $\mathcal{O}_G$-algebra structure in the sense that one has a coherent system of natural morphisms $\mathrm{Spec}(\mathcal{A}_n) \to G$ ("Taylor expansion at e to order n of functions on G"). Thus, if, for instance, $\mathcal{F}$ is a locally free coherent sheaf on G, it makes sense to write $\mathcal{A} \otimes_{\mathcal{O}_G} \mathcal{F}$. Moreover, $\mathcal{A} \otimes_{\mathcal{O}_G} \mathcal{F}$ will be a locally free $\mathcal{A}$-module of finite rank. In the following, we shall denote $\mathcal{A} \otimes_{\mathcal{O}_G} \wedge^i \Omega_{G/S}$ by $\Omega^i_{\mathcal{A}}$.

Sometimes, we shall want to consider the completed PD-envelope of G at an S-valued point $x \in G(S)$ of G which is not equal to e. Note, however, that *translation by* x induces an automorphism

$$T_x : G \to G$$

which maps the completed PD-envelope of G at e (i.e., $\mathcal{A}$) isomorphically onto the completed PD-envelope of G at x. Thus, in the following, we shall identify these two PD-envelopes by means of this isomorphism.

Now observe that by formally differentiating power series, we obtain a de Rham complex of "PD-functions"

$$\mathcal{A} \xrightarrow{d} \Omega^1_{\mathcal{A}} \xrightarrow{d} \Omega^2_{\mathcal{A}} \xrightarrow{d} \ldots$$

Composing the morphism $\Omega \to f_*\Omega_{G/S}$ constructed above with restriction to $\Omega^1_{\mathcal{A}}$ then gives us a morphism

$$\iota : \Omega \to \Omega^1_{\mathcal{A}}$$

Now we have the following well-known result:

Lemma A.1. *Suppose that the map $\mathcal{O}_S \to \mathcal{O}_S$ given by multplication by 2 is injective. Then the composite of ι with the exterior derivative d is zero.*

Proof. Since G is abelian, it follows that the automorphism $\nu : G \to G$ given by inversion is a *group homomorphism*. Note that ν induces natural actions ν^* on Ω and on the $\Omega^i_{\mathcal{A}}$. Indeed, ν^* acts as -1 on Ω and as $(-1)^i$ on $\Omega^i_{\mathcal{A}}$. Since d and ι are *natural*, ν^* acts on elements in the image of $d \circ \iota$ by multiplication by -1. On the other hand, since these elements are sections of $\Omega^2_{\mathcal{A}}$, ν^* must act on these elements as the identity. Thus, it follows that $\mathrm{Im}(d \circ \iota)$ is annihilated by multiplication by 2. On the other hand, since $\Omega^2_{\mathcal{A}}$ is an inverse limit of locally free coherent $\mathcal{O}_S$-modules, it follows from the assumption of the Lemma that $\mathrm{Im}(d \circ \iota) = 0$, as desired. ◯

Let us assume for the rest of this Appendix that the assumption of Lemma A.1 is in force. Thus, the differentials in the image of ι are closed, and hence, by formal integration, *exact* (see, e.g., [BO] for a discussion of the *Poincaré Lemma* in the context of PD-functions). Moreover, it is clear (from the Poincaré Lemma) that d induces an isomorphism:

$$d : \mathcal{I} \ (\subseteq \mathcal{A}) \cong \{\text{exact differentials of } \Omega^1_{\mathcal{A}}\}$$

Thus, we see that ι lifts to a natural morphism

$$\lambda : \Omega \hookrightarrow \mathcal{I}$$

whose composite with the projection to $\mathcal{I}/\mathcal{I}^{[2]} = \Omega$ is the identity.

Definition A.2. Suppose that S is a noetherian scheme such that multiplication by 2 on $\mathcal{O}_S$ is injective. Let $G \to S$ be a smooth, abelian group scheme over S. Then we shall refer to the natural morphism

$$\lambda : \Omega \hookrightarrow \mathcal{I}$$

just constructed as the *logarithmic uniformization (morphism) of* G.

Proposition A.3. *The morphism $\lambda : \Omega \hookrightarrow \mathcal{I}$ is functorial (in the obvious sense) with respect to homomorphisms $G \to H$ of smooth, abelian group schemes over S.*

Proof. Indeed, λ is determined by its composite with the exterior differential operator d. Thus, it suffices to observe that the morphism $\Omega \to f_*\Omega_{G/S}$ constructed above is functorial in G, but this is clear from its construction. ◯

Example A.4. Suppose that

$$G = (\mathbf{G}_{\mathrm{m}})_S$$

i.e., the *multiplicative group scheme* over S. One may think of G as the spectrum of $\mathcal{O}_S[U, U^{-1}]$, where U is an indeterminate. Then the image of the natural morphism

$\Omega \to f_*\Omega_{G/S}$ constructed above is generated by dU/U. Moreover, $\lambda(dU/U)$ is the *natural logarithm*

$$\log(U) \stackrel{\text{def}}{=} -\sum_{i=1}^{\infty} \frac{(1-U)^i}{i}$$

Example A.5. Suppose that

$$G = (\mathbf{G}_a)_S$$

i.e., the *additive group scheme* over S. One may think of G as the spectrum of $\mathcal{O}_S[T]$, where T is an indeterminate, and $T = 0$ is the section "e." Then the image of the natural morphism $\Omega \to f_*\Omega_{G/S}$ constructed above is generated by dT. Moreover, $\lambda(dT)$ is just the function T.

Finally, before proceeding, we recall the analogue of the construction above *at the infinite prime*. Thus, let $\mathbf{C}$ denote the field of complex numbers. Let $\mathcal{G}$ be an *abelian complex Lie group*. Thus, $\mathcal{G}$ is "an abelian group object" in the category of complex (i.e., holomorphic) manifolds. Let Θ (respectively, Ω) denote the tangent (respectively, cotangent) space to $\mathcal{G}$ at the origin. Thus, Θ and Ω are finite-dimensional complex vector spaces which are dual to one another.

Now one knows from the elementary theory of complex Lie groups (see, e.g., [Vara]) that there exists a unique holomorphic morphism, called *the exponential map*

$$\exp_{\mathcal{G}} : \Theta \to \mathcal{G}$$

compatible with the additive structures of Θ and $\mathcal{G}$ and whose derivative at the origin is the identity map $\Theta \to \Theta$. Let us denote by $\mathcal{A}^{\text{hol}}$ the local ring of holomorphic functions in a neighborhood of the origin of $\mathcal{G}$. Thus, $\mathcal{A}^{\text{hol}}$ is equipped with an ideal

$$\mathcal{I}^{\text{hol}} \subseteq \mathcal{A}^{\text{hol}}$$

of functions vanishing at the origin of $\mathcal{G}$. Since $\exp_{\mathcal{G}}$ induces an isomorphism between the local rings of holomorphic functions in a neighbhorhood of the origin of Θ and $\mathcal{G}$, we thus see that the local coordinate functions (i.e., elements of Ω) on Θ define elements of $\mathcal{I}^{\text{hol}}$, i.e., we have a natural morphism

$$\lambda^{\text{hol}} : \Omega \hookrightarrow \mathcal{I}^{\text{hol}}$$

which is the holomorphic analogue of the algebraic λ constructed above in the following sense: Suppose $\mathcal{G}$ is the complex Lie group defined by some abelian group scheme G over

C. Then both $\mathcal{A}^{\text{hol}}$ and $\mathcal{A}$ embed naturally in the ring $\mathcal{A}^{\text{for}}$ of *formal functions* at the origin of G (i.e., the ring obtained by completing G at e). Write $\mathcal{I}^{\text{for}} \subseteq \mathcal{A}^{\text{for}}$ for the ideal of functions vanishing at the origin. Thus, since both $\mathcal{I}^{\text{hol}}$ and $\mathcal{I}$ may be regarded as subsets of $\mathcal{I}^{\text{for}}$, it follows that both λ^{hol} and λ define natural morphisms $\Omega \hookrightarrow \mathcal{I}^{\text{for}}$.

Proposition A.6. *These two morphisms $\Omega \hookrightarrow \mathcal{I}^{\text{for}}$ coincide.*

Proof. Indeed, it is clear (from the theory of the exponential map) that the exterior derivative d of the image of λ^{hol} gives rise to (the power series expansions at the origin of) invariant differentials on $\mathcal{G}$. Since d is injective on $\mathcal{I}^{\text{for}}$, this completes the proof. $\bigcirc$

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